Lecture 10

The size-Ramsey number $\hat{r}(H)$ is the smallest natural number m such that there exists a graph G with m edges which is Ramsey with respect to H, that is, such that any two-colouring of the edges of G produces a monochromatic copy of H. For $H = K_t$, we simply write $\hat{r}(t)$.

The following theorem, attributed to Chvátal, says that the size-Ramsey number of the complete graph K_t is at least the number of edges in the complete graph on $r(t)$ vertices.

Theorem 1

$$\hat{r}(t) = \binom{r(t)}{2}.$$

Proof: Fix a colouring C of the complete graph on vertices $\{1, 2, \ldots, r(t)-1\}$ which does not contain a monochromatic K_t. Suppose that G is a graph which is Ramsey with respect to K_t. If the chromatic number $\chi(G)$ of G is smaller than $r(t)$, then we may colour edges of G by mapping uv to the edge $\chi(u)\chi(v)$ in the complete graph on vertices $\{1, 2, \ldots, r(t) - 1\}$ and colouring the edge with the colour of $\chi(u)\chi(v)$ in C. This colouring of G does not contain a monochromatic K_t and so we may assume that $\chi(G) \geq r(t)$. But if this is the case, G must have at least $\binom{r(t)}{2}$ edges. □

Consider the following game between two players, Builder and Painter. Builder draws edges one at a time and Painter colours them, in either red or blue, as each appears. Builder's aim is to force Painter to draw a monochromatic copy of a fixed graph H. The minimum number of edges which Builder must draw, regardless of Painter's strategy, in order to guarantee that this happens is known as the on-line Ramsey number $\tilde{r}(H)$ of H. For $H = K_t$, we simply write $\tilde{r}(t)$.

The following theorem says that infinitely often the on-line Ramsey number $\tilde{r}(t)$ is exponentially smaller than $\binom{r(t)}{2}$. This is in sharp contrast with Theorem 1.

Theorem 2 *For infinitely many t,*

$$\tilde{r}(t) \leq 1.001^{-t}\binom{r(t)}{2}.$$

Proof: Let $\alpha = 0.01$. Consider the following strategy. To begin, Builder draws $n-1$ edges emanating from a single vertex v_1. Painter must paint at least $(n-1)/2$ of these the same colour. Let V_1 be the neighbourhood of v_1 in such a colour. We also define a string s in terms of the colours chosen. We initialise this string by writing $s = R$ if the majority colour was red and $s = B$ if it were blue.

Suppose now that we are looking at a set V_i. We choose any given vertex v_{i+1} and draw all the neighbours of v_{i+1} within V_i. If, in the string s, there are more Rs than Bs, we choose V_{i+1} to be the neighbourhood of v_{i+1} in red if $|V_{i+1}| \geq (1-\alpha)(|V_i| - 1)$ and the neighbourhood in blue otherwise. Similarly, if there are more Bs than Rs, one chooses V_{i+1} to be the neighbourhood of v_{i+1} in blue if and only if $|V_{i+1}| \geq (1-\alpha)(|V_i| - 1)$. If the number of Rs and Bs in the string are the same, then we follow whichever has more neighbours. The string s then has whichever colour we followed appended to it. If, for example, our string were $a_1 \cdots a_i$, with each $a_j = R$ or B, and we followed red, the new string would be $a_1 \cdots a_i R$.

Let $\mu = 0.99$ and $\nu = 0.01$. The process stops when the string s contains either μt Rs, μt Bs or νt Rs and νt Bs. Suppose, at that stage, that we have chosen m vertices and m neighbourhoods. Builder's strategy now is to fill in a complete subgraph of V_m of size p equal to the maximum of $r((1-\mu)t,t)$ and $r((1-\nu)t)$. It is easy to see that Painter will be forced to draw a complete graph in one colour or the other. Suppose, for example, that our string contains μt blues. Since $p \geq r(t,(1-\mu)t)$, V_m contains either a red clique of size t, in which case we are done, or a blue clique of size $(1-\mu)t$. This latter clique may be appended to the μt vertices which correspond to B in the string to form a blue clique of size t. The other cases follow similarly.

