PARTICLES IN STRING THEORY

Daniel Wagenaar

ABSTRACT

A low energy effective action is derived for a single D-particle. We find that at open string tree level, it behaves like a free particle with mass $\sim 1/g_c$. A system of many D-particles is described by a Yang-Mills action after dimensional reduction to 0+1 dimensions. Physically, the degrees of freedom are the relative positions of the particles, and a set of harmonic oscillators corresponding to strings that may stretch between them. Upon compactification, the quantum mechanics of many D-particles turns out to be equivalent to Yang-Mills theory on a single multiply wound D-string, a fact that is known as T-duality.

Scattering of two D-particles is considered. It is found that supersymmetry is essential, since purely bosonic D-particles are confined by a potential that grows linearly with the distance between them. In the supersymmetric theory, D-particle scattering is found to contain broad resonances, which offers evidence for the existence of bound states of N D-particles, as needed for the conjectured equivalence of M-theory and D-particle quantum mechanics.

A short general introduction to string theory, and to the way D-branes appear in it, is included.

Particles in string theory — doctoraalscriptie by Daniel Wagenaar
Instituut voor Theoretische Fysica, Universiteit van Amsterdam, August 1997
Supervisors: prof dr H. L. Verlinde and prof dr R. H. Dijkgraaf

Piglet came a little closer to see what it was. Eeyore had three sticks on the ground, and was looking at them. Two of the sticks were touching at one end, but not at the other, and the third stick was laid accross them. Piglet thought that perhaps it was a Trap of some kind.

'Oh, Eeyore,' he began again, 'I just —'

'Is that little Piglet?' said Eeyore, still looking hard at his sticks.

'Yes, Eeyore, and I —'

'Do you know what this is?'

'No,' said Piglet.

'It's an A.'

'Oh,' said Piglet.

'Not O – A,' said Eeyore severely. 'Can't you hear, or do you think you have more education than Christopher Robin?'

'Yes,' said Piglet. 'No,' said Piglet very quickly. And he came closer still.

'Do you know what A means, little Piglet?'

'No, Eeyore, I don't.'

'It means Learning, it means Education, it means all the things that you and Pooh haven't got. That's what A means.'

'Oh,' said Piglet again. 'I mean, does it?' he explained quickly. He stepped back nervously, and looked round for help.

'Here's Rabbit,' he said gladly. 'Hallo, Rabbit.'

'What's this that I'm looking at?' said Eeyore still looking at it.

'Three sticks,' said Rabbit promptly.

'You see?' said Eeyore to Piglet. He turned to Rabbit. 'I will now tell you what Christopher Robin does in the mornings. He learns. He becomes Educated. He instigorates – I think that is the word he mentioned, but I may be referring to something else – he instigorates Knowledge. In my small way I also, if I have the word right, am – am doing what he does. That, for instance, is —'

'An A,' said Rabbit, 'but not a very good one. Well I must get back and tell the others.'

Eeyore looked at his sticks and then he looked at Piglet.

'What did Rabbit say it was?' he asked.

'An A,' said Piglet.

'Did you tell him?'

'No, Eeyore, I didn't. I expect he just knew.'

'He knew? You mean this A thing is a thing Rabbit knew?'

'Yes, Eeyore. He's clever, Rabbit is.'

'Clever!' said Eeyore scornfully, putting a foot heavily on his three sticks. 'Education!' said Eeyore bitterly, jumping on his six sticks. 'What is Learning?' asked Eeyore as he kicked his twelve sticks into the air. 'A thing Rabbit knows! Ha!'

— A. A. Milne, 'The house at Pooh corner' (slightly condensed)

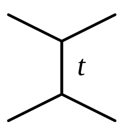
CONTENTS

HISTORICAL INTRODUCTION TO STRING THEORY AND D-BRANES

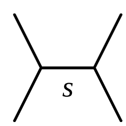
People will one day look back
on our epoch as the period
when it all began.

— *Edward Witten*

String theory was first introduced as a model for understanding the abundance of resonances found in strong interaction experiments. Resonances had already been found with spins ranging upto $^{11}/_{2}$, and – quite apart from the aesthetic unpleasantness of a theory with a large number of fundamental particles – no consistent theory was known for describing fundamental particles with spin higher than one. In 1968 Veneziano found a model for two particle scattering that produced amplitudes with contributions from the exchange of particles with any (integer) spin. This model had two very important characteristics: first, it predicted very soft high energy behaviour, whereas ordinary field theories containing high spin exchanges necessarily have divergent amplitudes in the ultraviolet[1)]. Second, it has the amazing property of *duality*: the expansion of the amplitude in terms of t-channel poles is exactly the same as the expansion in s-channel poles.



t*-channel scattering*



s*-channel scattering*

The Veneziano model offered a description of mesons in terms of strings[2)], with charges on either end. These charges are equivalent to the flavour charges of the modern description of mesons in QCD. By picking charges in the fundamental ($\mathbf{3}$) and anti-fundamental ($\bar{\mathbf{3}}$) representations of $SU(3)$ for the string endpoint charges, it modelled the then-known mesons (upto η and ϕ; J/ψ had not yet been discovered): $SU(3)$ is the ('Eightfold Way') symmetry group of u, d and s quarks.

