Threshold Ramsey multiplicity for odd cycles

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Abstract

The Ramsey number $r(H)$ of a graph H is the minimum n such that any two-coloring of the edges of the complete graph K_n contains a monochromatic copy of H. The threshold Ramsey multiplicity $m(H)$ is then the minimum number of monochromatic copies of H taken over all two-edge-colorings of $K_{r(H)}$. The study of this concept was first proposed by Harary and Prins almost fifty years ago. In a companion paper, the authors have shown that there is a positive constant c such that the threshold Ramsey multiplicity for a path or even cycle with k vertices is at least $(ck)^k$, which is tight up to the value of c. Here, using different methods, we show that the same result also holds for odd cycles with k vertices.

1 Introduction

The *Ramsey number* $r(H)$ of a graph H is the minimum positive integer n such that any two-coloring of the edges of the complete graph K_n on n vertices contains a monochromatic copy of H. Ramsey in 1930 proved that these numbers exist. However, determining or even estimating Ramsey numbers remains a formidable challenge for most graphs. For instance, the Ramsey number of K_5 is already not known, while the longstanding bounds $2^{k/2} \leq r(K_k) \leq 4^k$ have only been improved by lower-order factors [2, 23, 28].

To date, there are only a few non-trivial families of graphs for which the Ramsey number is known exactly, including stars, paths, and cycles. Let P_k and C_k denote the path and cycle on k vertices, respectively. In 1967, Gerencsér and Gyárfás [10] determined the Ramsey number of paths, namely,

$$r(P_k) = k - 1 + \lfloor k/2 \rfloor.$$

For cycles, the general case was solved independently by Rosta [21] and by Faudree and Schelp [9], who showed that

$$r(C_k) = 3k/2 - 1 \text{ if } k \geq 6 \text{ is even} \quad \text{and} \quad r(C_k) = 2k - 1 \text{ if } k \geq 5 \text{ is odd.}$$

A more general problem than computing Ramsey numbers is to determine the *Ramsey multiplicity* $M(H, n)$, the minimum number of monochromatic copies of H guaranteed in any two-edge-coloring of K_n. Indeed, it is easy to check that $M(H, n) = 0$ if and only if $n < r(H)$.

The asymptotic behaviour of $M(H, n)$ when H is fixed and n tends to infinity has attracted considerable attention. This is in part because of a famous conjecture of Erdős [8] stating that if H is a clique, then the value of $M(H, n)$ is asymptotically equal to the expected number of monochromatic copies of

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H in a uniformly random two-edge-coloring of K_n. Unfortunately, this conjecture (and a later generalization to all graphs [1]) is false already for $H = K_4$, as first shown by Thomason [30] (see also [14, 24]). However, it remains an interesting open problem to determine which graphs satisfy the conjecture, known in the literature as *common graphs*. For instance, the non-three-colorable 5-wheel is known to be common [13] and some hope remains that all bipartite graphs are common because of a connection to a celebrated conjecture of Sidorenko and Erdős–Simonovits [25, 26, 27] (see [3, 6, 7, 16, 18, 29] for some recent results towards this conjecture). We refer the interested reader to [4, Section 2.6] for more on this fascinating subject.

Another much-studied problem concerns the value of $M(H, n)$ when it first becomes positive, i.e., when $n = r(H)$. As in our companion paper [5], we refer to this value as the *threshold Ramsey multiplicity*.

Definition 1. The *threshold Ramsey multiplicity* $m(H)$ of a graph H is the minimum number of monochromatic copies of H in any two-coloring of the edges of K_n with $n = r(H)$. In other words,

$$m(H) = M(H, r(H)).$$

The threshold Ramsey multiplicity was first studied systematically by Harary and Prins [12] almost fifty years ago. The exact value of the threshold Ramsey multiplicity is known for all graphs with at most 4 vertices [11, 12, 20], but, in general, determining or even providing a non-trivial lower bound on the threshold Ramsey multiplicity appears to be quite challenging. In fact, the behavior of $m(H)$ can be rather erratic. For instance, Harary and Prins [12] proved that $m(K_2) = 1$ and $m(K_{1,k}) = 1$ for k even, but $m(K_{1,k}) = 2k$ for $k \geq 3$ odd.

In the same paper [12], Harary and Prins asked for a determination of $m(P_k)$ and $m(C_k)$. It is this question that concerns us in this paper and its companion [5]. Indeed, in [5], not only did we provide the first non-trivial bound for the Ramsey multiplicity of paths and even cycles, but the bound is tight up to a lower-order factor.

Theorem 2 ([5])**.** *There is a positive constant c such that, for every positive integer k, the threshold Ramsey multiplicity of the path with k vertices satisfies $m(P_k) \geq (ck)^k$ and, if k is even, the threshold Ramsey multiplicity of the cycle on k vertices satisfies $m(C_k) \geq (ck)^k$.*

In this paper, we address Harary and Prins' question for odd cycles. Unlike the cases studied in [5], this odd-cycle case has received considerable previous attention, with Rosta and Surányi [22] already proving the exponential lower bound $m(C_k) \geq 2^{ck}$ in the 1970's. This was later improved to a superexponential bound in an unpublished work of Rosta (see [15]). More recently, Károlyi and Rosta [15] improved the lower bound to $m(C_k) \geq k^{ck}$. To the best of our knowledge, this was the state-of-the-art prior to our result, which we now state.

Theorem 3. *There is a positive constant c such that, for every odd positive integer k, the threshold Ramsey multiplicity of the cycle on k vertices satisfies $m(C_k) \geq (ck)^k$.*

As for paths and even cycles, this bound is tight up to the constant c. However, it is proved using rather different methods to those employed in [5], because the Ramsey numbers, and the associated extremal colorings, are quite different for odd cycles and for paths and even cycles. To describe the extremal colorings in the odd setting, consider the red/blue edge-coloring $\chi(a, b)$ of the complete graph on $n = a + b$ vertices with vertex set $A \cup B$, $|A| = a$ and $|B| = b$, where A and B form blue cliques and all edges between A and B are red. Let $k \geq 5$ be an odd positive integer. Then $\chi(k-1, k-1)$ is a coloring of the complete graph on $r(C_k) - 1 = 2k - 2$ vertices with no monochromatic C_k, while $\chi(k, k-1)$ is a coloring of the complete graph on $r(C_k) = 2k - 1$ vertices with exactly $(k-1)!/2$ monochromatic copies of C_k, as all monochromatic C_k are in the blue clique of order k. This provides an upper bound on $m(C_k)$ showing that the bound in Theorem 3 is tight apart from a lower-order factor. It also suggests that our bound can be strengthened, as follows.

Conjecture 4. *For any sufficiently large odd integer k, $m(C_k) = (k-1)!/2$.*

2 Proof of Theorem 3

2.1 Preliminaries

As in our proof of Theorem 2 in [5], we will use Szemerédi's regularity lemma, an important tool which gives a rough structural decomposition for all graphs. Roughly speaking, for any graph, the regularity lemma outputs a vertex partition of the graph into a small number of parts, where the bipartite graph between almost every pair of parts behaves like a random graph. Among its many applications (see, for example, [17]), this decomposition is useful for embedding and counting copies of sparse graphs, such as the cycles that concern us here.

To formally state the regularity lemma, we first need some definitions to quantify what is meant by a "random-like" bipartite graph. For a pair of disjoint vertex subsets (X, Y) of a graph, let $d(X, Y) = e(X, Y)/|X||Y|$ denote the density of edges between X and Y.

Definition 5 (ϵ-regular pair)**.** A pair (X, Y) of disjoint vertex subsets of a graph is said to be ϵ-regular if, for all subsets $U \subset X, V \subset Y$ such that $|U| \geq \epsilon|X|$ and $|V| \geq \epsilon|Y|$, $|d(U, V) - d(X, Y)| \leq \epsilon$.

The next lemma sets out two basic facts about ϵ-regular pairs that will be useful later.

Lemma 6. *If (X, Y) is an ϵ-regular pair and $d(X, Y) = d$, then the following hold:*

(i) If $Y' \subset Y$ satisfies $|Y'| \geq \epsilon|Y|$, then the number of vertices in X with degree in Y' greater than $(d+\epsilon)|Y'|$ is less than $\epsilon|X|$ and the number of vertices in X with degree in Y' less than $(d-\epsilon)|Y'|$ is less than $\epsilon|X|$.

