Graph Ramsey Theory

David Conlon

Lecture 1

Definition The Ramsey number $r(t)$ is the smallest natural number n such that, in any 2-colouring of the edges of the complete graph K_n, there is guaranteed to be a monochromatic copy of K_t.

Theorem 1

$$r(t) \leq 2^{2t-3}.$$

Proof: Suppose that we have a complete graph on $n = 2^{2t-3}$ vertices whose edges have been 2-coloured, in red and blue, say. Fix a vertex v_1. By the pigeonhole principle, either the red or blue neighborhood of v_1 has size 2^{2t-4}. We let V_1 be this neighborhood. Moreover, we initiate a string S, letting it be R if it was the red neighborhood or B if it was blue.

At step i, we have a set V_{i-1} of size 2^{2t-3-i}. Fix a vertex $v_i \in V_{i-1}$. By the pigeonhole principle, either the red or blue neighborhood of v_1 has size $2^{2t-3-(i+1)}$. Let V_i be this neighborhood. Moreover, depending on whether the neighborhood was red or blue, we append an R or a B to the string, which now reads $S = a_1 a_2 \dots a_i$, where each a_j is either R or B.

We terminate this process after $2t-3$ steps. Note that $|V_{2t-3}| \geq 1$ and let v_{2t-2} be an element of this set. Also, at this stage, the string S has length $2t-3$, so there must be some subsequence of length $t-1$ which consists only of Rs or only of Bs. Suppose that $a_{j_1}, a_{j_2}, \dots, a_{j_{t-1}}$ is this sequence and that it consists entirely of Rs. We claim that $\{v_{j_1}, v_{j_2}, \dots, v_{j_{t-1}}\} \cup \{v_{2t-2}\}$ is a monochromatic clique of size t. Clearly it has size t. Moreover, by the choice of $a_{j_i} = R$ each v_{j_i} is connected to every future vertex in red. This completes the proof. □

To sharpen this bound, we need to expand our horizons and consider the following definition.

Definition The Ramsey number $r(s,t)$ is the smallest natural number n such that, in any 2-colouring of the edges of the complete graph K_n, in red and blue, say, there is guaranteed to be a red copy of K_s or a blue copy of K_t.

Lemma 1

$$r(s+1, t+1) \leq r(s, t+1) + r(s+1, t).$$

Proof: Suppose that the edges of K_n have been 2-coloured, in red and blue, say, so that there is no red clique of size s and no blue clique of size t. Then, for a given vertex v, the size of its red neighborhood is at most $r(s, t+1) - 1$. For if it were $r(s, t+1)$, the graph would contain either a blue

K_{t+1}, in which we would be done, or a red K_s. But adding the vertex v to this clique would give a red K_{s+1}. Similarly, the blue neighborhood of v is at most $r(s+1,t)-1$. Therefore, we must have

$$n \leq 1 + (r(s,t+1)-1) + (r(s+1,t)-1) = r(s,t+1) + r(s+1,t) - 1.$$

This implies the required result. □

In particular, this lemma implies that $r(3,3) \leq 6$. To see that this is sharp, consider the cycle on five vertices. Its complement is also a five-cycle, so neither contains a triangle. The lemma also implies that $r(3,4) \leq 10$, but this is not sharp. To improve this bound, note that any vertex in a 2-coloured K_9 must have red degree 5 and blue degree 3. But this is not possible. You cannot have a graph with an odd number of vertices, where every vertex has odd degree. So there must be some vertex for which the red degree is 6 or the blue degree is 4. The proof then proceeds the same.

The following corollary, proved by Erdős and Szekeres in 1935, remained the state of the art for some fifty years.

Theorem 2

$$r(s+1,t+1) \leq \binom{s+t}{t}.$$

Proof: We prove this result by induction. Note that $r(s+1,2) = s+1 = \binom{s+1}{1}$. Similarly, the bound is sharp for $r(2,t+1)$. Suppose now that $r(i+1,j+1) \leq \binom{i+j}{i}$ for all (i,j) with either $i < s$ or $j < t$. Then, by Lemma 1, we have

$$\begin{aligned} r(s+1,t+1) &\leq r(s,t+1) + r(s+1,t) \\ &\leq \binom{s+t-1}{s-1} + \binom{s+t-1}{s} = \binom{s+t}{s}. \end{aligned}$$

□

Corollary 1

$$r(t) = O\left(\frac{4^t}{\sqrt{t}}\right).$$

Proof: This follows from an application of Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We get

$$r(t) \leq \binom{2t}{t} = \frac{(2t)!}{(t!)^2} \approx \frac{\sqrt{4\pi t}\left(\frac{2t}{e}\right)^{2t}}{2\pi t\left(\frac{t}{e}\right)^{2t}} = \frac{4^t}{\sqrt{\pi t}}.$$

□

The formula $r(s+1,t+1) \leq r(s,t+1) + r(s+1,t)$ is not the only elementary relation between Ramsey numbers that is known. In order to prove another, we need the following simple lemma, due to Goodman, which tells one how many monochromatic triangles you may expect in a 2-edge-colouring of K_n. For a graph G, let $d_G(v)$ be the degree of v in G.

