Lecture 11

The goal of this lecture is to prove that the size-Ramsey number for the path P_t with t edges is linear in t. That is, there exists a constant c such that $\hat{r}(P_t) \leq ct$.

We begin with a well-known lemma of Pósa. To state it, suppose that, for a given vertex x_0, the path $P = x_0 x_1 \ldots x_h$ is a longest path starting from x_0 within a given graph G. A simple transform P' of P is a path of the form $P' = x_0 x_1 \ldots x_i x_h x_{h-1} \ldots x_{i+1}$. A transform of P is anything obtained by a sequence of simple transforms. Let U be the set of endpoints of transforms of P and set

$$N = \{x_i : 0 \leq i \leq h-1, \{x_{i-1}, x_{i+1}\} \cap U \neq \emptyset\},$$

and $R = V(P) - U \cup N$. Thus U is the set of final vertices, N is the set of neighbours and R the rest of the vertices. Pósa's lemma is now the following. In short, it says that the set U of final points doesn't really expand.

Lemma 1 *There are no edges between U and R.*

Proof: Suppose $x \in U$ and $xy \in E(G)$. Let P_x be a transform of P ending in x. Since $xy \in E(G)$, P_x has a simple transform P_z ending in a vertex z which is a neighbour of y in P_x.

If $yz \in E(P)$, then $y \in N$. Otherwise, an edge $yw \in E(P)$ had to be erased in one step of the sequence $P \to P' \to \cdots \to P_x$. When yw was erased for the first time, one of y or w became an endvertex and, therefore, $\{y, w\} \subset U \cup N$. Hence $y \in U \cup N$. □

This yields, as a corollary, the following simple condition for a graph to contain a long path. Basically, if every small set expands, then, in light of Lemma 1, the set of endpoints must be quite large. This then implies that the length of the longest path is quite large.

Lemma 2 *Suppose $u \geq 1$ is such that for every set $U \subset V(G)$, $|U| \leq u$,*

$$|U \cup N(U)| \geq 3|U|.$$

Then G contains a path of length at least $3u - 1$.

Proof: Let $P = x_0 x_1 \ldots x_h$ be a longest path in G. By Lemma 1, there are no edges between U and R. Note also that, because P is a longest path, there are no edges between U and $V \backslash V(P)$. Therefore, $N(U) \subset U \cup N$, implying

$$|U \cup N(U)| \leq 3|U| - 1,$$

since

$$U \cup N \subset U \cup \{x_{h-1}\} \bigcup_{x_j \in U, j<h} \{x_{j-1}, x_{j+1}\}.$$

This implies that $|U| > u$. Choose a $U_0 \subset U$ with $|U_0| = u$. Then, by assumption,

$$h \geq |U_0 \cup N(U_0)| \geq 3u,$$

as required. □

For any two subsets $X, Y \subset V = V(G)$, let $e_G(X,Y)$ be the number of edges between X and Y. Let

$$d_G(X) = \left(\frac{1}{|X|}\right) e_G(X, V).$$

This is distinct from $d(G)$, which is the average degree of G. Note that, if $X = \{x\}$, $d(X)$ is the degree of x and, for $X = V$, $d(V) = \frac{e(G)}{|V|} = \frac{1}{2}d(G)$. The next lemma says that any graph G has a subgraph H for which $d_H(X)$ is at least half the average degree for every set $X \subset V(H)$.

Lemma 3 *Every graph G contains a subgraph H for which*

$$d_H(X) \geq \frac{1}{2}d(G)$$

for every non-empty subset X of $V(H)$.

Proof: Let $W \subset V$ be a minimal non-empty subset satisfying $e(G[W]) \geq \frac{1}{2}d(G)|W|$ and set $H = G[W]$. Take a non-empty proper subset $X \subset W$ and let $Y = W \backslash X \neq \emptyset$. Then

$$\begin{aligned} d_H(X) &= \frac{1}{|X|}\left(e(H) - e(H[Y])\right) > \frac{1}{|X|}\left(e(H) - \frac{1}{2}d(G)|Y|\right) \\ &\geq \frac{1}{|X|}\left(\frac{1}{2}d(G)|W| - \frac{1}{2}d(G)|Y|\right) = \frac{1}{2}d(G). \end{aligned}$$

□

The next lemma says that if between any two small sets $X \subset Y \subset V$, the density is quite small, then any subgraph G' of G with at least half the edges of G contains long paths. It works by first applying Lemma 3 to find a subgraph H of G' where every subset X has large average degree. This implies that $U \cup N_H(U)$ must always be large. Then one can apply Lemma 2 to show that H contains a long path.