(ii) If $X' \subset X$ and $Y' \subset Y$ are such that $|X'| \geq \alpha|X|$ and $|Y'| \geq \alpha|Y|$, then (X', Y') is $\max(\epsilon/\alpha, 2\epsilon)$-regular.

A vertex partition is called *equitable* if each pair of parts differ in size by at most one. We are now ready to state the regularity lemma.

Lemma 7 (Szemerédi's regularity lemma)**.** *For every $\epsilon > 0$ and positive integer l, there are positive integers n_0 and M_0 such that every graph G with at least n_0 vertices admits an equitable vertex partition $V(G) = V_1 \cup \cdots \cup V_M$ into M parts with $l \leq M \leq M_0$ where all but at most $\epsilon\binom{M}{2}$ pairs of parts (V_i, V_j) with $1 \leq i < j \leq M$ are ϵ-regular.*

In practice, we will use the following standard colored version of the regularity lemma.

Lemma 8 (Colored regularity lemma)**.** *For every $\epsilon > 0$ and positive integer l, there are positive integers n_0 and M_0 such that every two-edge-coloring of the complete graph K_n with $n \geq n_0$ in red and blue admits an equitable vertex partition $V(G) = V_1 \cup \cdots \cup V_M$ into M parts with $l \leq M \leq M_0$ where all but at most $\epsilon\binom{M}{2}$ pairs of parts (V_i, V_j) with $1 \leq i < j \leq M$ are ϵ-regular in both the red and blue subgraphs.*

Lemmas 7 and 8 are in fact equivalent, since a pair (V_i, V_j) in an edge-coloring of K_n with colors red and blue is ϵ-regular in red if and only if it is ϵ-regular in blue.

2.2 The stability lemma

The main ingredient in the proof of Theorem 3 is a stability lemma, Lemma 10 below, which implies that any two-edge-coloring of K_n (where, for us, n will be $r(C_k) = 2k - 1$ for some sufficiently large odd k) either has a regularity partition whose reduced graph contains a long monochromatic path or the edge-coloring of K_n is close to the coloring $\chi(k, k-1)$ described before Conjecture 4. In either case, we can show that the conclusion of Theorem 3 must hold.

Before introducing the stability lemma, we need to make precise what we mean by saying that a two-edge-coloring of the complete graph on $2k-1$ vertices is close to $\chi(k, k-1)$. In the definition, we will refer to the density of a set X, given by $d(X, X) = 2e(X)/|X|^2$.

Definition 9 (Extremal coloring with parameter λ)**.** A two-edge-coloring of the complete graph on n vertices is an *extremal coloring with parameter* λ if there exists a partition $A \cup B$ of the vertex set such that

- $|A| \geq (1/2 - \lambda)n$ and $|B| \geq (1/2 - \lambda)n$;
- the graph induced on A has density at least $(1 - \lambda)$ in some color, the graph induced on B has density at least $(1 - \lambda)$ in the same color, and the bipartite graph between A and B has density at least $(1 - \lambda)$ in the other color.

Our key stability lemma is now as follows.

Lemma 10. *For any $0 < \epsilon < 10^{-20}$, there is a constant $M_0 = M_0(\epsilon)$ such that if $\alpha = 20\sqrt{\epsilon}$, then, for n sufficiently large in terms of ϵ, any two-edge-coloring of the complete graph K_n falls into one of the following two cases:*

- ***Case 1:*** *There is a positive integer $\epsilon^{-1} \leq M \leq M_0$ such that if t is the odd integer with*
$$(1/2 + \alpha)M \geq t > (1/2 + \alpha)M - 2, \tag{1}$$
then there are disjoint vertex sets $V_0, \ldots, V_{t-1}$, indexed by the elements of $\mathbb{Z}/t\mathbb{Z}$, and a color χ such that, for each $i \in \mathbb{Z}/t\mathbb{Z}$, $|V_i| \geq \lfloor n/M \rfloor$, the pair (V_i, V_{i+1}) is ϵ-regular in the color χ, and the edge density between V_i and V_{i+1} in the color χ is at least $11\epsilon^{1/2}$.
- ***Case 2:*** *The graph is an extremal coloring with parameter $300\sqrt{\alpha}$.*

We will hold off on proving Lemma 10 until Section 2.5, first showing, across the next two sections, how Theorem 3 follows from either of the conclusions in the lemma.

2.3 Theorem 3 for colorings satisfying Case 1 of Lemma 10

In this section, we prove Theorem 3 for colorings satisfying Case 1 of Lemma 10. We will repeatedly work in a setting where we have disjoint vertex sets $V_0, \ldots, V_{t-1}$ from a graph where the indices of the V_i are the elements of $\mathbb{Z}/t\mathbb{Z}$. We say that a path P of length ℓ with vertices $w_0, w_1, \ldots, w_\ell$ and edges $w_0w_1, w_1w_2, \ldots, w_{\ell-1}w_\ell$ is $(V_0, \ldots, V_{t-1})$*-transversal* if $w_i \in V_i$ for each $0 \leq i \leq \ell$. Note that we will typically have $\ell > t$, so the path may pass through each of the vertex sets multiple times.

Lemma 11. *Suppose that $0 < \epsilon < 10^{-5}$ and t and n are integers with $t \geq 2$ and $n \geq \epsilon^{-2}$. Suppose also that $V_0, \ldots, V_{t-1}$ are disjoint vertex sets in a graph where the indices of the V_i are the elements of $\mathbb{Z}/t\mathbb{Z}$ and, for each $i \in \mathbb{Z}/t\mathbb{Z}$, $|V_i| \geq n$, (V_i, V_{i+1}) is ϵ-regular, and $d(V_i, V_{i+1}) \geq d$ for some $d \geq 5\epsilon^{1/2}$. Then the following hold:*

1. *For any integer ℓ with $2 \leq \ell \leq t(1 - \sqrt{\epsilon})n$ and any vertex $w_0 \in V_0$ with at least $(d - \epsilon)|V_1|$ neighbors in V_1, the number of $(V_0, \ldots, V_{t-1})$-transversal paths of length ℓ starting from w_0 is at least $(d - \epsilon - \sqrt{\epsilon})^\ell \prod_{i=1}^{\ell}(n - \lfloor i/t \rfloor)$.*
2. *For any integer ℓ with $4 \leq \ell \leq t(1 - 3\sqrt{\epsilon})n$ which is divisible by t and any two (not necessarily distinct) vertices $w_0, w_0' \in V_0$ such that w_0 has at least $(d - \epsilon)|V_1|$ neighbors in V_1 and w_0' has at least $(d - \epsilon)|V_{t-1}|$ neighbors V_{t-1}, the number of $(V_0, \ldots, V_{t-1})$-transversal paths of length ℓ with end vertices w_0 and w_0' is at least $(d - 5\sqrt{\epsilon})^{\ell-1}(1 - 2\sqrt{\epsilon})^{\ell-2}(\epsilon n) \prod_{i=1}^{\ell-2}(n - \lfloor i/t \rfloor)$.*

Proof. For any integer $0 \le l \le \ell$, let N_l be the number of good paths of length l starting from w_0, where a $(V_0, \dots, V_{t-1})$-transversal path $w_0, w_1, \dots, w_l$ of length l starting from w_0 is *good* if there are at least $(d-\epsilon)\big(|V_{l+1}| - \lfloor (l+1)/t \rfloor\big)$ ways to extend the path to V_{l+1}. We will prove by induction that $N_l \ge (d - \epsilon - \sqrt{\epsilon})^l \prod_{i=1}^{l} (n - \lfloor i/t \rfloor)$ for $0 \le l \le \ell$, which will settle Part 1.

For the base case, note that $N_0 = 1$, since the path with zero edges starting from w_0 is w_0 itself and, by assumption, the vertex $w_0 \in V_0$ has at least $(d-\epsilon)|V_1|$ neighbors in V_1. Suppose now that the required lower bound holds for N_{l-1} and we wish to deduce the lower bound for N_l.

Fix any good path P of length $l-1$. By the definition of goodness, there are at least $(d-\epsilon)(|V_l| - \lfloor l/t \rfloor)$ choices of $w_l \in V_l$ that extend P. To bound N_l, we need a lower bound on the number of vertices $w_l \in V_l$ such that the path formed by extending P to w_l is also good.