[1)]Two scalar particles exchanging a spin m particle give rise to an t-channel amplitude roughly of the form $A \sim \frac{s^J}{t-M^2}$. The factor s^J can be understood from the fact that the vertex of two scalar particles ϕ and a spin J particle $\chi_{\mu_1\mu_2\ldots\mu_J}$ must be proportional to $\phi^* \overleftrightarrow{\partial}_{\mu_1} \cdots \overleftrightarrow{\partial}_{\mu_J} \phi\chi^{\mu_1\mu_2\ldots\mu_J}$. This gives $2J$ factors of momentum for the t-channel diagram, or (very roughly) J factors of s.

[2)]Historically, the interpretation in terms of open strings was discovered only a some years later.

The model's high energy behaviour fitted experimental data quite well, but there were some problems. Most importantly, the theory could not describe baryons, because it contained only particles with integer spin. Then, in 1969, experiments at SLAC showed that the structure functions of baryons agreed very well with the notion of point-like constituents, or partons[1)]. The discovery of the J/ψ particle and its explanation in terms of a fourth quark ended the debate in favour of the parton model of gluons and quarks, or quantum chromodynamics. The Veneziano model disappeared from the particle physicists' view of the world.

Theoretical physicists, however, did not lose interest completely, and investigated whether the model could be used in another context where high spin had been causing major problems: quantum gravity. The graviton of general relativity is a spin two particle, which in quantum field theory has a badly diverging high energy scattering amplitude: $A \sim s^2/t$. String theory could solve this problem, because of the beautiful scaling behaviour of the Veneziano model. However, such optimism was dealt a heavy blow when the theory was found to predict a tachyon, a particle with negative mass squared, that really belongs to the realm of science-fiction.

Luckily, a way to get rid of the tachyon was developed during the '70s: in 1971, Ramond [2], and Neveu and Schwarz [3] had found methods for introducing fermions to string theory. Their models contained a symmetry between bosonic and fermionic degrees of freedom, which was to become known as supersymmetry. In 1974, Wess and Zumino extended these ideas to supersymmetry in four dimensions [4]. In 1977 Gliozzi, Scherk and Olive showed [5] that it was possible to truncate the spectrum of the Ramond-Neveu-Schwarz model in such a way that it becomes supersymmetric in space-time. Their technique, which has become known as the GSO projection, expunges the tachyon from the string spectrum, and states that the massless graviton and its superpartners, the gravitinos, form the ground state multiplet of the string spectrum. Finally, in 1981 Green and Schwarz found a different formulation for this theory, in which space-time supersymmetry is evident right from the beginning.

By this time, theoretical interest had increased considerably: string theory consistently described massless spin two particles, and related them to other massless particles by supersymmetry. Thus, for the first time in the history of physics, this was a model that might just possibly be found to describe all of the known particles and forces at the same time. In supersymmetric string theory (or 'superstring theory'), the particles occurring in the standard model are represented by excitations of strings. This does not happen in a completely straight-forward manner, though. Considerations of quantum mechanical consistency show that string theory works best in ten space-time dimensions. In order to make contact with the four dimensions of every day experience, it is suggested that six of these should be compactified, that is, replace $\mathbb{R}^{10}$ by $\mathbb{R}^4 \times \mathrm{C}_6$ with C_6 a compact manifold, eg T^6, and take the radii of this manifold to (near) zero. Since a reasonable energy scale for string theory turns out to be near the Planck scale (10^{19} GeV), the particles in the standard model should come forth – through compactification and a Higgs-like mechanism of spontaneous symmetry breaking [6] – from the massless modes of the string. This scheme is an extension of an old attempt to unify gravity and electromagnetism, namely Kaluza-Klein theory, in which a five-dimensional space-time is proposed, with one compact direction. The graviton g_{MN} on this space then splits into the graviton $g_{\mu\nu}$ of four-dimensional space-time, the electromagnetic potential $A_\mu \equiv g_{\mu 4}$ and a scalar, the dilaton

[1)] An historic account of these events may be found in [1], chapter 21.

$\phi \equiv g_{44}$[1].

Gradually, a picture emerged of five consistent superstring theories, which were called type I, with open and closed strings; types IIA and IIB, with closed strings only[2]; and two 'heterotic' theories, which are a mixture of superstrings and bosonic strings[3]. This was all very nice and pretty, but, as many people noted, only one theory-of-everything is really needed: 'As it was in the 'eighties, there were five consistent string theories, leading to the mystery: if one of them describes our world, who lives in the other four worlds?' (Witten, in a 1997 interview).

This mystery has largely been solved, since over the last five or ten years, people have found various connections between these theories, the so-called *dualities*, which relate different regimes of different theories, for example equating the strong and weak coupling constant limits of the heterotic string after compactification to four dimensions, and equating the IIB theory compactified on a cylinder with radius R to the IIA theory compactified on a cylinder with radius $\sim 1/R$. Whereas at first these connections seemed to be independent of one another, in recent years strong evidence has appeared that in fact all of the five string theories[4] are connected [8], leading to the idea that there should be one underlying theory, which is usually referred to as M-theory, although nobody really knows what the 'M' is supposed to stand for: suggestions vary between 'membrane', 'matrix', or even 'magic', or 'mystery'.

Overview

In this thesis we shall be concerned with particles in string theory, which are a consequence of one of the dualities mentioned above: in a 1989 article [9] (see also [10]), Dai, Leigh and Polchinski showed that a theory of open and closed strings in a space-time with some compact directions is equivalent to a seemingly different theory in which the open strings must have their endpoints fixed on a single hyperplane, which is called a D-brane[5],[6]. As will be shown in chapter two, it is possible to have D-branes of any dimension, running from the dimension of space all the way down to zero.

