Lower bounds for multicolor Ramsey numbers

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Abstract

We give an exponential improvement to the lower bound on diagonal Ramsey numbers for any fixed number of colors greater than two.

1 Introduction

The Ramsey number $r(t; ℓ)$ is the smallest natural number $n$ such that every ℓ-coloring of the edges of the complete graph $K_n$ contains a monochromatic $K_t$. For $ℓ = 2$, the problem of determining $r(t) := r(t; 2)$ is arguably one of the most famous in combinatorics. The bounds $\sqrt{2^t} < r(t) < 4^t$ have been known since the 1940s, but, despite considerable interest, only lower-order improvements [2, 7, 8] have been made to either bound. In particular, the lower bound $r(t) > (1 + o(1)) \frac{t}{\sqrt{2e}}\sqrt{2^t}$, proved by Erdős [3] as one of the earliest applications of the probabilistic method, has only been improved [8] by a factor of 2 in the intervening 70 years.

If we ignore lower-order terms, the best known upper bound for $ℓ ≥ 3$ is $r(t; ℓ) < ℓ^t$, proved through a simple modification of the Erdős–Szemerédi neighborhood-chasing argument [4] that yields $r(t) < 4^t$. For $ℓ = 3$, the best lower bound, $r(t; 3) > \sqrt{3^t}$, again comes from the probabilistic method. For higher $ℓ$, the best lower bounds come from the simple observation of Lefmann [5] that

$$r(t; ℓ_1 + ℓ_2) - 1 ≥ (r(t; ℓ_1) - 1)(r(t; ℓ_2) - 1).$$

To see this, we blow up an $ℓ_1$-coloring of $K_{r(t; ℓ_1) - 1}$ with no monochromatic $K_t$ so that each vertex set has order $r(t; ℓ_2) - 1$ and then color each of these copies of $K_{r(t; ℓ_2) - 1}$ separately with the remaining $ℓ_2$ colors so that there is again no monochromatic $K_t$. By using the bounds $r(t; 2) - 1 ≥ 2t/2$ and $r(t; 3) - 1 ≥ 3t/2$, we can repeatedly apply this observation to conclude that

$$r(t; 3k) > 3^{kt/2}, \quad r(t; 3k + 1) > 2^t3^{(k-1)t/2}, \quad r(t; 3k + 2) > 2^{t/2}3^{kt/2}.$$ 

Our main result is an exponential improvement to all these lower bounds for three or more colors.

Our principal contribution is the following theorem, proved via a construction which is partly deterministic and partly random. The deterministic part shares some characteristics with a construction of Alon and Krivelevich [1], in that we consider a graph whose vertices are vectors over a finite field where adjacency is determined by the value of their scalar product, while randomness comes in through both random coloring and random sampling.

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**Theorem 1.** For any prime $q$, $r(t; q + 1) > 2^{t/2}q^{3t/8+o(t)}$.

In particular, the cases $q = 2$ and $q = 3$ yield exponential improvements over the previous bounds for $r(t; 3)$ and $r(t; 4)$, both of which came from the probabilistic method (in fact, Lefmann’s observation gives an additional polynomial factor in the four-color case, but this is of lower order than the exponential improvements that are our concern).

**Corollary 2.** $r(t; 3) > 2^{7t/8+o(t)}$ and $r(t; 4) > 2^{t/2}3^{3t/8+o(t)}$.

For the sake of comparison, we note that the improvement for three colors is from $1.732^t$ to $1.834^t$, while, for four colors, it is from $2^t$ to $2.135^t$. Improvements for all $\ell \geq 5$ now follow from repeated applications of Lefmann’s observation, yielding

$$
r(t; 3k) > 2^{7kt/8+o(t)}, \quad r(t; 3k + 1) > 2^{(k-1)t/8+t/2}3^{3t/8+o(t)}, \quad r(t; 3k + 2) > 2^{7kt/8+t/2+o(t)},
$$

where we used, for instance,

$$r(t; 3k + 1) - 1 \geq (r(t; 3(k-1)) - 1)(r(t; 4) - 1) \geq (r(t; 3) - 1)^{k-1}(r(t; 4) - 1).$$