Let U be the set of vertices in V_l which have degree less than $(d-\epsilon)|V_{l+1} \setminus V(P)| = (d-\epsilon)(|V_{l+1}| - \lfloor (l+1)/t \rfloor)$ in $V_{l+1} \setminus V(P)$. Note that, since $\ell \le t(1-\sqrt{\epsilon})n$,

$$\begin{aligned} |V_{l+1} \setminus V(P)| &= |V_{l+1}| - \lfloor (l+1)/t \rfloor \ge |V_{l+1}| - \ell/t - 1 \ge |V_{l+1}| - t(1-\sqrt{\epsilon})n/t - 1 \\ &\ge |V_{l+1}| - (1-\sqrt{\epsilon})|V_{l+1}| - 1 = \sqrt{\epsilon}|V_{l+1}| - 1 \ge \epsilon |V_{l+1}|. \end{aligned}$$

Together with the fact that (V_l, V_{l+1}) is ϵ-regular with density at least d, Lemma 6 (i) implies that $|U| \le \epsilon |V_l|$. Hence, the number of choices for w_l such that P extended to w_l is also good is at least

$$(d-\epsilon)(|V_l| - \lfloor l/t \rfloor) - |U| \ge (d-\epsilon)(|V_l| - \lfloor l/t \rfloor) - \epsilon|V_l| \ge (d - \epsilon - \sqrt{\epsilon})(|V_l| - \lfloor l/t \rfloor).$$

The last inequality is equivalent to $(\sqrt{\epsilon} - \epsilon)|V_l| \ge \sqrt{\epsilon} \lfloor \ell/t \rfloor$, which again follows from $\ell \le t(1-\sqrt{\epsilon})n$. Thus,

$$N_l \ge (d - \epsilon - \sqrt{\epsilon})(n - \lfloor l/t \rfloor) N_{l-1} \ge (d - \epsilon - \sqrt{\epsilon})^l \prod_{i=1}^{l} (n - \lfloor i/t \rfloor),$$

establishing Part 1.

For Part 2, we pass from the V_i to a collection of subsets V_i'. By assumption, $|N_{V_{t-1}}(w_0')| \ge (d-\epsilon)|V_{t-1}| \ge \epsilon |V_{t-1}|$, so we may set aside a subset W_{t-1} of $N_{V_{t-1}}(w_0')$ of size $\epsilon|V_{t-1}|$ and let $V_{t-1}' = V_{t-1} \setminus W_{t-1}$. If w_0' is distinct from w_0, we let $V_0' = V_0 \setminus \{w_0'\}$, while, in all remaining cases, we let $V_i' = V_i$, noting that $|V_i'| \ge (1-\epsilon)|V_i|$ for all i. Therefore, if we set $\epsilon' = 2\epsilon$ and $d' = d - \epsilon$, Lemma 6 (ii) now tells us that for each $i \in \mathbb{Z}/t\mathbb{Z}$ the pair of sets (V_i', V_{i+1}') is ϵ'-regular with edge density at least d'.

As in Part 1, for any positive integer $0 \le l \le \ell - 3$, let N_l be the number of good paths of length l starting from w_0, though with the condition now reading that there are at least $(d' - \epsilon')(|V_{l+1}'| - \lfloor (l+1)/t \rfloor)$ ways to extend the path to a vertex in V_{l+1}'. Since w_0 has at least $(d-\epsilon)|V_1| - |V_1 \setminus V_1'| \ge (d - 2\epsilon)|V_1| > (d' - \epsilon')|V_1'|$ neighbors in V_1' and $\ell \le t(1-3\sqrt{\epsilon})n \le t(1-\sqrt{2\epsilon})((1-\epsilon)n) = t(1-\sqrt{\epsilon'})((1-\epsilon)n)$, we may apply Part 1 to conclude that

$$N_{\ell-3} \ge (d' - \epsilon' - \sqrt{\epsilon'})^{\ell-3} \prod_{i=1}^{\ell-3} ((1-\epsilon)n - \lfloor i/t \rfloor). \tag{2}$$

Fix any such path P of length $\ell - 3$. Suppose its vertices are $w_0, w_1, \dots, w_{\ell-3}$ in order, where $w_j \in V_j'$. By definition, there are at least $(d' - \epsilon')(|V_{\ell-2}'| - \lfloor (\ell-2)/t \rfloor)$ ways to extend the path to a vertex $w_{\ell-2} \in V_{\ell-2}'$. Denote this set of candidates for $w_{\ell-2}$ by C. Since ℓ is divisible by t, we have that $C \subset V_{t-2}$. Using that $d \ge 5\sqrt{\epsilon}$ and $\ell/t \le (1-3\sqrt{\epsilon})n$, we have that $|C| \ge (d' - \epsilon')(|V_{\ell-2}'| - \lfloor (\ell-2)/t \rfloor) \ge \epsilon' |V_{t-2}'| \ge \epsilon |V_{t-2}|$. Since $|W_{t-1}| \ge \epsilon |V_{t-1}|$ and (V_{t-2}, V_{t-1}) is ϵ-regular with density at least d, the number of edges between C and W_{t-1} is at least

$$(d-\epsilon)|C||W_{t-1}| \ge (d-\epsilon)(d'-\epsilon')(|V_{\ell-2}'| - \lfloor (\ell-2)/t \rfloor) \cdot \epsilon |V_{t-1}|. \tag{3}$$

Note now that P together with the two end vertices of any edge in $E(C, W_{t-1})$ results in a path of length $\ell - 1$ that can be extended to w_0'. Therefore, using (2) and (3), we see that the number of paths with end vertices w_0 and w_0' is at least

$$\begin{aligned} N_{\ell-3} \cdot (d-\epsilon)|C||W_{t-1}| &\geq (d' - \epsilon' - \sqrt{\epsilon'})^{\ell-3} \prod_{i=1}^{\ell-3}((1-\epsilon)n - \lfloor i/t \rfloor) \\ &\quad \cdot (d-\epsilon)(d'-\epsilon')(|V'_{\ell-2}| - \lfloor (\ell-2)/t \rfloor) \cdot \epsilon |V_{t-1}| \\ &\geq (d - 3\epsilon - 2\sqrt{\epsilon})^{\ell-1}(\epsilon n) \prod_{i=1}^{\ell-2}((1-\epsilon)n - \lfloor i/t \rfloor) \\ &\geq (d - 3\epsilon - 2\sqrt{\epsilon})^{\ell-1}(1-\epsilon-\sqrt{\epsilon})^{\ell-2}(\epsilon n) \prod_{i=1}^{\ell-2}(n - \lfloor i/t \rfloor) \\ &\geq (d - 5\sqrt{\epsilon})^{\ell-1}(1-2\sqrt{\epsilon})^{\ell-2}(\epsilon n) \prod_{i=1}^{\ell-2}(n - \lfloor i/t \rfloor), \end{aligned}$$

where the second to last inequality holds because $(1-\epsilon)n - \lfloor i/t \rfloor \geq (1-\epsilon-\sqrt{\epsilon})(n - \lfloor i/t \rfloor)$ for $i \leq t(1-2\sqrt{\epsilon})n$. $\square$

Part 2 of Lemma 11 with $w_0 = w_0'$ implies that there are many cycles of length ℓ when ℓ is divisible by t. The next lemma shows that the same result holds even when ℓ is not divisible by t.

Lemma 12. *Suppose that $0 < \epsilon < 10^{-5}$, t is an odd integer with $t \geq 3$, and n is a positive integer with $n \geq t\epsilon^{-2}$. Suppose also that $V_0, \ldots, V_{t-1}$ are disjoint vertex sets in a graph where the indices of the V_i are the elements of $\mathbb{Z}/t\mathbb{Z}$ and, for each $i \in \mathbb{Z}/t\mathbb{Z}$, $|V_i| \geq n$, (V_i, V_{i+1}) is ϵ-regular, and $d(V_i, V_{i+1}) \geq d$ for some $d \geq 10\epsilon^{1/2}$. Then, for any odd positive integer p with*

$$2t + 6 \leq p \leq t(1 - 5\sqrt{\epsilon})n, \tag{4}$$

the number of cycles of length p is at least $\frac{\epsilon^2}{4} n^4 (d - 10\sqrt{\epsilon})^{p-2}(1-3\sqrt{\epsilon})^{2p} \prod_{i=1}^{p-4}(n - \lfloor i/t \rfloor)$.