At first D-branes were seen mostly as yet another interesting phenomenon in string theory, that did not have too much importance. However, in 1995 Polchinski showed [11] that D-branes carry a charge, which identified them as a particular kind of object that was required to exist in type II theory because of a conjectured duality, but which no-one had found as yet. Since then, D-branes have been the subject of many studies. In particular the zero dimensional ones, the *D-particles*, have recently been the focus of much interest, since a proposal [12] has been

[1] In the original formulation this extra particle was considered a nuisance, and people went into many twists and turns to explain it away. In string theory, the coupling constant for string–string scattering is determined by the vacuum expectation value of the dilaton, and so nobody would want to sweep under the theoretical rug.

[2] The difference between IIA and IIB is that the IIA theory contains both left- and right-handed fermions, and treats them symmetrically, while IIB theory contains fermions of one handedness only.

[3] It is not easy to explain heterotic strings in a completely qualitative way, but after reading chapter 1 of this thesis, the treatment presented in [7] §6.3 should be accessible.

[4] and a sixth theory, supergravity in eleven dimensions.

[5] 'D' stands for Dirichlet: a string that has its endpoints fixed, satisfies Dirichlet boundary conditions, and 'brane' is a neologism meaning 'an *n*-dimensional generalization of a membrane'.

[6] Actually, strings with Dirichlet boundary conditions had been considered before (as early as the 1970s), but Day, Leigh and Polchinksi were the first to realize that the Dirichlet hyperplane is a dynamical object.

forwarded which suggests that they may play a much more fundamental role in string theory than was understood when they were first discovered. In this proposal, D-particles are perceived as fundamental building blocks in M-theory: the eleven dimensional theory that is conjectured to be the underlying theory from which all string theories may be found in various limits.

D-particles will be the main subject of the present work: after two introductory chapters on string theory and the way D-branes appear in it, we shall derive an effective action for D-particles. This will be done in two steps: first (in chapter three) we shall compute an effective action for a single D-particle, using techniques developed by Fradkin and Tseytlin in 1985 [13, 14, 15, 16] in the context of finding a low energy effective action for various fields coupling to the string. In chapter four we shall generalize this action to a system of many D-particles, and we shall see how the large N limit of N D-particles on a compact space is connected to open string theory.

Finally, in chapter five we shall be concerned with some aspects of the spectrum of the D-particle action, and we shall investigate a scattering experiment of two supersymmetric D-particles. By means of a calculation that is quite interesting in its own right, because of its relative simplicity and its hardly depending on specifically string theoretic techniques, we find that in such scattering, resonances are found with a long lifetime. This is evidence in favour of the existence of bound states of D-particles, a *sine qua non* for M-theory.

Acknowledgements

I wish to express my gratitude to a number of people:
to Robbert Dijkgraaf, whose enthousiasm during his mathematics course got me interested in string theory in the first place;
to Herman Verlinde for helping me meet a sudden deadline;
to my family and Steven for showing confidence in me all through my work on this thesis, even when I felt overwhelmed by it;
to Mischa, with whom I've had many talks about the good and the hard parts of graduation work. It helps a great deal to know you're not alone.
To all the students of rooms 363 and 259 for allowing me to bother them with nasty equations, and for joining me for coffee breaks at unusual times;
to Peter de Paepe and Gerard Bäuerle for helping me with some points of mathematics;
to Claire for coming into my life at this most unexpected time,
and to all for bearing with me while I had so little time and attention left for the world outside.
Thank you all.

CHAPTER 1

PARTICLES AND STRINGS

If a string has one end,
then it has another end.

— Unix fortune cookie

This chapter will introduce the basics of string theory as needed for the rest of this thesis. Since most of the calculations to come will be done in the context of bosonic string theory, that will be the main focus of this chapter. Even so, supersymmetry will play an important role in the final chapter, and a short introduction to this important subject will be given at the end.

The structure of this chapter is as follows: We will first find an action for the string which is an extension of the action for a relativistic particle. We will then proceed to find mode expansions for the fields occurring in this action. These form the basis from which to quantize the theory. We shall be extremely brief about quantization proper. After some discussion about the possibility to add charges to the ends of an open string, as mentioned in the introduction, the final section will present a minimal introduction to supersymmetry in string theory.

About all of these subjects, we shall necessarily be brief. A proper introduction to the major concepts of string theory could easily fill a book. This book has in fact been written [7], and anyone who intends to learn about string theory should read it.

1.1 Strings from particles

As promissed, we start by finding an action for the free relativistic string as an extension of the point particle action. After reviewing the latter, we shall first consider a generalization to objects of any dimension, before focussing on the string action itself.

1.1.1 The relativistic point particle

The action of a free particle in special or general relativity is proportional to the length of its world-line in Minkowski space:

$$S_{\text{particle}} \sim \int_{\text{world}-\text{line}} \mathrm{d}s\,.$$

The form which correctly reduces to the non-relativistic case $S = \int \mathrm{d}t\ \frac{1}{2} m v^2$ is

$$S_{\text{particle}} = -m \int\limits_{\text{world-line}} \mathrm{d}\tau \sqrt{-g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}} \equiv -m \int\limits_{\text{world-line}} \mathrm{d}\tau \sqrt{-\dot{x}^2}, \tag{1.1}$$

where τ is any parametrization of the world-line, $x^\mu(\tau)$ are the coordinates of the particle and $g_{\mu\nu}(x)$ is the metric of space-time. We will be using a Minkowski signature $(-1, +1, \ldots, +1)$.

