

Extremal numbers of cycles revisited

David Conlon*

Abstract

We give a simple geometric interpretation of an algebraic construction of Wenger that yields n -vertex graphs with no cycle of length 4, 6 or 10 and close to the maximum number of edges.

What is the maximum number of edges in an n -vertex graph containing no cycle C_{2k} of length $2k$? This is one of the most closely studied yet elusive problems in combinatorics. Despite many decades of intense interest in this and related problems in extremal graph theory [11], the answer is only reasonably well understood for cycles of length 4, 6 and 10.

If we write $\text{ex}(n, H)$ for the maximum number of edges in an n -vertex graph containing no copy of the graph H , then a result attributed to Erdős [2, 5] says that $\text{ex}(n, C_{2k}) \leq Cn^{1+1/k}$ for some constant C depending only on k . For C_4 , a matching lower bound, due to Klein, already appeared in a paper of Erdős [4] from 1938. For C_6 and C_{10} , constructions matching the upper bound were found by Benson [1] and Singleton [9] in 1966, though several alternative constructions have since been found [7, 8, 13]. None of these constructions can be described as simple, though that of Wenger [13] has the best claim. Here we show that this claim is justified, by rephrasing his algebraic construction in geometric terms that make its properties manifest.

Let q be a prime power and $\mathbb{F}_q$ the finite field of order q . For each $x \in \mathbb{F}_q^k$ and $z \in \mathbb{F}_q$, we form the line

$$\{x + y \cdot (1, z, z^2, \dots, z^{k-1}) : y \in \mathbb{F}_q\}.$$

The number of distinct lines of this form is exactly q^k , since there are q distinct directions, determined by the different values for z , and exactly q^{k-1} parallel lines in each direction. We form a bipartite graph $D_k(q)$ between two sets P and L of order q^k , where the vertices of P are indexed by the points of $\mathbb{F}_q^k$, the vertices of L are indexed by the q^k lines described above and there is an edge between $p \in P$ and $\ell \in L$ if and only if $p \in \ell$. Our main result is as follows.

Theorem 1. *For $k = 2, 3$ and 5, $D_k(q)$ is a C_{2k} -free bipartite graph between sets P and L of order $n = q^k$ with $n^{1+1/k}$ edges.*

Proof. The bound on the number of edges follows from the fact that each of the q^k lines contains exactly q points, so there are $q^{k+1} = n^{1+1/k}$ edges in total. Note now that any cycle in $D_k(q)$ is of the form $p_1\ell_1p_2\ell_2\dots p_t\ell_tp_1$, where $p_i \in P$ and $\ell_i \in L$ for all $1 \leq i \leq t$. We make two simple observations about these cycles:

- For any $1 \leq i \leq t$ (taken mod t), ℓ_i and ℓ_{i+1} cannot be parallel.

*Department of Mathematics, California Institute of Technology, Pasadena, CA 91125. Email: dconlon@caltech.edu.

- If $t \leq k$, then, for any $1 \leq i \leq t$, there must be another line $\ell_{i'}$ with $1 \leq i' \leq t$ parallel to ℓ_i .

The first observation is obvious, since two parallel lines, both passing through the point p_{i+1} , must coincide. For the second observation, note that if ℓ_i has direction determined by z_i for each $1 \leq i \leq t$, then the difference $p_{i+1} - p_i$ is a non-zero multiple of $(1, z_i, z_i^2, \dots, z_i^{k-1})$. Adding over all i , we have that

$$0 = \sum_{i=1}^t (p_{i+1} - p_i) = \sum_{i=1}^t a_i (1, z_i, z_i^2, \dots, z_i^{k-1})$$

for some collection of non-zero coefficients a_i . But it is a well-known fact, proved by considering the Vandermonde determinant, that any k distinct vectors of the form $(1, z, z^2, \dots, z^{k-1})$ are linearly independent. Hence, for the sum $\sum_{i=1}^t a_i (1, z_i, z_i^2, \dots, z_i^{k-1})$ to be zero, each direction must appear at least twice. That is, every line ℓ_i has at least one parallel line $\ell_{i'}$. We can now dispense with each case as a bulletpoint:

- No $D_k(q)$ with $k \geq 2$ contains a cycle of length 4, since any such cycle $p_1 \ell_1 p_2 \ell_2 p_1$ must be such that ℓ_1 and ℓ_2 are both parallel and not parallel by the two observations above.
- No $D_k(q)$ with $k \geq 3$ contains a cycle of length 6, since any such cycle $p_1 \ell_1 p_2 \ell_2 p_3 \ell_3 p_1$ must again be such that the three lines ℓ_1, ℓ_2 and ℓ_3 are all parallel and all not parallel by our two observations.
- No $D_k(q)$ with $k \geq 5$ contains a cycle of length 10. Indeed, any such cycle $p_1 \ell_1 p_2 \ell_2 p_3 \ell_3 p_4 \ell_4 p_5 \ell_5 p_1$ must have one group of two parallel lines and another group of three parallel lines. But then it is impossible to enforce the condition that ℓ_i not be parallel to ℓ_{i+1} for all $1 \leq i \leq 5$ (taken mod 5). $\square$

Suppose now that $\theta_{k,\ell}$ is the graph consisting of ℓ internally disjoint paths of length k , each with the same endpoints. In particular, $\theta_{k,2}$ is just C_{2k} , so the problem of determining $\text{ex}(n, \theta_{k,\ell})$ extends the problem of determining $\text{ex}(n, C_{2k})$. The extremal numbers for these theta graphs were first studied by Faudree and Simonovits [6], who proved that $\text{ex}(n, \theta_{k,\ell}) \leq Cn^{1+1/k}$ for a constant C depending only on k and ℓ . On the other hand, a result of Conlon [3] says that for any k there exists a natural number ℓ and a positive constant c such that $\text{ex}(n, \theta_{k,\ell}) \geq cn^{1+1/k}$.

The following question therefore becomes valid: for each k , what is the smallest ℓ such that $\text{ex}(n, \theta_{k,\ell}) \geq cn^{1+1/k}$ for some positive c ? The results on $\text{ex}(n, C_{2k})$ show that for $k = 2, 3$ or 5 we can take $\ell = 2$. A much more recent result of Verstraëte and Williford [12] says that for $k = 4$ it suffices to take $\ell = 3$. As a bonus, we show that the graph $D_4(q)$ has a one-sided version of the property satisfied by the Verstraëte–Williford graph, thus giving a natural interpolation between the cases $k = 3$ and 5 .

Theorem 2. $D_4(q)$ is a bipartite graph between sets P and L of order $n = q^4$ with $n^{5/4}$ edges such that any two vertices in P have at most 2 paths of length 4 between them.

Proof. A theta graph $\theta_{4,\ell}$ whose endpoints are $p, p' \in P$ consists of paths $p\ell_{1,j}p_2\ell_{2,j}p'$ with $1 \leq j \leq \ell$. However, any two such paths yield a cycle, so we can still apply our basic observations:

- No $D_k(q)$ with $k \geq 4$ contains a theta graph $\theta_{4,3}$ with both endpoints $p, p' \in P$. Indeed, any such graph consists of paths $p\ell_{1,j}p_2\ell_{2,j}p'$ for $1 \leq j \leq 3$ where $\ell_{1,j}$ and $\ell_{2,j}$ are not parallel,