Proof. Suppose $p \equiv r \bmod t$ for some $0 \leq r \leq t-1$. If $r = 0$, then we can choose $w_0 = w_0'$ in Part 2 of Lemma 11 in at least $(1-2\epsilon)n$ ways. Then this lemma easily implies that the number of cycles of length p is at least $(1-2\epsilon)\epsilon n^2 (d-5\sqrt{\epsilon})^{p-1}(1-2\sqrt{\epsilon})^{p-2}\prod_{i=1}^{p-2}(n - \lfloor i/t \rfloor)$, which is stronger than the required bound. We may therefore assume that p is not divisible by t, i.e., that $r > 0$. Define an auxiliary constant h by $h = r + t$ if r is odd, $h = r$ if $r > 2$ and even, and $h = 2 + 2t$ if $r = 2$. Note that h is always a positive even integer with $4 \leq h \leq 2t + 2$, so a cycle of length p can be constructed by combining an even path L_1 of length h alternating between V_0 and V_{t-1} with a $(V_0, \ldots, V_{t-1})$-transversal path L_2 of length $p - h$ with the same end vertices as L_1. Since $(p-h)$ is divisible by t, the path L_2 will use exactly $(p-h)/t$ vertices in each of the V_i other than V_0.

By Lemma 6 (i), the set U_0 of vertices in V_0 with at least $(d-\epsilon)|V_1|$ neighbors in V_1 and at least $(d-\epsilon)|V_{t-1}|$ neighbors in V_{t-1} has size at least $(1-2\epsilon)|V_0|$. We now fix two vertices $u, v \in U_0$ and bound the number of paths of the types L_1 and L_2 described above with end vertices u and v.

We first bound the number of paths of type L_1 with end vertices u and v. Since $h \geq 4$ and $h \leq 2t + 2 \leq 2(1 - 3\sqrt{\epsilon})n$, we may apply Part 2 of Lemma 11 to (V_0, V_{t-1}) to conclude that the number of paths of length h alternating between V_0 and V_{t-1} with end vertices u and v is at least

$$(d - 5\sqrt{\epsilon})^{h-1}(1 - 2\sqrt{\epsilon})^{h-2}(\epsilon n) \prod_{i=1}^{h-2}(n - \lfloor i/2 \rfloor). \tag{5}$$

We now bound the number of paths of type L_2 available for each such L_1. For a given L_1, remove its $h-1$ interior vertices from V_0 and V_{t-1}, calling the updated vertex sets V_0' and V_{t-1}', respectively. Since $h \leq 2t+2$, we have $|V_0'| \geq |V_0| - (t+1) \geq (1-\epsilon/2)|V_0|$ and, similarly, $|V_{t-1}'| \geq (1-\epsilon/2)|V_{t-1}|$. Hence, by Lemma 6 (ii), each of the pairs (V_0', V_{t-1}'), (V_0', V_1), and (V_{t-1}', V_{t-2}) is 2ϵ-regular with density at least $d-\epsilon$. Furthermore, u and v each have at least $(d-\epsilon)|V_{t-1}| - (t+1) \geq (d-2\epsilon)|V_{t-1}| \geq (d-2\epsilon)|V_{t-1}'|$ neighbors in V_{t-1}'. Since also

$$4 \leq p - h \leq t(1 - 5\sqrt{\epsilon})n \leq t(1 - 3\sqrt{2\epsilon})(1 - \epsilon/2)n,$$

we may apply Part 2 of Lemma 11 with ϵ, ℓ, d, and n replaced by 2ϵ, $p-h$, $d-\epsilon$, and $(1-\epsilon/2)n$, respectively, to conclude that the number of $(V_0', V_1, \ldots, V_{t-2}, V_{t-1}')$-transversal paths of length $p-h$ with end vertices u and v, each of which is a valid choice for L_2, is at least

$$\begin{aligned}
&(d - \epsilon - 5\sqrt{2\epsilon})^{p-h-1}(1 - 2\sqrt{2\epsilon})^{p-h-2}(2\epsilon(1-\epsilon/2)n) \prod_{i=1}^{p-h-2} ((1-\epsilon/2)n - \lfloor i/t \rfloor) \\
&\geq (d - 10\sqrt{\epsilon})^{p-h-1}(1 - 3\sqrt{\epsilon})^{p-h-2}(\epsilon n) \prod_{i=1}^{p-h-2} ((1-\epsilon/2)n - \lfloor i/t \rfloor) \\
&\geq (d - 10\sqrt{\epsilon})^{p-h-1}(1 - 3\sqrt{\epsilon})^{p-h-2}(\epsilon n)(1 - \epsilon/2 - \sqrt{\epsilon/2})^{p-h-2} \prod_{i=1}^{p-h-2} (n - \lfloor i/t \rfloor) \\
&\geq (d - 10\sqrt{\epsilon})^{p-h-1}(1 - 3\sqrt{\epsilon})^{p-h-2}(\epsilon n)(1 - 2\sqrt{\epsilon})^{p-h-2} \prod_{i=1}^{p-h-2} (n - \lfloor i/t \rfloor) \\
&\geq (d - 10\sqrt{\epsilon})^{p-h-1}(1 - 3\sqrt{\epsilon})^{2(p-h-2)}(\epsilon n) \prod_{i=1}^{p-h-2} (n - \lfloor i/t \rfloor),
\end{aligned} \tag{6}$$

where the second inequality holds because $(1-\epsilon/2)n - \lfloor i/t \rfloor \geq (1 - \epsilon/2 - \sqrt{\epsilon/2})(n - \lfloor i/t \rfloor)$ for $i \leq t(1 - 2\sqrt{\epsilon/2})n$.

Therefore, since the number of choices for u and v is at least $\frac{1}{2}(1-2\epsilon)|V_0|((1-2\epsilon)|V_0| - 1)$, we may combine (5) and (6) to conclude that the number of cycles of length p is at least

$$\begin{aligned}
&\frac{1}{2}(1-2\epsilon)|V_0|((1-2\epsilon)|V_0| - 1) \cdot (d - 5\sqrt{\epsilon})^{h-1}(1 - 2\sqrt{\epsilon})^{h-2}(\epsilon n) \prod_{i=1}^{h-2}(n - \lfloor i/2 \rfloor) \\
&\qquad \cdot (d - 10\sqrt{\epsilon})^{p-h-1}(1 - 3\sqrt{\epsilon})^{2(p-h-2)}(\epsilon n) \prod_{i=1}^{p-h-2} (n - \lfloor i/t \rfloor) \\
&\geq \frac{1}{4}(1-2\epsilon)^2 n^2 (d - 10\sqrt{\epsilon})^{p-2}(1 - 3\sqrt{\epsilon})^{2p-h-2}(\epsilon n)^2 \prod_{i=1}^{p-4}(n - \lfloor i/t \rfloor) \\
&\geq \frac{\epsilon^2}{4} n^4 (d - 10\sqrt{\epsilon})^{p-2}(1 - 3\sqrt{\epsilon})^{2p} \prod_{i=1}^{p-4}(n - \lfloor i/t \rfloor),
\end{aligned}$$

as required. □

Finally, we can show that Theorem 3 holds for colorings satisfying Case 1 of Lemma 10.

Proof of Theorem 3 for colorings satisfying Case 1 of Lemma 10. Suppose, for concreteness, that $\epsilon = 10^{-30}$ and let $V_0, \ldots, V_{t-1}$ be as in Case 1 of Lemma 10 with $n = 2k-1$, where k (and hence n) is a sufficiently large odd integer. We wish to apply Lemma 12 to show there are many cycles of length $k=(n+1)/2$. To confirm that the conditions of Lemma 12 hold, we only have to check (4), i.e., that

$$2t+6 \leq k \leq t(1-5\sqrt{\epsilon})\lfloor n/M \rfloor. \tag{7}$$

By (1), $(1/2+\alpha)M \geq t > (1/2+\alpha)M-2$, so it suffices to show that

$$k=(n+1)/2 \geq 2(1/2+\alpha)M+6$$

and

$$k \leq \left((1/2+\alpha)M-2\right)(1-5\sqrt{\epsilon})(n/M-1).$$

The first inequality easily holds for n sufficiently large in terms of ϵ, while the second inequality holds because

$$k=(n+1)/2 \leq (1+\epsilon)n/2 \leq (1/2+\alpha/2)M(1-5\sqrt{\epsilon})(1-\epsilon)(n/M),$$

where we used that $\alpha=20\sqrt{\epsilon}$ and $(1+\alpha)(1-5\sqrt{\epsilon})(1-\epsilon) \geq (1+20\sqrt{\epsilon})(1-6\sqrt{\epsilon}) > 1+\epsilon$. Hence, since $M \geq 4/\alpha$ and assuming $n \geq M/\epsilon$,

$$k \leq (1/2+\alpha/2)M(1-5\sqrt{\epsilon})(1-\epsilon)(n/M) \leq \left((1/2+\alpha)M-2\right)(1-5\sqrt{\epsilon})(n/M-1),$$

as required.