While this form of the action shows its origin very clearly, it is not well suited for use in calculations, since the square root makes for unpleasant equations of motion. A more manageable form can be acquired by introducing an *einbein* e on the world-line, and writing

$$S_{\text{particle}} = \frac{1}{2} \int\limits_{\text{world-line}} \mathrm{d}\tau \left(e^{-1} \dot{x}^2 - e m^2 \right). \tag{1.2}$$

This reduces to the previous form (1.1) upon solving the equation of motion for e:

$$-e^{-2}\dot{x}^2 - m^2 = 0 \qquad \Leftrightarrow \qquad e = \frac{\sqrt{-\dot{x}^2}}{m}.$$

The action (1.2) is much more useful in calculations, particularly since it is possible to get rid of e by gauge fixing: the action is invariant under reparametrization $\tau \to \tilde{\tau}(\tau)$ for general functions $\tilde{\tau}$. This invariance can be exploited to set e to a convenient value, eg. $e = m^{-1}$. This reduces the action to the simple form

$$S_{\text{particle}} = \frac{m}{2} \int\limits_{\text{world-line}} \mathrm{d}\tau\, \dot{x}^2,$$

since the second term is a constant, which is (classically) unimportant. In this form the action looks quite a lot like the classical form, but this is slightly deceptive: it contains a term $\dot{x}_0^2$, which is absent classically. The gauge choice $e = m^{-1}$ is incompatible with globally setting $x^0 = \tau$, so $\dot{x}_0^2$ is not just a constant.

1.1.2 Extension to *n* dimensions

Just like the path of a point particle – a zero dimensional object – is specified by writing down a mapping from a one dimensional *world-line* into space-time, the path of an n-dimensional object is specified by a mapping from a $(n+1)$-dimensional *world-volume* into space-time. We could take the volume of this path as the action, generalizing (1.1), but it turns out that in fact (1.2) is more easily generalized:

$$S_n = -\frac{T^{(n)}}{2} \int\limits_{\text{world-volume}} \mathrm{d}^{n+1}\sigma \sqrt{-h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \cdot \partial_\beta X_\mu, \tag{1.3}$$

where: $T^{(n)}$ is a parameter of dimension $(\text{mass})^{n+1}$, the ‘tension’ of the object,
σ^α parametrizes theworld-volume ($\alpha = 0 \ldots n$),
$h(\sigma)$ is the determinant of the metric on the world-volume,

$h^{\alpha\beta}(\sigma)$ is the inverse of this metric, and

$X^\mu(\sigma)$ are the space-time coordinates of the object.

Note that this action is again invariant under reparametrizations of the world-volume: as is well known from general relativity, $\mathrm{d}^{n+1}\sigma\sqrt{-h}$ represents the invariant volume element, while $h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$ is manifestly invariant since it contains no free world-sheet indices.

We can ask whether it is possible to gauge $h_{\alpha\beta}$ away in a similar manner as the *einbein* for a particle. In general, this is not the case: whereas in the point particle case e contained exactly the right number of degrees of freedom to be eliminated by the reparametrization invariance of the world-line, its generalization $h_{\alpha\beta}$ contains many more degrees of freedom. In fact, $h_{\alpha\beta}$, being a symmetrical $(n+1)\times(n+1)$-matrix, contains $\frac{1}{2}(n+1)(n+2)$ gauge degrees of freedom, while only $(n+1)$ can be fixed using the invariance. This means that for higher dimensional objects, some degrees of freedom are left in the metric.

For the very important special case $n=1$ – in other words, for strings – there is however a simple way out of this problem: in this case – and only in this case – the action is invariant under *Weyl scaling*:

$$h_{\alpha\beta}(\sigma) \to \Lambda(\sigma)\, h_{\alpha\beta}(\sigma).$$

$\sqrt{-h}$ scales like $\Lambda^{(n+1)/2}$ under this scaling when $h_{\alpha\beta}$ is an $(n+1)\times(n+1)$-matrix, so $\sqrt{-h}\,h^{\alpha\beta}$ is Weyl invariant for $n=1$. Together, the reparametrization and Weyl invariances are exactly enough to gauge fix the three degrees of freedom of $h_{\alpha\beta}$ in the string case.

1.1.3 Action for a free relativistic string

The rest of this chapter will deal only with strings. We will use the term *world-sheet* for the two dimensional world-volume of a string, and we will parametrize it using (τ,σ), τ being a time-like coordinate and σ space-like. The tension T is then associated with a fundamental length, which – in units $\hbar = c = 1$ – is given by $l = 1/\sqrt{\pi T}$.

As noted above, the reparametrization invariance can be used to bring the metric to the form $h_{\alpha\beta} = \mathrm{e}^{-\Phi}(h_0)_{\alpha\beta}$, where h_0 is some standard form. In two dimensions the action (1.3) is Weyl-invariant and the conformal factor $\mathrm{e}^{-\Phi}$ scales out of the problem. $\Phi = \Phi(\tau,\sigma)$ may thus be used to re-shape awkward diagrams to standard shape and to bring external particle lines down to points on a sphere or n-torus, as depicted in figure 1.1.