Lemma 4 *Suppose G is such that if $X \subset Y \subset V(G)$ and $|Y| \leq 3|X| - 1 \leq 3u - 1$, then*

$$e(X,Y) < \frac{1}{4}d(G)|X|.$$

Let G' be a subgraph of G with $e(G') \geq \frac{1}{2}e(G)$. Then G' contains a path of length $3u-1$.

Proof: By Lemma 3, there is a subgraph H of G' with

$$d_H(X) \geq \frac{1}{2}d(G') \geq \frac{1}{4}d(G)$$

for every non-empty $X \subset V(H)$. Let $U \subset V(H)$ with $|U| \leq u$. Then

$$|U \cup N_H(U)| \geq 3|U|.$$

Suppose otherwise. Let $X = U$ and $Y = U \cup N_H(U)$. Then

$$d_H(X)|X| = e_H(X,Y) \leq e(X,Y) < \frac{1}{4}d(G)|X|,$$

which is a contradiction. Therefore, by Lemma 2, H contains a path of length at least $3u - 1$. □

All that's left to prove is that there are sparse random graphs satisfying the requirement of Lemma 4.

Theorem 1 *There exists a constant c such that*

$$\hat{r}(P_t) \leq ct.$$

Proof: We will consider the probability space $G(n,p)$, where $p = \frac{c}{n}$, for c to be chosen later. Let d be such that $16 \leq 4d < c < c' < 4.001d$. Note that, almost surely, $d(G_p) \geq 4d$ and $e(G_p) \leq c'n/2$. Moreover, let $0 < \alpha < 0.03$, $\rho > c'/6\alpha$, $u = \lfloor \alpha n \rfloor$ and $t = 3u - 1$.

Our aim is to show that, with high probability, for all $X \subset Y \subset V$, $|Y| = 3|X| = 3s \leq 3u$, we have $e(X,Y) < ds$. Since either the red or blue graph contains half the edges of G, Lemma 4 will then imply that the graph contains a monochromatic path of length t and so $\hat{r}(P_t) \leq \rho t$.

For each $1 \leq s \leq u$, let Z_s be the random variable counting the number of pairs (X,Y) with $X \subset Y \subset V$, $|Y| = 3|X| = 3s \leq 3u$ and $e(X,Y) \geq ds$. We want to prove that

$$\mathbb{P}\left(\sum_{s=1}^{u} Z_s \geq 1\right) = o(1).$$

For a fixed pair (X,Y), $X \subset Y$, $|Y| = 3|X| = 3s \leq 3u$, the number of edges between X and Y is binomial with probability p and $s^* = \binom{s}{2} + 2s^2$ possible edges. Therefore,

$$\begin{aligned}
\mathbb{E}(Z_s) &\leq \binom{n}{s}\binom{n}{2s}\binom{5s^2/2}{ds} p^{ds}(1-p)^{5s^2/2-ds} \\
&\leq c_0 s^{-3/2}\left(\frac{n}{s}\right)^s \left(\frac{n}{2s}\right)^{2s}\left(\frac{n}{n-3s}\right)^{n-3s}\left(\frac{5esc}{2dn}\right)^{ds} e^{-5s^2c/2n} \\
&\leq c_0 s^{-3/2}(s/n)^{(d-3)s} 2^{-2s} e^{(d+3)s}(10.01)^{ds} e^{-5s^2c/2n}.
\end{aligned}$$

Hence,

$$\mathbb{E}(Z_s) \leq c_0 \left(sn^{-1}(10.01)^{d/(d-3)} e^{(d+3)/(d-3)} 2^{-2/(d-3)}\right)^{(d-3)s}.$$

With $c = 72.001$, $c' = 72.002$, $d = 18$, $\alpha = 1/59$ and $\rho = 720$, we find that the base is bounded away from one. Therefore,

$$\sum_{s=1}^{u} \mathbb{E}(Z_s) = o(1),$$

as required. □

It was an open question for some time to determine whether a similar theorem held for graphs of maximum degree 3. Rödl and Szemerédi showed that this is not the case by finding a 3-regular graph H for which $\hat{r}(H) \geq cn(\log n)^{c'}$. On the other hand, it was shown recently that for any graph H with maximum degree Δ, $\hat{r}(H) \leq n^{2-1/\Delta}(\log n)^{1/\Delta}$. This difficult result is due to Kohayakawa, Rödl, Schacht and Szemerédi. A very large gap still remains between the upper and lower bounds.