We may therefore apply Lemma 12 with parameters ϵ, p, d, and n replaced by ϵ, k, d, and $\lfloor n/M \rfloor$, respectively, to conclude that the number of cycles of length k is at least

$$\frac{\epsilon^2}{4}(\lfloor n/M \rfloor)^4(d-10\sqrt{\epsilon})^{k-2}(1-3\sqrt{\epsilon})^{2k}\prod_{i=1}^{k-4}(\lfloor n/M \rfloor - \lfloor i/t \rfloor). \tag{8}$$

Since $\lfloor n/M \rfloor \geq k/t$ by (7), the last term in (8) satisfies

$$\prod_{i=1}^{k-4}(\lfloor n/M \rfloor - \lfloor i/t \rfloor) \geq \prod_{i=1}^{k-4}(\lfloor n/M \rfloor - i/t) \geq t^{-(k-4)}(k-4)!.$$

Therefore, (8) is lower bounded by

$$\begin{aligned}
&\frac{\epsilon^2}{4}(\lfloor n/M \rfloor)^4(d-10\sqrt{\epsilon})^{k-2}(1-3\sqrt{\epsilon})^{2k}t^{-(k-4)}(k-4)! \\
&\geq \frac{\epsilon^2}{4}\frac{n^4}{(2M)^4}(d-10\sqrt{\epsilon})^{k-2}(1-3\sqrt{\epsilon})^{2k}t^{-(k-4)}((k-4)/e)^{k-4} \\
&\geq \frac{\epsilon^2}{4(2M)^4}(d-10\sqrt{\epsilon})^{k-2}(1-3\sqrt{\epsilon})^{2k}t^{-(k-4)}((k-4)/e)^{k}.
\end{aligned}$$

Hence, since ϵ is a constant, $d \geq 11\sqrt{\epsilon}$, and M and t are bounded in terms of ϵ, there is a constant c_1 depending only on ϵ such that the number of cycles of length k is at least $(c_1k)^k$, as required. □

2.4 Theorem 3 for colorings satisfying Case 2 of Lemma 10

The proof of Theorem 3 for colorings satisfying Case 2 of Lemma 10 has several cases. To make the presentation cleaner, we first prove some simple claims.

Claim 13. *Let S and T be two disjoint vertex sets in a graph F such that any two vertices in S have at least s common neighbors in T. If there is an edge within S, then the number of cycles of length l is at least $(s-l/2+3/2)^{(l-1)/2}\left(|S|-(l-1)/2\right)^{(l-3)/2}$ for any odd integer $3 \leq l \leq \min(2s+1, 2|S|-1)$.*

Proof. Suppose that (u, u') is an edge in S. To construct cycles of length l, we will find paths P of the form $u, v_1, u_1, v_2, u_2, \ldots, v_{(l-1)/2}, u_{(l-1)/2} = u'$ alternating between S and T with end vertices u and u'. Each such path together with the edge (u, u') clearly gives rise to a cycle of length l.

To estimate the number of paths P of the required form, suppose that $u, u_1, u_2, \ldots, u_{(l-1)/2-1}$, $u_{(l-1)/2} = u'$ is a fixed sequence of distinct vertices in S. Note that any of the s common neighbors of u and u_1 in T can be chosen as v_1. More generally, given choices for $v_1, \ldots, v_{i-1}$, the number of choices for v_i is at least $s-(i-1)$ for $1 \le i \le (l-1)/2$, since we may pick any vertex in the common neighborhood of u_{i-1} and u_i in T except $v_1, \ldots, v_{i-1}$. Therefore, given $u, u_1, u_2, \ldots, u_{(l-1)/2-1}$, $u_{(l-1)/2} = u'$, the number of choices for P is at least

$$\prod_{i=1}^{(l-1)/2} (s-(i-1)) \ge (s-((l-1)/2-1))^{(l-1)/2} = (s-l/2+3/2)^{(l-1)/2}.$$

Since the number of choices for $u_1, \ldots, u_{(l-1)/2-1}$ is

$$(|S|-2)(|S|-3)\cdots(|S|-(l-1)/2) \ge (|S|-(l-1)/2)^{(l-3)/2},$$

the claim follows. □

Claim 14. *Let S and T be two disjoint cliques in a graph F. Suppose that there are two vertex-disjoint paths P_1 and P_2 such that each path has length at most 2, has one end vertex in S, the other end vertex in T, and the rest of the vertices outside $S \cup T$. Then the number of cycles of length l is at least $(((l-1)/2-3)/e)^{l-6}$ for any integer $7 \le l \le \min(2|S|-1, 2|T|-1)$.*

Proof. Suppose that P_1 has length l_1 and endpoints $a_1 \in S$ and $b_1 \in T$, while P_2 has length l_2 and endpoints $a_2 \in S$ and $b_2 \in T$. Let $s = \lfloor l/2 \rfloor$. We will construct cycles of length l by concatenating the following four paths: (1) a path L_1 of length $s-l_1$ in S with end vertices a_1 and a_2, (2) the path P_2, (3) a path L_2 in T of length $l-s-l_2$ with end vertices b_1 and b_2, and (4) the path P_1. This process clearly yields a cycle of length l.

Since S is a clique, we can always find paths L_1 of length $s-l_1$ in $S \setminus \{a_1, a_2\}$ with end vertices a_1 and a_2. Indeed, the number of such L_1 is exactly the number of length-$(s-l_1-1)$ ordered sequences of vertices in $S \setminus \{a_1, a_2\}$. Since $|S|-2 \ge s-l_1-1$, such a sequence exists. Furthermore, the number of such sequences is exactly $(|S|-2)!/(|S|-2-(s-l_1-1))! \ge (s-l_1-1)! \ge ((s-l_1-1)/e)^{s-l_1-1}$, where we used the inequality $(x-1)\cdots(x-y) \ge y! \ge (y/e)^y$ for positive integers x, y with $x \ge y+1$. Similarly, the number of choices for L_2 is at least $((l-s-l_2-1)/e)^{l-s-l_2-1}$. In total, the number of cycles of length l is at least $((s-l_1-1)/e)^{s-l_1-1}((l-s-l_2-1)/e)^{l-s-l_2-1}$. Since $l_1, l_2 \le 2$ and $l-s \ge s$, the quantity above is at least $((s-3)/e)^{s-l_1-1}((s-3)/e)^{l-s-l_2-1}$. If $s-3 \ge e$, then the previous quantity is at least $((s-3)/e)^{l-6}$. Otherwise, we counted a positive integer number of cycles of length l, which is at least the bound in the claim. □

Claim 15. *Let S and T be two disjoint vertex sets in a graph F. Suppose that $w \in S$ and there is a complete bipartite graph between $S \setminus \{w\}$ and T and at least one edge between w and T (so, in particular, there may be a complete bipartite graph between S and T). If there is a path P' of length two with one end vertex in S, the other end vertex in T, and the rest of the vertices outside $S \cup T$, then the number of cycles of length l is at least $((l-5)/2e)^{l-5}$ for any odd integer l with $7 \le l \le \min(2|S|+1, 2|T|+1)$.*

Proof. Suppose the two end vertices of P' are $a \in S$ and $b \in T$. We will construct cycles of length l by concatenating P' with paths P of length $l-2$ alternating between S and T with end vertices a and b. Fix a neighbor x of w in T. If $a = w$, each path P will start with w and then x before returning to some $a' \in S$, while if $a \ne w$, we will avoid w while building our paths. In either case, a lower estimate for the number of cycles of length l is given by estimating the number of paths of length $l-4$ starting at a fixed $a' \ne w$ and ending at b alternating between $S \setminus \{w\}$ and $T \setminus \{x\}$.