by our first observation. But, since $p\ell_{1,1}p_{2,1}\ell_{2,1}p'\ell_{2,2}p_{2,2}\ell_{1,2}p$ is a cycle, we must have, by our second observation, that $\ell_{1,1}$ is parallel to $\ell_{2,2}$ and $\ell_{2,1}$ is parallel to $\ell_{1,2}$. Similarly, considering the cycle $p\ell_{1,1}p_{2,1}\ell_{2,1}p'\ell_{2,3}p_{2,3}\ell_{1,3}p$, we must have that $\ell_{1,1}$ is parallel to $\ell_{2,3}$ and $\ell_{2,1}$ is parallel to $\ell_{1,3}$. But then the cycle $p\ell_{1,2}p_{2,2}\ell_{2,2}p'\ell_{2,3}p_{2,3}\ell_{1,3}p$ violates the first observation, since $\ell_{2,2}$ and $\ell_{2,3}$ are parallel. $\square$

We conclude with a conjecture. It is now quite commonplace to believe that the true value of certain extremal numbers lie below what the classical arguments give. We suspect that this should already be the case for C_8 , that is, that $\text{ex}(n, C_8) = o(n^{5/4})$. Proving this is likely to be exceptionally difficult, but a first step might be to show that no construction of the type studied in this paper can yield the lower bound. Concretely, we have the following conjecture. The definition of a line in $\mathbb{F}_q^k$ agrees with that used throughout, namely, a set of the form $\{x + y \cdot z : y \in \mathbb{F}_q\}$ for some $x, z \in \mathbb{F}_q^k$.

Conjecture 1. *The maximum number of lines in $\mathbb{F}_q^4$ containing no C_4 of lines, that is, four distinct lines $\ell_1, \ell_2, \ell_3, \ell_4$ such that ℓ_i and ℓ_{i+1} intersect in distinct points for all $1 \leq i \leq 4$ (taken mod 4), is $o(q^4)$.*

A similar conjecture can be made for any cycle of length at least 6.

Note added. After this article was accepted, Tibor Szabó drew my attention to his lecture notes [10], where Wenger's construction is also described in very similar terms.

References

- [1] C. T. Benson, Minimal regular graphs of girths eight and twelve, *Canad. J. Math.* **18** (1966), 1091–1094.
- [2] J. A. Bondy and M. Simonovits, Cycles of even length in graphs, *J. Combin. Theory Ser. B* **16** (1974), 97–105.
- [3] D. Conlon, Graphs with few paths of prescribed length between any two vertices, *Bull. Lond. Math. Soc.* **51** (2019), 1015–1021.
- [4] P. Erdős, On sequences of integers no one of which divides the product of two others and on some related problems, *Mitt. Forsch.-Inst. Math. Mech. Univ. Tomsk* **2** (1938), 74–82.
- [5] P. Erdős, Extremal problems in graph theory, in *Theory of Graphs and its Applications* (Proc. Sympos. Smolenice, 1963), 29–36, Publ. House Czechoslovak Acad. Sci., Prague, 1964.
- [6] R. J. Faudree and M. Simonovits, On a class of degenerate extremal graph problems, *Combinatorica* **3** (1983), 83–93.
- [7] F. Lazebnik and V. A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, *Discrete Appl. Math.* **60** (1995), 275–284.
- [8] K. E. Mellinger and D. Mubayi, Constructions of bipartite graphs from finite geometries, *J. Graph Theory* **49** (2005), 1–10.
- [9] R. R. Singleton, On minimal graphs of maximum even girth, *J. Combin. Theory* **1** (1966), 306–332.

- [10] T. Szabó, lecture notes available at <http://discretemath.imp.fu-berlin.de/DMI-2018/notes-evencycles.pdf>.
- [11] J. Verstraëte, Extremal problems for cycles in graphs, in Recent trends in combinatorics, 83–116, IMA Vol. Math. Appl., 159, Springer, Cham, 2016.
- [12] J. Verstraëte and J. Williford, Graphs without theta subgraphs, *J. Combin. Theory Ser. B* **134** (2019), 76–87.
- [13] R. Wenger, Extremal graphs with no C^4 's, C^6 's, or C^{10} 's, *J. Combin. Theory Ser. B* **52** (1991), 113–116.