We need now to estimate the number of edges that Builder has to draw. In order to guarantee that the process works, we need to start with an n which will guarantee that $|V_m| \geq p$. If we made the most expensive choice at each point as we were choosing the v_i, we may have to choose n to be as large as $(2/\alpha)^{\nu t}(1-\alpha)^{-\mu t}p$. Since $m \leq (\mu+\nu)t = t$, it is then elementary to see that the number of edges Builder draws is at most

$$mn + \binom{p}{2} \leq t(2/\alpha)^{\nu t}(1-\alpha)^{-t}p + \binom{p}{2}.$$

To estimate the value of this expression, we must first understand something about p. By the choice of μ and the standard bound $r(s,t) \leq \binom{s+t}{s}$,

$$\begin{aligned} r((1-\mu)t,t) = r(0.01t,t) &\leq \binom{1.01t}{t} \leq \left(\frac{1.01et}{0.01t}\right)^{0.01t} \\ &\leq 1.06^t \leq 1.25^{-t}r(t), \end{aligned}$$

since $r(t) \geq 2^{t/2}$.

On the other hand, there must be infinitely many values t for which

$$r((1-\nu)t) = r(0.99t) \leq 1.001^{-t}r(t).$$

Suppose otherwise. Then there exists t_0 such that, for all $t \geq t_0$,

$$\frac{r(t)}{r(0.99t)} \leq 1.001^t.$$

By telescoping, this would imply that, for every positive integer A,

$$\begin{aligned} r(0.99^{-A}t_0) &\leq (1.001)^{(0.99^{-1}+\cdots+0.99^{-A})t_0}r(t_0) \\ &\leq (1.001)^{100(0.99)^{-A}t_0}r(t_0). \end{aligned}$$

If we rewrite this equation, with $t = 0.99^{-A}t_0$ and $C = r(t_0)$, we see that this would imply

$$r(t) \leq C(1.001)^{100t} \leq C(1.106)^t,$$

which, since $r(t) \geq 2^{t/2}$, is plainly a contradiction for large t.

We may therefore conclude that $p \leq 1.001^{-t}r(t)$ infinitely often. At such values of t we have that the number of edges Builder must draw to force Painter to draw a monochromatic K_t is less than

$$\begin{aligned} t(2/\alpha)^{\nu t}(1-\alpha)^{-t}p + \binom{p}{2} &\leq t(200)^{0.01t}(0.99)^{-t}p + \binom{p}{2} \\ &\leq t(1.066)^t p + \binom{p}{2} \leq \frac{r(t)-1}{4}p + \binom{p}{2} \\ &\leq 1.001^{-t}\binom{r(t)}{2}, \end{aligned}$$

provided that t is sufficiently large. $\square$

For complete bipartite graphs, the size-Ramsey number is known to within surprisingly accurate bounds:

$$c' t^2 2^t \leq \hat{r}(K_{t,t} \leq c t^3 2^t.$$

While the upper bound may be improved to $\tilde{r}(K_{t,t}) \leq c 2^t t^{5/2} \log^{1/2} t$ for on-line Ramsey numbers, the only lower bound known is $\tilde{r}(K_{t,t}) \geq \frac{r(K_{t,t})-1}{2} \geq ct\sqrt{2}^t$. This same problem arises for most on-line Ramsey numbers and it remains a major open question to show that $\tilde{r}(t)$ is exponentially larger than $\sqrt{2}^t$.

For simpler graphs like the path, it is easy to show that $\tilde{r}(P_t)$ is linear in t. What is perhaps more surprising is that the same is true for the size-Ramsey number $\hat{r}(P_t)$. We shall prove both these results in the next lecture.