Since there is a complete bipartite graph between $S\setminus\{w\}$ and $T\setminus\{x\}$, any length-$(l-5)/2$ sequence of ordered vertices in $S\setminus\{a',w\}$ and any length-$(l-5)/2$ sequence of ordered vertices in $T\setminus\{b,x\}$ give rise to a relevant path by alternating between the two sequences as interior vertices. Such sequences exist because $|S|-2\geq(l-5)/2$ and $|T|-2\geq(l-5)/2$. Thus, the number of choices for the path is at least the product of the number of such sequences in $S\setminus\{a',w\}$ and $T\setminus\{b,x\}$, which is

$$\begin{aligned}\frac{(|S|-2)!}{(|S|-(l-5)/2)-2)!}\cdot\frac{(|T|-2)!}{(|T|-(l-5)/2)-2)!}&\geq((l-5)/2)!((l-5)/2)!\\&\geq((l-5)/2e)^{l-5},\end{aligned}$$

where we again used that $(x-1)\cdots(x-k)\geq k!\geq(k/e)^k$ for positive integers x,k with $x\geq k+1$. □

Claim 16. *Let S and T be two disjoint vertex sets in a graph F. If there are no two vertex-disjoint edges between S and T, then, by removing at most one vertex from $S\cup T$, there is no edge between S and T.*

Proof. If there is no vertex in S with a neighbor in T, the claim trivially holds. If there is exactly one vertex $a\in S$ with neighbors in T, then there is no edge between $S\setminus\{a\}$ and T. If there is more than one vertex in S with a neighbor in T, then all of them have the same neighbor $b\in T$ and there is no edge between S and $T\setminus\{b\}$. In each case, the claim follows. □

We are now ready to prove Theorem 3 for colorings satisfying Case 2 of Lemma 10.

Proof of Theorem 3 for colorings satisfying Case 2 of Lemma 10. Suppose again that $\epsilon=10^{-30}$ and $n=2k-1$ for k a sufficiently large odd integer, but we now have an extremal coloring of K_n with parameter $\lambda=300\sqrt{\alpha}$ and vertex partition $A\cup B$, as in Case 2 of Lemma 10. Without loss of generality, we will assume that the red densities within A and B are both at least $1-\lambda$ and the blue density between A and B is at least $1-\lambda$, where $|A|,|B|\geq(1/2-\lambda)n$.

We first conduct a simple cleaning-up procedure.

Claim 17. *There is a vertex partition of K_n as $A'\cup B'\cup X\cup Y$ satisfying the following conditions:*

1. *$A=A'\cup X$ and $B=B'\cup Y$.*
2. *$|X|\leq2\sqrt{\lambda}|A|$ and $|Y|\leq2\sqrt{\lambda}|B|$.*
3. *$|A'|\geq(1/2-2\sqrt{\lambda})n$ and $|B'|\geq(1/2-2\sqrt{\lambda})n$.*
4. *Each vertex in A' has red degree at least $(1-3\sqrt{\lambda})|A|$ in A' and blue degree at least $(1-3\sqrt{\lambda})|B|$ in B'. Similarly, each vertex in B' has red degree at least $(1-3\sqrt{\lambda})|B|$ in B' and blue degree at least $(1-3\sqrt{\lambda})|A|$ in A'.*

Proof. Suppose that there are $x|A|$ vertices in A whose red degree in A is at most $(1-\sqrt{\lambda})|A|$. Then,

$$x|A|(1-\sqrt{\lambda})|A|+(1-x)|A||A|\geq(1-\lambda)|A|^2,$$

which implies that $x\leq\sqrt{\lambda}$. Similarly, there are at most $\sqrt{\lambda}|A|$ vertices in A whose blue degree in B is at most $(1-\sqrt{\lambda})|B|$. Letting X be the union of these two bad sets of vertices, we see that $|X|\leq2\sqrt{\lambda}|A|$. We define $Y\subset B$ similarly, again noting that $|Y|\leq2\sqrt{\lambda}|B|$. Letting $A'=A\setminus X$ and $B'=B\setminus Y$, we see that items 1 and 2 hold. To verify item 3, note that $|A'|=|A|-|X|\geq(1-2\sqrt{\lambda})|A|$. Since $|A|\geq(1/2-\lambda)n$, we have

$$|A'|\geq(1-2\sqrt{\lambda})(1/2-\lambda)n\geq(1/2-2\sqrt{\lambda})n,$$

as required. Similarly, $|B'|\geq(1/2-2\sqrt{\lambda})n$. Finally, for item 4, note, for example, that each vertex in $A'=A\setminus X$ has red degree at least $(1-\sqrt{\lambda})|A|-|X|\geq(1-3\sqrt{\lambda})|A|$ in A', while each vertex in A' has blue degree at least $(1-\sqrt{\lambda})|B|-|Y|\geq(1-3\sqrt{\lambda})|B|$ in B'. □

The following claim allows us to assume that all the edges in A' and B' are red, i.e., that A' and B' are both red cliques, as otherwise we would be done.

Claim 18. *If there is a blue edge within A' or B', then the number of blue cycles of length k with $k = (n+1)/2$ is at least $(n/5)^{k-2}$.*

Proof. Without loss of generality, suppose that there is a blue edge (u, u') in A'. We will apply Claim 13 to the blue subgraph with $S = A'$, $T = B'$, and $l = k$. Since, by Claim 17, every vertex in A' has at least $(1 - 3\sqrt{\lambda})|B|$ blue neighbors in B', the size of the common blue neighborhood in B' of any two vertices in A' is at least

$$2(1 - 3\sqrt{\lambda})|B| - |B'| \geq (1 - 6\sqrt{\lambda})|B'|.$$

Thus, again in reference to Claim 13, we may take $s = (1 - 6\sqrt{\lambda})|B'|$.

It remains to verify that the conditions of Claim 13 hold, that is, that $k \leq \min(2(1 - 6\sqrt{\lambda})|B'| + 1, 2|A'| - 1)$. But this is simple, since

$$(1 - 6\sqrt{\lambda})|B'| - (k-1)/2 \geq (1 - 6\sqrt{\lambda})(1/2 - 2\sqrt{\lambda})n - n/4 > (1/4 - 5\sqrt{\lambda})n \tag{9}$$

and

$$|A'| - (k+1)/2 \geq (1/2 - 2\sqrt{\lambda})n - (k+1)/2 \geq (1/4 - 5\sqrt{\lambda})n. \tag{10}$$

Therefore, by Claim 13 and the estimates (9) and (10), the number of cycles of length k is at least

$$((1/4 - 5\sqrt{\lambda})n)^{(k-1)/2} \cdot ((1/4 - 5\sqrt{\lambda})n)^{(k-3)/2} = ((1/4 - 5\sqrt{\lambda})n)^{k-2} \geq (n/5)^{k-2},$$

as required. □

Our next claim is as follows.

Claim 19. *Suppose that A' and B' are both red cliques. If there are two vertex-disjoint red paths P_1 and P_2 such that each has length at most 2 and each has one end vertex in A' and the other in B', then there are at least $(n/5e)^{k-6}$ red cycles of length k.*

Proof. We will apply Claim 14 to the red subgraph with $S = A'$, $T = B'$, and $l = k$. To check that the condition $7 \leq k \leq \min(2|A'| - 1, 2|B'| - 1)$ holds, note that

$$2|A'| - 1 \geq (1 - 4\sqrt{\lambda})n - 1 > (n+1)/2 = k$$

for n sufficiently large and, similarly, $k \leq 2|B'| - 1$. Therefore, we may apply Claim 14 to conclude that the number of red cycles of length k is at least

$$(((k-1)/2 - 3)/e)^{k-6} = (((n-1)/4 - 3)/e)^{k-6} \geq (n/5e)^{k-6}$$

for n sufficiently large. □

Therefore, we are done if the assumptions of Claim 19 are satisfied, so we can and will assume that if A' and B' are both red cliques, then there are no two vertex-disjoint red edges between A' and B'. By applying Claim 16 to the red subgraph with $S = A'$ and $T = B'$, we see that we may remove at most one vertex from $A' \cup B'$ to make all the edges between A' and B' blue. Without loss of generality, we may therefore assume that there is a vertex v in A' such that all the edges between $A' \setminus \{v\}$ and B' are blue. In what follows, we let $A'' = A' \setminus \{v\}$.