\section{Proof of Theorem 1}

Let $q$ be a prime. Suppose $t \neq 0 \mod q$ and let $V \subseteq \mathbb{F}_q^t$ be the set consisting of all vectors $v \in \mathbb{F}_q^t$ for which $\sum_{i=1}^tv_i^2 = 0 \mod q$, noting that $q^{t-2} \leq |V| \leq q^t$. Here the lower bound follows from observing that we may pick $v_1, \ldots, v_{t-2}$ arbitrarily and, since every element in $\mathbb{F}_q$ can be written as the sum of two squares, there must then exist at least one choice of $v_{t-1}$ and $v_t$ such that $v_{t-1}^2 + v_t^2 = -\sum_{i=1}^{t-2}v_i^2$.

We will first color all the pairs $\binom{V}{2}$ and then define a coloring of $E(K_n)$ by restricting our attention to a random sample of $n$ vertices in $V$. Formally:

**Coloring all pairs in $\binom{V}{2}$.** For every pair $uv \in \binom{V}{2}$, we define its color $\chi(uv)$ according to the following rules:

- If $u \cdot v = i \mod q$ and $i \neq 0$, then set $\chi(uv) = i$.
- Otherwise, choose $\chi(uv) \in \{q, q + 1\}$ uniformly at random, independently of all other pairs.

**Mapping $[n]$ into $V$.** Take a random injective map $f: [n] \to V$ and define the color of every edge $ij$ as $\chi(f(i)f(j))$.

Our goal is to upper bound the orders of the cliques in each color class.

**Colors** $1 \leq i \leq q - 1$. There are no $i$-monochromatic cliques of order larger than $t$ for any $1 \leq i \leq q - 1$. Indeed, suppose that $v_1, \ldots, v_s$ form an $i$-monochromatic clique. We will try to show that they are linearly independent and, therefore, that there are at most $t$ of them. To this end, suppose that

$$u := \sum_{j=1}^s \alpha_jv_j = 0$$
and we wish to show that $\alpha_j = 0 \mod q$ for all $j$. Observe that since $v_j \cdot v_j = 0 \mod q$ for all $j$ (our ground set $V$ consists only of such vectors) and $v_k \cdot v_j = i \mod q$ for each $k \neq j$, by considering all the products $u \cdot v_j$, we obtain that the vector $\tilde{\alpha} = (\alpha_1, \ldots, \alpha_s)$ is a solution to

$$M \tilde{\alpha} = 0$$

with $M = iJ - iI$, where $J$ is the $s \times s$ all 1 matrix and $I$ is the $s \times s$ identity matrix. In particular, we obtain that the eigenvalues of $M$ (over $\mathbb{Z}$) are $is - i$ with multiplicity 1 and $-i$ with multiplicity $s - 1$. Therefore, if $s \neq 1 \mod q$, the matrix is also non-singular over $\mathbb{Z}_q$, implying that $\tilde{\alpha} = 0$, as required. On the other hand, if $s = 1 \mod q$, we can apply the same argument with $v_1, \ldots, v_{s-1}$ to conclude that $s - 1 \leq t$. But, we cannot have $s - 1 = t$, since this would imply that $t = 0 \mod q$, contradicting our assumption. Therefore, we may also conclude that $s \leq t$ in this case.

**Colors $q$ and $q + 1$.** We call a subset $X \subseteq V$ a potential clique if $|X| = t$ and $u \cdot v = 0 \mod q$ for all $u, v \in X$. Given a potential clique $X$, we let $M_X$ be the $t \times t$ matrix whose rows consist of all the vectors in $X$. Observe that $M_X \cdot M_X^t = 0$, where we use the fact that each vector is self-orthogonal. First we wish to count the number of potential cliques and later we will calculate the expected number of cliques that survive after we color randomly and restrict to a random subset of order $n$.

