Lecture 2

A perfect matching in a bipartite graph with two sets of equal size is a collection of edges such that every vertex is contained in exactly one of them.

Hall’s (marriage) theorem is a necessary and sufficient condition which allows one to decide if a given bipartite graph contains a matching. Suppose that the two parts of the bipartite graph $G$ are $A$ and $B$. Then Hall’s theorem says that if, for every subset $U$ of $A$, there are at least $|U|$ vertices in $B$ with neighbours in $U$ then $G$ contains a perfect matching. The condition is clearly necessary. To prove that it is sufficient we use the following notation.

For any subset $X$ of a graph $G$, let $N_G(X)$ be the set of neighbours of $X$, that is, the set of vertices with a neighbour in $X$.

**Theorem 1 (Hall’s theorem)** Let $G$ be a bipartite graph with parts $A$ and $B$ of equal size. If, for every $U \subset A$, $|N_G(U)| \geq |U|$ then $G$ contains a perfect matching.

**Proof** Let $|A| = |B| = n$. We will prove the theorem by induction on $n$. Clearly, the result is true for $n = 1$. We therefore assume that it is true for $n - 1$ and prove it for $n$.

If $|N_G(U)| \geq |U| + 1$ for every non-empty proper subset $U$ of $A$, pick an edge $\{a, b\}$ of $G$ and consider the graph $G' = G - \{a, b\}$. Then every non-empty set $U \subset A \setminus \{a\}$ satisfies

$$|N_{G'}(U)| \geq |N_G(U)| - 1 \geq |U|.$$  

Therefore, there is a perfect matching between $A \setminus \{a\}$ and $B \setminus \{b\}$. Adding the edge from $a$ to $b$ gives the full matching.

Suppose, on the other hand, that there is some non-empty proper subset $U$ of $A$ for which $|N(U)| = |U|$. Let $V = N(U)$. By induction, since Hall’s condition holds for every subset of $U$, there is a matching between $U$ and $V$. But Hall’s condition also holds between $A \setminus U$ and $B \setminus V$. If this weren’t the case, there would be some $W$ in $A \setminus U$ with fewer than $|W|$ neighbours in $B \setminus V$. Then $W \cup U$ would be a subset of $A$ with fewer than $|W \cup U|$ neighbours in $B$, contradicting our assumption. Therefore, there is a perfect matching between $A \setminus U$ and $B \setminus V$. Putting the two matchings together completes the proof. □

A Hamiltonian cycle in a graph $G$ is a cycle which visits every vertex exactly once and returns to its starting vertex. Dirac’s theorem says that if the minimum degree $\delta(G)$ of a graph $G$ is such that $\delta(G) \geq n/2$ then $G$ contains a Hamiltonian cycle. This is sharp since, if one takes a complete bipartite graph with one part of size $\lceil \frac{n}{2} - 1 \rceil$ (and the other the complement of this), then it cannot contain a Hamiltonian cycle. This is simply because one must pass back and forth between the two sets.

**Theorem 2 (Dirac’s theorem)** If a graph $G$ satisfies $\delta(G) \geq \frac{n}{2}$, then it contains a Hamiltonian cycle.

**Proof** First, note that $G$ is connected. If it weren’t, the smallest component would have size at most $n/2$ and no vertex in this component could have degree $n/2$ or more.

Let $P = x_0x_1 \ldots x_k$ be a longest path in $G$. Since it can’t be extended, every neighbour of $x_0$ and $x_k$ must be contained in $P$. Since $\delta(G) \geq n/2$, we see that $x_0x_{i+1}$ is an edge for at least $n/2$ values of $i$. 

with $0 \leq i \leq k - 1$. Similarly, $x_ix_k$ is an edge for at least $n/2$ values of $i$. There are at most $n - 1$ values of $i$ with $0 \leq i \leq k - 1$. Therefore, since the total number of edges of the form $x_0x_{i+1}$ or of the form $x_ix_k$ with $0 \leq i \leq k - 1$ is at least $n$, there must be some $i$ for which both $x_0x_{i+1}$ and $x_i x_k$ are edges in $G$.

We claim that

$$ C = x_0x_{i+1}x_{i+2} \ldots x_kx_ix_{i-1} \ldots x_0 $$

is a Hamiltonian cycle. Suppose not and that there is a set of vertices $Y$ which are not contained in $C$. Then, since $G$ is connected, there is a vertex $x_j$ and a vertex $y$ in $Y$ such that $x_j y$ is in $E(G)$. But then we may define a path $P'$ starting at $y$, going to $x_j$ and then around the cycle $C$ which is longer than $P$. This would contradict our assumption about $P$. $\square$

A tree $T$ is a connected graph containing no cycles. The Erdős-Sós conjecture states that if a tree $T$ has $t$ edges, then any graph $G$ with average degree $t$ must contain a copy of $T$. This conjecture has been proven, for sufficiently large graphs $G$, by Ajtai, Komlós, Simonovits and Szemerédi. Here we prove a weaker version of this conjecture.

**Theorem 3** If a graph $G$ has average degree $2t$, it contains every tree $T$ with $t$ edges.

**Proof** We start with a standard reduction, by showing that a graph of average degree $2t$ has a subgraph of minimum degree $t$. If the number of vertices in $G$ is $n$, the number of edges in $G$ is at least $tn$. If there is a vertex of degree less than $t$, delete it. This will not decrease the average degree. Moreover, the process must end, since any graph with fewer than $2t$ vertices cannot have average degree $2t$.

We now use this condition to embed the vertices of the tree greedily. Suppose we have already embedded $j$ vertices, where $j < t + 1$. We will try to embed a new vertex which is connected to some already embedded vertex. By the minimum degree condition, there are at least $t$ possible places to embed this vertex. At most $t - 1$ of these are blocked by already embedded vertices, so the embedding may always proceed. $\square$