Claim 20. *Suppose that all the edges between A'' and B' are blue. If there is a blue path P' of length two with one end vertex in A'' and the other in B', then there are at least $(n/8e)^{k-5}$ blue cycles of length k.*

Proof. We will apply Claim 15 to the blue subgraph with $S = A''$, $T = B'$, and $l = k$, using the fact that the bipartite graph between A'' and B' is complete in blue. To check that the condition $7 \le k \le \min(2|A''|+1, 2|B'|+1)$, note, for instance, that

$$2|A''| + 1 \ge 2|A'| - 1 \ge (1 - 4\sqrt{\lambda})n - 1 \ge (n+1)/2 = k$$

for n sufficiently large. Therefore, by Claim 15, the number of blue cycles of length k is at least $((k-5)/2e)^{k-5} \ge (n/8e)^{k-5}$, as required. □

Since we are done if the assumptions of Claim 20 hold, we can now assume that there is no blue path P' of length two with one end vertex in A'' and the other in B'. This means that any vertex in $\{v\} \cup X \cup Y$ is either completely red to A'' or completely red to B'. Therefore, there is a vertex partition of $\{v\} \cup X \cup Y$ into two sets $Z_1 \cup Z_2$ such that each vertex in Z_1 is completely red to A'' and each vertex in Z_2 is completely red to B'.

By another application of Claim 19, we can also assume that there are no two vertex-disjoint red paths of length at most two each with one end vertex in A'' and the other in B'. Therefore, if Z_1 and Z_2 are both non-empty, either Z_1 is completely blue to B' or Z_2 is completely blue to A''. Without loss of generality, suppose that Z_1 is completely blue to B'. If now $|Z_2| \ge 1$, either there is at most one vertex in Z_2 with red neighbors in A'' or there is a vertex $a \in A''$ such that this is the only red neighbor of vertices in Z_2. Therefore, by removing at most one vertex from $V(G)$, it will also be completely blue between Z_2 and A''.

In summary, we have disjoint sets $Z_1' \subset Z_1$, $Z_2' \subset Z_2$, $A''' \subset A''$, and $B'' \subset B'$ (at most one of which differs from its superset) such that $|Z_1' \cup Z_2' \cup A''' \cup B''| \ge n - 1$ and the following conditions hold:

1. Z_1' is completely red to A''' and completely blue to B''.

2. Z_2' is completely blue to A''' and completely red to B''.

3. It is completely blue between A''' and B''.

4. A''' and B'' are both red cliques.

Let $\tilde{A} = A''' \cup Z_1'$ and $\tilde{B} = B'' \cup Z_2'$. Then it is completely blue between $\tilde{A}$ and B'' and between $\tilde{B}$ and A'''. Furthermore,

$$|\tilde{A}| \ge |A'''| \ge |A'| - 2 \ge (1/2 - 2\sqrt{\lambda})n - 2 > (1/2 - 3\sqrt{\lambda})n$$

and, similarly, $|\tilde{B}| \ge (1/2 - 3\sqrt{\lambda})n$. Finally,

$$|\tilde{A}| + |\tilde{B}| \ge n - 1 = 2k - 2. \tag{11}$$

By following the proofs of Claims 18 and 19, we obtain the following two results.

Claim 21. *If there is a blue edge within either $\tilde{A}$ or $\tilde{B}$, then there are at least $(n/5)^{k-2}$ blue cycles of length k.*

Claim 22. *Suppose that $\tilde{A}$ and $\tilde{B}$ are both red cliques, each with $k-1$ vertices. If there are two vertex-disjoint red edges between $\tilde{A}$ and $\tilde{B}$, then there are at least $(n/5e)^{k-6}$ red cycles of length k.*

By Claim 21, we can assume that there is no blue edge within $\tilde{A}$ or $\tilde{B}$. That is, $\tilde{A}$ and $\tilde{B}$ are both red cliques. If either of these cliques has order at least k we are done, as we then get at least $(k-1)!/2 \ge ((k-1)/2e)^{k-1}$ red cycles of length k. Hence, by (11), we can assume that $|\tilde{A}| = |\tilde{B}| = k-1$.

By Claim 22, we can assume that there are no two vertex-disjoint red edges between $\tilde{A}$ and $\tilde{B}$. Therefore, applying Claim 16 to the red subgraph with $S = \tilde{A}$ and $T = \tilde{B}$, we see that by removing at most one vertex, say w, all the edges between $\tilde{A}$ and $\tilde{B}$ are blue. Without loss of generality, we will assume that $w \in \tilde{A}$, noting that w must have at least one blue neighbor in $\tilde{B}$, since otherwise $\tilde{B} \cup \{w\}$ would be a red clique of order k, again completing the proof.

We require one final observation, proved in the same manner as Claim 20.

Claim 23. *Suppose that $w \in \tilde{A}$ and all the edges between $\tilde{A} \setminus \{w\}$ and $\tilde{B}$ are blue, while at least one edge between w and $\tilde{B}$ is blue. If there is a blue path P' of length two with one end vertex in $\tilde{A}$ and the other in $\tilde{B}$, then there are at least $(n/8e)^{k-5}$ blue cycles of length k.*

Suppose that u is the single vertex of K_n which is not in $\tilde{A} \cup \tilde{B}$. If u has a blue neighbor in both $\tilde{A}$ and $\tilde{B}$, then Claim 23 implies that we are done. Therefore, we can assume that u is completely red to either $\tilde{A}$ or $\tilde{B}$. If u is completely red to $\tilde{A}$, then $\tilde{A} \cup \{u\}$ is a red clique with k vertices, in which case there are at least $(k-1)!/2$ red cycles of length k. Since this is also true if u is completely red to $\tilde{B}$, this completes the proof. $\square$

2.5 Proof of Lemma 10

The following stability lemma of Nikiforov and Schelp [19] is an essential ingredient in our proof.

Lemma 24 (Theorem 13, [19])**.** *Let $0 < \alpha < 5 \cdot 10^{-6}$, $0 \leq \beta \leq \alpha/25$, and $n \geq \alpha^{-1}$. If G is a graph with n vertices and $e(G) > (1/4 - \beta)n^2$, then one of the following holds:*

1. *There are cycles $C_t \subset G$ for every $t \in [3, \lceil (1/2 + \alpha)n \rceil]$.*

2. *There exists a partition $V(G) = U_0 \cup U_1 \cup U_2$ such that*

$$|U_0| < 2000\alpha n,$$

$$\left(1/2 - 10\sqrt{\alpha + \beta}\right) n < |U_1| \leq |U_2| < \left(1/2 + 10\sqrt{\alpha + \beta}\right) n,$$

and the induced subgraph $G - U_0$ on vertex set $V(G) \setminus U_0$ is a subgraph of either the complete bipartite graph between U_1 and U_2 or its complement.

With this preliminary in place, we can begin the proof of Lemma 10. We apply the colored regularity lemma, Lemma 8, with parameters ϵ and $l = \lceil \epsilon^{-1} \rceil$ to the given red/blue coloring of K_n. This implies that there exist $n_0(\epsilon)$ and $M_0(\epsilon)$ such that, for any $n \geq n_0$, there is a regular partition of K_n into M parts $V_1, \ldots, V_M$ with $\epsilon^{-1} \leq M \leq M_0$. We now consider a reduced graph H with M vertices $v_1, \ldots, v_M$ corresponding to $V_1, \ldots, V_M$, placing an edge between v_i and v_j if and only if the pair (V_i, V_j) is ϵ-regular. We then color the edge (v_i, v_j) red if the density of red edges between V_i and V_j is at least $d = 12\epsilon^{1/2}$ and we color an edge blue under the analogous condition with blue in place of red. By the regularity lemma, all but at most $\epsilon\binom{M}{2}$ pairs of distinct vertices of H are edges and, since $d < 1/2$, all edges of H are colored red, blue, or, perhaps, both red and blue. We say an edge is red-only if it is colored in red and not blue, while blue-only is defined similarly.

Let the subgraph of H induced by edges containing the color red be H_R and the subgraph induced by edges containing the color blue be H_B. Hence,

$$|E(H_R)| + |E(H_B)| \geq (1 - \epsilon)\binom{M}{2} > (1 - 2\epsilon)M^2/2,$$

where we used that $M \geq \epsilon^{-1}$. Thus, without loss of generality, we can assume that

$$|E(H_R)| > (1 - 2\epsilon)M^2/4.$$

We now apply Lemma 24 to H_R with $\beta = \epsilon/2$ and $\alpha = 20\sqrt{\epsilon}$. There are two cases:

Case 1 of Lemma 24. In this case, we can find a red cycle C_t for every $t \in [3, \lceil (1/2+\alpha)M \rceil]$. In particular, we can find an odd cycle C_t with

$$(1/2+\alpha)M \geq t > (1/2+\alpha)M - 2.$$

But this means that there are disjoint vertex sets $V_{k_0}, \ldots, V_{k_{t-1}}$ such that, for each $0 \leq i \leq t-1$, $|V_{k_i}| \geq \lfloor n/M \rfloor$ and each pair $(V_{k_i}, V_{k_{i+1}})$ (with addition taken mod t) is ϵ-regular in red with red density at least $12\epsilon^{1/2}$. Thus, we are in Case 1 of Lemma 10.