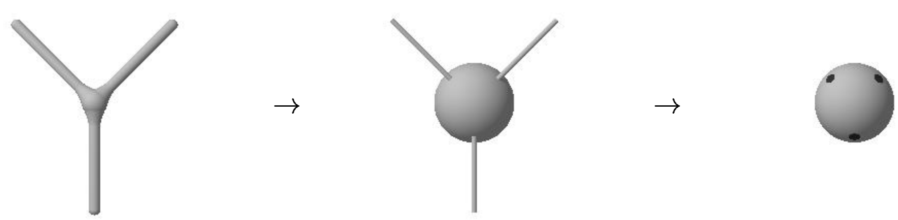


Figure 1.1: Scattering of three strings in mass eigenstates. Transformation of an awkward shape to an easier one using Weyl scaling.

We may thus make the gauge choice $h_{\alpha\beta} = \eta_{\alpha\beta}$, which reduces the action to

$$S_{\text{string}} = \frac{T}{2} \int\limits_{\text{world-sheet}} \mathrm{d}\tau \, \mathrm{d}\sigma \, \left\{ (\partial_\tau X)^2 - (\partial_\sigma X)^2 \right\}. \tag{1.4}$$

1.2 Mode expansion

Mode expansion is the basis for quantization: it is possible to expand the fields X^μ of (1.4) in terms of creation and annihilation operators α_n^μ and $\alpha_n^{\dagger\mu}$. These can be used to build an infinite tower of states with increasing energy, or mass in space-time.

The action (1.4) can be used to describe two kinds of string: *open strings* (with two endpoints), or closed strings (without endpoints). Since closed strings have simpler boundary conditions, we will consider them first.

1.2.1 Mode expansion for closed strings

For closed strings the boundary conditions are periodicity in the σ-direction. We will take $\sigma \equiv \sigma + \pi$. The most general solution to the equation of motion $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$ which also satisfies the boundary condition $X^\mu(\sigma, \tau) = X^\mu(\sigma + \pi, \tau)$ can then be written as the sum of an arbitrary function depending on $\sigma + \tau$ only, and one depending on $\sigma - \tau$ only:

$$X^\mu(\sigma, \tau) = X_L^\mu(\sigma + \tau) + X_R^\mu(\sigma - \tau). \tag{1.5}$$

Mode expansion for the right- and left-moving parts can be written as:

$$X_R^\mu(\sigma - \tau) = \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau - \sigma) + \frac{\mathrm{i}}{2} l \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau - \sigma)} \tag{1.6a}$$

and

$$X_L^\mu(\sigma + \tau) = \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau + \sigma) + \frac{\mathrm{i}}{2} l \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau + \sigma)}. \tag{1.6b}$$

Note that x^μ and p^μ are the centre of mass position and space-time momentum.

Reality of X^μ implies that x^μ and p^μ are also real, while the oscillator coefficients obey

$$(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu, \qquad \text{and} \qquad (\tilde{\alpha}_n^\mu)^\dagger = \tilde{\alpha}_{-n}^\mu.$$

The equations of motion for the metric, ($\partial_\alpha X \cdot \partial_\beta X = \frac{1}{2} \eta_{\alpha\beta} (\partial X)^2$ in our gauge), can be shown to be equivalent to two constraint equations:

$$\dot{X}_R^2 = \dot{X}_L^2 = 0.$$

The mode expansions for these equations read

$$L_m \equiv \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n = 0,$$

$$\tilde{L}_m \equiv \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n = 0.$$

In these expressions we have introduced $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} l p^\mu$. The mass of an object in space-time is always given by $M^2 = -p^\mu p_\mu$. Together with the L_0 and $\tilde{L}_0$ equations the mass of a closed string in a given oscillation state is thus found to be given by

$$M_{\text{clas}}^2 = \frac{4}{l^2} \sum_{n=1}^{\infty} \left(\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n\right), \tag{1.7}$$

where we have used that in the present – classical – context everything commutes, so in particular $\alpha_n \alpha_{-n} = \alpha_{-n} \alpha_n$. It is important to realize that this restricts the validity of (1.7) to *classical* closed string theory. We shall come back to this in the next section.

1.2.2 Mode expansion for open strings

For open strings the boundary conditions are found by partially integrating the string action with respect to τ and σ:

$$\begin{aligned} S_{\text{string}} = & -\frac{T}{2} \int\limits_{\text{world-sheet}} \mathrm{d}\tau\, \mathrm{d}\sigma\, X_\mu \left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu \\ & + \frac{T}{2} \int_0^\pi \mathrm{d}\sigma\, X_\mu \partial_\tau X^\mu \Big|_{\tau=-\infty}^{+\infty} - \frac{T}{2} \int_{-\infty}^{+\infty} \mathrm{d}\tau\, X_\mu \partial_\sigma X^\mu \Big|_{\sigma=0}^{\pi}. \end{aligned}$$

We will ignore the boundary term at $\tau = \pm\infty$, but after imposing $\left(\partial_\tau^2 - \partial_\sigma^2\right) X^\mu = 0$ the boundary terms at $\sigma = 0, \pi$ remain. The boundary equation of motion is found to be

$$\partial_\sigma X^\mu = 0 \qquad \text{for } \sigma = 0 \text{ and } \sigma = \pi.$$

Physically, this corresponds to the fact that no momentum can flow past the ends of the string.