Lemma 2 (Goodman's formula) *Suppose that the edges of K_n are 2-coloured, in red and blue. Let G be the red graph. Then the number of monochromatic triangles Δ is given by the formula*

$$\Delta = \frac{1}{2}\left(\sum_v \binom{d_G(v)}{2} + \sum_v \binom{n-1-d_G(v)}{2} - \binom{n}{3}\right).$$

Proof: Let $\overline{G}$ be the complement of G, that is, the blue graph. The formula that we are trying to prove says that

$$\Delta = \frac{1}{2}\left(\sum_v \binom{d_G(v)}{2} + \sum_v \binom{d_{\overline{G}}(v)}{2} - \binom{n}{3}\right).$$

To prove the formula, note that a red triangle will contribute 3 to $\sum_v \binom{d_G(v)}{2}$ and 0 to $\sum_v \binom{d_{\overline{G}}(v)}{2}$. Therefore, since every triangle contributes 1 to $\binom{n}{3}$, the total contribution to the sum is 1. Similarly, any blue triangle contributes 1 to the sum. A triangle with 2 red edges and 1 blue edge will contribute 1 to $\sum_v \binom{d_G(v)}{2}$ and 0 to $\sum_v \binom{d_{\overline{G}}(v)}{2}$, so the total contribution is zero. Similarly, triangles with 1 red edge and 2 blue edges will contribute zero to the sum. □

One corollary of this result is the following estimate for the number of monochromatic triangles. The random graph $G(n, \frac{1}{2})$ shows that it is essentially sharp.

Corollary 2 *Suppose that the edges of K_n are 2-coloured. Then the number of monochromatic triangles Δ satisfies*

$$\Delta \geq \frac{1}{4}\frac{n(n-1)(n-5)}{6} = \frac{1}{4}\binom{n}{3} + O(n^2).$$

Proof: For any given v, the sum $\binom{d_G(v)}{2} + \binom{n-1-d_G(v)}{2}$ is minimised by taking $d_G(v) = (n-1)/2$. Putting this into the formula yields

$$\begin{aligned}
\Delta &\geq \frac{1}{2}\left(2\sum_v \binom{(n-1)/2}{2} - \binom{n}{3}\right) \\
&= \frac{1}{2}\left(\frac{2n(\frac{n-1}{2})(\frac{n-3}{2})}{2} - \frac{n(n-1)(n-2)}{6}\right) \\
&= \frac{n(n-1)}{2}\left(\frac{n-3}{4} - \frac{n-2}{6}\right) \\
&= \frac{n(n-1)(n-5)}{24},
\end{aligned}$$

as required. □

The minimum number of K_4s in a 2-colouring is not so well-behaved, as observed by Thomason. In a random graph, one would expect 1/32 of the K_4s to be monochromatic. Thomason gives an example in which less than 1/33 of the K_4s are monochromatic. The best lower bound, due to Giraud, says that at least 1/46 of them are monochromatic.

With this corollary, we are now ready to prove the promised formula.

Corollary 3

$$r(t,t) \leq 4r(t,t-2) + 2.$$

Proof: Suppose that we have a 2-colouring of K_n which contains no monochromatic clique of size t. For every red edge ab, there are at most $r(t-2,t) - 1$ vertices v which are connected to both a and b by red edges. Otherwise, this set of vertices would contain either a blue K_t or a red K_{t-2}. But adjoining a and b to this K_{t-2} would yield a red K_t. Similarly, for every blue edge ab, there are at most $r(t,t-2)$ vertices v which are connected to both a and b by blue edges. Therefore, the number of monochromatic triangles is at most $\frac{1}{3}\binom{n}{2}(r(t,t-2)-1)$. Comparing this with Corollary 2, we see that

$$\frac{1}{4}\frac{n(n-1)(n-5)}{6} \leq \frac{1}{3}\binom{n}{2}(r(t,t-2)-1).$$

Simplifying, we see that $n \leq 4r(t,t-2)+1$. This implies the result. □

A careful application of these ideas was used by Thomason to prove a bound of the form $r(t) = O\left(4^t/t\right)$. The current best bound, due to the author, is

$$r(t) = t^{-c\frac{\log t}{\log\log t}} 4^t.$$

This uses ideas from the theory of quasirandomness.