Case 2 of Lemma 24. In this case, there exists a partition $V(H_R) = U_0 \cup U_1 \cup U_2$ such that $|U_0| < 2000\alpha M$ and

$$\left(1/2 - 10\sqrt{2\alpha}\right) M < |U_1| \leq |U_2| < \left(1/2 + 10\sqrt{2\alpha}\right) M. \tag{12}$$

Furthermore, the induced subgraph $H_R - U_0$ is a subgraph of the disjoint cliques on U_1 and U_2 or a subgraph of the complete bipartite graph between U_1 and U_2. We will assume that the induced subgraph $H_R - U_0$ is a subgraph of the graph consisting of disjoint cliques on U_1 and U_2. The other case, where all edges of $H_R - U_0$ are between U_1 and U_2, can be handled similarly.

Thus, by assumption, any edges between U_1 and U_2 are blue-only. Moreover, since the number of non-adjacent pairs is at most $\epsilon\binom{M}{2}$, the number of blue-only edges between U_1 and U_2 is at least

$$|U_1||U_2| - \epsilon\binom{M}{2}. \tag{13}$$

Let $U_1' \subset U_1$ be the set of vertices in U_1 that have blue degree at least $(1-\sqrt{\epsilon})|U_2|$ in U_2. Suppose $|U_1 \setminus U_1'| = x|U_1|$. Then

$$(1-\sqrt{\epsilon})|U_2|x|U_1| + |U_2|(1-x)|U_1| \geq |U_1||U_2| - \epsilon M^2/2,$$

which implies that $x \leq \sqrt{\epsilon}M^2/(2|U_1||U_2|)$. Since $|U_1|, |U_2| \geq (1/2 - 10\sqrt{2\alpha})M$, we have

$$x \leq \sqrt{\epsilon}M^2/(2|U_1||U_2|) \leq \sqrt{\epsilon}M^2/(2(1/2-10\sqrt{2\alpha})^2M^2) < \sqrt{\epsilon}(2+200\sqrt{\alpha}) < 3\sqrt{\epsilon},$$

where we used that $\alpha < 5 \cdot 10^{-6}$. Defining $U_2' \subset U_2$ analogously, we therefore have

$$|U_1 \setminus U_1'| \leq 3\sqrt{\epsilon}|U_1|, \quad |U_2 \setminus U_2'| \leq 3\sqrt{\epsilon}|U_2|. \tag{14}$$

Thus, each vertex in U_1' has at least $(1-\sqrt{\epsilon})|U_2| - |U_2 \setminus U_2'| \geq (1-4\sqrt{\epsilon})|U_2|$ blue neighbors in U_2' and, similarly, each vertex in U_2' has at least $(1-4\sqrt{\epsilon})|U_1|$ blue neighbors in U_1'.

Claim 25. *If there is a blue edge within U_1' or U_2', then Case 1 of Lemma 10 holds.*

Proof. It will suffice to show that there is a blue cycle C_t in H, where t is the odd integer with $(1/2+\alpha)M \geq t > (1/2+\alpha)M - 2$. Suppose that there is a blue edge (u, u') in U_1'. We will apply Claim 13 to H_B with (S, T, l) being (U_1', U_2', t). Since each vertex in U_1' has at least $(1-4\sqrt{\epsilon})|U_2|$ blue neighbors in U_2', any two vertices in U_1' have blue common neighborhood in U_2' of order at least

$$2(1-4\sqrt{\epsilon})|U_2| - |U_2'| \geq (1-8\sqrt{\epsilon})|U_2'|.$$

Thus, we can let s in Claim 13 be $(1-8\sqrt{\epsilon})|U_2'|$. To check that Claim 13 applies, we need to show that $(1/2+\alpha)M \leq \min(2|U_1'|-1, 2s+1)$. First, by (12), (14), and the fact that $M \geq \epsilon^{-1}$,

$$2|U_1'| - 1 \geq 2(1-3\sqrt{\epsilon})|U_1| - 1 \geq 2(1-3\sqrt{\epsilon})(1/2-10\sqrt{2\alpha})M - 1 > 0.6M > (1/2+\alpha)M.$$

Similarly,

$$2s+1 \geq 2(1-8\sqrt{\epsilon})(1-3\sqrt{\epsilon})|U_2| \geq 2(1-11\sqrt{\epsilon})(1/2-10\sqrt{2\alpha})M > 0.6M > (1/2+\alpha)M.$$

Thus, by Claim 13, there is a cycle of length t in H_B, as required. □

We may therefore assume that there is no blue edge inside U_1' or U_2'. That is, all the edges within U_1' and U_2' are red-only. We move the vertices in U_0 arbitrarily to U_1 and U_2 to obtain $\tilde{U}_1$ and $\tilde{U}_2$. Thus, $\tilde{U}_1 \cup \tilde{U}_2$ is a vertex partition of $V(H)$. Let $X_1 \subset V(K_n)$ be the vertices in K_n corresponding to $\tilde{U}_1$ in H and let $X_2 \subset V(K_n)$ be the vertices corresponding to $\tilde{U}_2$. We will conclude the proof by showing that the partition $X_1 \cup X_2$ induces an extremal coloring.

Claim 26. *The vertex partition $X_1 \cup X_2$ induces an extremal coloring with parameter λ, where $\lambda = 300\sqrt{\alpha}$.*

Proof. By Claim 25, any edge in U_1' is red-only and at most $\epsilon\binom{M}{2}$ pairs of distinct vertices in U_1' are non-adjacent. Moreover, for any red-only edge (i,j) in U_1', the red density between V_i and V_j is at least $1-d$, since otherwise (i,j) would also be colored blue. Since $n/M - 1 < |V_i| \le n/M + 1$, the number of red edges in X_1 is at least

$$(1-d)(n/M-1)^2\binom{|U_1'|}{2} - (n/M+1)^2\epsilon\binom{M}{2}.$$

Note also, by (12), that

$$\begin{aligned}|X_1| &\le |\tilde{U}_1|\cdot(n/M+1) \le (|U_1|+|U_0|)(n/M+1) \le (|U_1| + 2000\alpha M)(n/M+1)\\ &\le \left(1/2 + 10\sqrt{2\alpha} + 2000\alpha\right) M(n/M+1) \le (1/2 + 20\sqrt{\alpha})n.\end{aligned}$$

Combining the two inequalities above with (12) and (14), we see that the red density in X_1 is at least

$$\begin{aligned}\frac{(1-d)\binom{|U_1'|}{2}(n/M-1)^2 - (n/M+1)^2\epsilon M^2/2}{|X_1|^2/2} &\ge \frac{2(1-d)\binom{(1-3\sqrt{\epsilon})|U_1|}{2}(n/M-1)^2 - (n/M+1)^2\epsilon M^2}{(1/2+20\sqrt{\alpha})^2 n^2}\\ &\ge 1 - d - 200\sqrt{\alpha} - 10\sqrt{\epsilon} > 1 - 300\sqrt{\alpha}.\end{aligned}$$

Similarly, $|X_2| \le (1/2 + 20\sqrt{\alpha})n$ and the red density in $X_2 \subset V(G)$ is at least $1 - 300\sqrt{\alpha}$.

It only remains to lower bound the blue density between X_1 and X_2. By (12) and (13), the number of blue edges between X_1 and X_2 is at least

$$\begin{aligned}(1-d)\left(|U_1||U_2| - \epsilon\binom{M}{2}\right)(n/M-1)^2 &\ge (1-d)\left((1/2 - 20\sqrt{\alpha})^2M^2 - \epsilon\binom{M}{2}\right)(n/M-1)^2\\ &> (1-d)(1/4 - 25\sqrt{\alpha})n^2.\end{aligned}$$

Thus, by a similar computation to before, the blue density between X_1 and X_2 is at least

$$\frac{(1-d)(1/4-25\sqrt{\alpha})n^2}{|X_1||X_2|} \ge \frac{(1-d)(1/4-25\sqrt{\alpha})n^2}{n^2/4} > 1 - 300\sqrt{\alpha},$$

as required. □

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