The open string mode expansion then reads:

$$X^\mu(\sigma,\tau) = x^\mu + l^2 p^\mu \tau + \mathrm{i} l \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-in\tau} \cos n\sigma.$$

For future reference, it is useful to rewrite this expansion in terms of left and right movers:

$$\begin{aligned} X_R^\mu(\sigma - \tau) &= \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau - \sigma) + \frac{\mathrm{i}}{2} l \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-in(\tau-\sigma)}, \\ X_L^\mu(\sigma + \tau) &= \frac{1}{2} x^\mu + \frac{1}{2} l^2 p^\mu (\tau + \sigma) + \frac{\mathrm{i}}{2} l \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu \mathrm{e}^{-in(\tau+\sigma)}. \end{aligned}$$

(Note that the coefficients α_n are now the same for left- and right movers.)

Introducing $\alpha_0^\mu = l p^\mu$, one can find constraint equations

$$L_m \equiv \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n = 0.$$

The equation for L_0 once again yields a mass formula:

$$M_{\text{clas}}^2 = \frac{2}{l^2} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n.$$

Again, this formula is valid for classical strings only.

1.3 Quantization

Although much of the work in this thesis is performed in a semi-classical context, a few words on the very important subject of string quantization seem to be in order.

String quantization is non-trivial, since at present no method is known to quantize the string while maintaining both manifest Lorentz invariance and unitarity: a Lorentz covariant scheme of quantization introduces states of negative norm, which have to be cancelled by introducing ‘ghost coordinates’[1)]. Alternatively, it is possible to follow a quantization method which is manifestly free of negative norm states, but only at the cost of giving up manifest Lorentz invariance. If one chooses to do so, one must explicitly check Lorentz invariance afterwards. This procedure is called light-cone quantization.

We shall not go into the details here, but simply state that the α_n^μ’s become bosonic creation and annihilation operators, that satisfy the following commutation relations:

$$\begin{aligned}
\left[x^\mu, p^\nu\right] &= \mathrm{i}\eta^{\mu\nu}, \\
\left[\alpha^\mu_{-m}, \alpha^\nu_n\right] &= -m\delta_{m-n}\eta^{\mu\nu}, \\
\left[\tilde\alpha^\mu_{-m}, \tilde\alpha^\nu_n\right] &= -m\delta_{m-n}\eta^{\mu\nu},
\end{aligned}$$

(all other commutators vanishing), which can be combined to build up an infinite tower of massive states with various (integer) spins.

Upon quantization the constraint that $L_m = 0$ for $m > 0$ becomes the demand that the L_m’s for $m > 0$ annihilite physical states. The equation $L_0 = 0$ should be treated with more care: the operators that occur in it do not commute, so there is an ordering ambiguity. This ambiguity is resolved by *defining*

$$L_0 \equiv \frac{1}{2}\alpha_0^2 + \sum_{n>0} \alpha_{-n}\cdot\alpha_n.$$

With such a definition it would seem unreasonable to expect that L_0 directly annihilates physical states. However, since the commutators of the α_n’s are c-numbers, we do know that there is a constant a such that $L_0 - a$ annihilates all physical states. Consistency demands, such as absence of negative-norm states (‘ghosts’) show that $a = 1$ when $D = 26$, which is the natural number of space-time dimensions for bosonic string theory[2)]. With these modifications, the mass formulas change into

$$M^2 = -8l^{-2} + 4l^{-2}\sum_{n=1}^{\infty}\left(\alpha_{-n}\cdot\alpha_n + \tilde\alpha_{-n}\cdot\tilde\alpha_n\right),$$

for closed strings and

$$M^2 = -2l^{-2} + 2l^{-2}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n,$$

for open strings.

There is a consistency condition which implies that the excitation level of the left- and the right-moving sector must be equal. This means that the ground state for bosonic closed string

[1)] See eg. [7] chapter 3 for an explanation.

[2)] Not only is bosonic string theory inconsistent for $D > 26$, for $D < 26$ the theory is much less rich. The considerations that lead to these conclusions may be found in [7], chapter 2.

theory is a scalar tachyon (with mass squared $M^2 = -8l^{-2}$, while the first excited level consists of massless fields: a traceless symmetric spin two field $G_{\mu\nu}$: the graviton, an anti-symmetric spin two field $B_{\mu\nu}$ and a scalar Φ, the dilaton. For the bosonic open string, the ground state is again a tachyon (with $M^2 = -2l^{-2}$), while the first excited states form a massless vector multiplet.

1.3.1 String field theory

Had this thesis dealt with particle theory, we would now have extended a first-quantized particle action such as $S = \frac{m}{2}\int d\tau\, \dot{x}^2$ to a second-quantized action such as $S = \int d^4x\, \left\{\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2\right\}$. We would certainly like to do something similar in string theory, extending the world-sheet action (1.4) to an action expressed as an integral over space-time. Although a lot of effort has always been directed at this subject,[1)], no serious candidates for a string field theory has been found in this way. However, recent research aimed at a completely different branch of string theory, has shown promise of shedding some light on string field theory from a different perspective altogether: work associated with the unravelling of the connection between D-particle mechanics and M-theory may well turn out to have important bearing on a field theoretic formulation of string theory.