Suppose that $X$ is a potential clique and let $r := \text{rank}(X)$ be the rank of the vectors in this clique, noting that $r \leq t/2$, since the dimension of any isotropic subspace of $\mathbb{F}_q^t$ is at most $t/2$. By assuming that the first $r$ elements are linearly independent, the number of ways to build a potential clique $X$ of rank $r$ is upper bounded by

$$\left(\prod_{i=0}^{r-1} q^{t-1-i}\right) \cdot q^{t(\ell-1)} = q^{t^2 - \frac{3t^2}{2} + \frac{1}{2}}.$$

Indeed, suppose that we have already chosen the vectors $v_1, \ldots, v_s \in X$ for some $s < r$. Then, letting $M_s$ be the $s \times t$ matrix with the $v_i$ as its rows, we need to choose $v_{s+1}$ such that $M_s \cdot v_{s+1} = 0$. Since the rank of $M_s$ is assumed to be $s$, there are exactly $q^{t-s}$ choices for $v_{s+1}$ in $\mathbb{F}_q^t$ and, therefore, at most that many choices for $v_{s+1} \in V$. If, instead, $s \geq r$, then we need to choose a vector $v_{s+1} \in \text{span}\{v_1, \ldots, v_r\}$ and there are at most $q^r$ such choices in $V$.

Now observe that the function $2tr - \frac{3t^2}{2} + \frac{1}{6}$ appearing in the exponent of the expression above is increasing up to $r = \frac{2t}{3} + \frac{1}{6}$, so the maximum occurs at $t/2$. Therefore, by plugging this into our estimate and summing over all possible ranks, we see that the number $N_t$ of potential cliques in $V$ is upper bounded by $q^{t^2 + o(t^2)}$.

The probability that a potential clique becomes monochromatic after the random coloring is $2^{1 - \left(\frac{t}{2}\right)}$. Suppose now that $p$ is such that $p|V| = 2n$ and observe that $p = nq^{-t+O(1)}$. If we choose a random subset of $V$ by picking each $v \in V$ independently with probability $p$, the expected number of monochromatic potential cliques in this subset is, for $n = 2^{t/2} q^{3t/8 + o(t)}$,

$$p^{2^{1 - \left(\frac{t}{2}\right)}} N_t \leq q^{t^2 + o(t^2)} n \cdot 2^{-\frac{3t^2}{2} + o(t^2)} q^{\frac{5t^2}{2} + o(t^2)} = \left(2^{-\frac{1}{2} \frac{3t^2}{2} + o(t^2)} n\right)^t < 1/2.$$ 

Since our random subset will also contain more than $n$ elements with probability at least $1/2$, there exists a choice of coloring and a choice of subset of order $n$ such that there is no monochromatic potential clique in this subset. This completes the proof.
Remark. Our method also gives a construction which matches Erdős’ bound \( r(t) > \sqrt{2^t} \) up to lower-order terms. To see this, we set \( V = \mathbb{F}_2^t \) and color edges red or blue depending on whether \( u \cdot v = 0 \) or 1 mod 2. If we then sample \( 2^{t/2+o(t)} \) vertices of \( V \) at random, we can show that w.h.p. the resulting set does not contain a monochromatic clique of order \( t \). We believed this to be new, but, after the first version of this article was made public, we learned that such a construction was already discovered by Pudlák, Rödl and Savický [6] in 1988. It was also pointed out to us by Jacob Fox that one can achieve the same end by starting with any pseudorandom graph on \( n \) vertices for which the count of cliques and independent sets of order \( 2c \log_2 n \) is approximately the same as in \( G(n, 1/2) \) and sampling \( n^c \) vertices. This can be applied, for instance, with the Paley graph.

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References