1.3.2 Vertex factors

Computing the amplitudes of diagrams such as the one shown in figure 1.1 is done by replacing external string lines by *vertex operators* that correspond to emission or absorption of a string in a particular mass eigenstate. To get some indication of how this works out, consider the following example. Suppose that all three external lines in figure 1.1 are massless gravitons, with momenta k_i^μ and polarizations $\zeta_i^{\mu\nu}$ ($i = 1\ldots3$). The amplitude for this diagram is then given by

$$A_{(3\ \text{gravitons})} = g_c \langle\zeta_2;k_2|V_G(\zeta_3;k_3)|\zeta_1;k_1\rangle\,,$$

with g_c a constant[2)]; $|\zeta;k\rangle$ the string state that describes the graviton, which is constructed from the tachyon state $|0;k\rangle$ by

$$|\zeta;k\rangle = \zeta_{\mu\nu}\alpha_{-1}^{\mu}\alpha_{-1}^{\nu}|0;k\rangle\,;$$

and V_G the graviton emission vertex operator, which is given by

$$V_G(\zeta,k) = \zeta_{\mu\nu}\int_{\text{world-sheet}} d\tau\, d\sigma\; \partial_\alpha X^\mu \partial^\alpha X^\nu e^{ik\cdot X}.$$

Qualitatively the form of the latter can be understood as follows: first, a particle that carries momentum k should certainly introduce a factor $e^{ik\cdot X}$ to the scattering amplitude, to ensure conservation of momentum. Secondly, a graviton is a spin two particle, so we should expect a factor build out of two of the X^μ's or their derivatives. (The fact that we have $\partial X \partial X$ and not some other combination can be understood from the requirement of *conformal invariance*, but we shall not go into it.) Finally, the integral over the world-sheet is the last remnant of

1) In [17] a basic introduction is presented.

2) g_c is called the string coupling constant. It is not a coupling constant in the normal field theoretical sense, since its value is given by the vacuum expectation value of a field: $g_c = \langle e^{\Phi}\rangle$, with Φ the dilaton.

the integration over all world-sheet shapes that are topologically equivalent to the ones shown in figure 1.1: when considering tree level three particle scattering, one is not interested in the actual shape of the blob in the centre, and so the shape dependence should be integrated away. Most of the integrals involved are killed by the gauge fixing, but the integral over the relative positions of the vertices remain. By stereographical projection of the sphere onto a plane, we see that only the position of one particle has physical meaning: by rotating and scaling it is possible to put the other two at positions (0,0) and (0,1).

Obviously, a lot more could be said about string quantization, but within the scope of this thesis, there is no room for such an expansion.

1.4 Chan-Paton factors

Before moving on to superstring theory, let us introduce one more concept that will be needed later on: *Chan-Paton factors*. Since open strings have two endpoints which are special points from the world-sheet point of view, it would seem possible to insert extra symmetry charges at those points. In fact this *is* possible, by augmenting string states $|\Lambda\rangle$ by group labels a and $\bar{b}$ to read $|\Lambda; a\bar{b}\rangle$. Similarly, the vertex operators become matrices acting on the group indices. If a transforms in a representation R of the gauge group G, $\bar{b}$ must transform in the $\bar{R}$ representation[1]. These charges are non-dynamical (their Hamiltonian vanishes), so the value of a and $\bar{b}$ cannot change along the length of the world-sheet.

The conservation of Chan-Paton charges has an important consequence for the amplitude of scattering diagrams such as the one sketched in figure 1.2: since there is no well defined point where the right-hand side of string 1 turns into the left-hand side of string 2, it must be the case that a for string 1 and $\bar{b}$ for string 2 are equal. Supposing the three strings are in states $|\Lambda_{(i)}; g_{(i)}\rangle = \sum_{a,\bar{b}} |\Lambda_{(i)}; a\bar{b}\rangle \lambda^{(i)}_{a\bar{b}}$, with $i = 1 \ldots 3$, the Chan-Paton degrees of freedom contribute a factor $\mathrm{Tr}\left[\lambda^{(1)}\lambda^{(2)}\lambda^{(3)}\right]$ to the scattering amplitude.

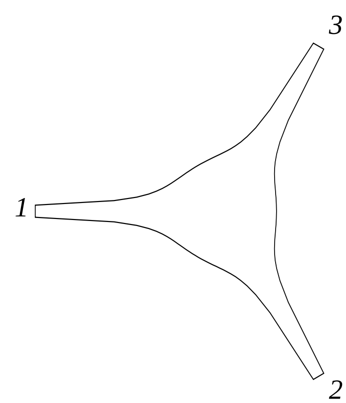


Figure 1.2: A disk-level open string diagram. Where does the end of string 1 turn into the end of string 2?

Quantum consistency considerations impose severe restrictions on the choice of gauge group and representations: an interacting theory with massless vector bosons can only be consistent if these transform in the adjoint of the gauge group. This gauge group must also obey certain conditions (see [7], p. 293). In particular, this implies that $R \times \bar{R}$ must be the adjoint representation of G. This proves to be a strong restriction: it turns out that for oriented strings – with distinguishable endpoints – the only possibilities are the $\mathrm{U}(n)$ groups, with an n charge[2] on

[1] If R is a real representation, $\bar{R}$ and R are equal, of course.
[2] ie a charge transforming in the fundamental representation.

the one end and an $\bar{n}$ charge on the other end of open strings, so string states transform in $n \times \bar{n}$, which is the adjoint of $\mathrm{U}(n)$. For unoriented strings there are two possibilities: $\mathrm{SO}(n)$ when the amplitude is invariant under orientation reversal, and $\mathrm{U\,Sp}(n)$ when the amplitude changes sign.

The above is a very qualitative sketch. Hopefully it suffices as an introduction to the use of Chan-Paton factors in the next chapter.

1.5 Superstring theory

The fact that the ground state is a tachyon – together with the absence of fermions – means that bosonic string theory is not a serious candidate for a TOE. Luckily, it can be supersymmetrized, yielding a theory with bosons and fermions, while at the same time removing the tachyon from the spectrum. Supersymmetrizing the theory of bosonic closed strings yields type II superstring theory, in which the lowest energy physical states are the massless bosons $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ, a set of bosonic forms, and left- and right-moving gravitinos $\chi^{\mu a}$. One can either choose the left- and right-moving fermions to have different chirality, in which case the gravitinos have different chirality as well and the forms will be $F^{(0)}$, $F^{(2)} = dA^{(1)}$, $F^{(4)} = dA^{(3)}$, or one can choose the fermions to have equal chirality. In that case the gravitinos will also have equal chirality and the forms will be $F^{(1)} = dA^{(0)}$, $F^{(3)} = dA^{(2)}$, $F^{(5)} = dC^{(4)}$ (with $C^{(4)}$ self-dual).

Supersymmetry is a subject that requires more introduction than can be given in these pages. The reader is referred to [18] for a general introduction. The concepts of supersymmetry most essential to string theory are also covered in [7]. Here we shall only present a supersymmetric extension of the string action of §1.1.3 and describe how the tachyon is dealt with.

1.5.1 World-sheet supersymmetry

We may augment the string action with a fermionic oscillator term:

$$S = -\int d\tau\, d\sigma \left\{ \frac{T}{2} \partial X \cdot \partial X - i\bar{\psi}^{\mu} \rho^{\alpha} \partial_{\alpha} \psi_{\mu} \right\}, \tag{1.8}$$

with

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

two dimensional Dirac matrices satisfying $\{\rho^{\alpha}, \rho^{\beta}\} = -2\eta^{\alpha\beta}$, and ψ a two-component Majorana spinor[1)]

$$\psi = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix},$$

while $\bar{\psi}$ is defined by $\bar{\psi} \equiv \psi^T \rho^0$.

Adding a term to the action in this way may seem to be a rather ad hoc thing to do, but (1.8) possesses a very special symmetry which makes it interesting: the action (1.8) is invariant under

$$\begin{aligned} \delta X^{\mu} &= \bar{\varepsilon} \psi^{\mu} \\ \delta \psi^{\mu} &= -i \rho^{\alpha} \varepsilon \partial_{\alpha} X^{\mu}, \end{aligned} \tag{1.9}$$

[1)] ie a spinor with real components

when ε is a constant (in both τ and σ) anticommuting spinor. This is a global symmetry of a very peculiar nature: it mixes bosons and fermions.

(1.8) is not invariant under local supersymmetry transformations, that is, transformations like (1.9) with general $\varepsilon = \varepsilon(\tau, \sigma)$. However, by adding some additional terms to the action, it is possible to arrive at a locally supersymmetric form. It is then possible to make a gauge choose which kills the additional terms, leaving (1.8) as the action to be considered for quantization. We shall not go into the details, but merely remark that even in this version of string theory, the ground state is a tachyon, with mass squared $M^2 = -\alpha'^{-1}$.

1.5.2 Space-time supersymmetry

This embarressment can be alleviated in a most elegant way. By eliminating all states that are annihilated by the GSO projection operator – which is roughly of the form $P = \frac{1+(-1)^F}{2}$, with F the fermion excitation level – the tachyon is expunged from the spectrum, leaving the massless states described above as the ground state. This projection has another virtue: the resulting theory is supersymmetric in a space-time sense as well as on the world-sheet: the action can be rewritten in terms of bosons X^μ and fermions θ_i, with $i = 1, 2$[1)]. Quantum consistency requires that $D = 10$, in which case the fermions are (32-component) Majorana-Weyl spinors.

[1)] For each boson, there are two fermions: this is called $N = 2$ supersymmetry.

CHAPTER 2

T-DUALITY AND D-BRANES

Surprisingly, string theory also contains other objects, which you might call membranes – or flying carpets.

— Brian Greene

In the previous chapter we found actions for open and closed strings, and we found mode expansions for their space-time coordinates. We will now consider what happens to these mode expansions when the space-time has compact directions. We shall find that in the closed string sector new winding modes arise. These cause a duality between the theory compactified on radius R, and the same theory compactified on radius $R' = \frac{\alpha'}{R}$ [1)]. In the open string sector, this duality has a striking consequence: in the dual theory, the open strings will no longer be completely free. Rather, their endpoints will be confined to lie on certain dynamical hyper-planes, the so called D-branes.

Again, this chapter does not aim to be a comprehensive review of the subject. A thorough discussion about D-branes and their properties may be found in [19]. Here, we shall merely introduce the main ingredients and set the stage for the next chapters.

2.1 Closed strings on a space-time with compact directions

We shall begin by considering closed string theory. Examining the zero modes of the string fields, we shall discover the duality mentioned above.

Consider a space-time with one compact direction: $X^{25} \equiv X^{25} + 2\pi R$. Upon quantization, this implies that the space-time momentum in this direction, p^{25}, no longer has a continuous spectrum. Rather,

$$p^{25} = \frac{n}{R},$$

with n integer.

Additionally, $X^{25}(\sigma)$ no longer needs to be strictly periodic: a string can be wound around the compact direction, so we may have $X^{25}(\sigma + \pi) = X^{25}(\sigma) + 2\pi w R$ (with w integer). This

[1)] Two theories are said to be dual to one another when the physical contents of the two theories are the same. The transformation that changes the one theory into the other is called a duality.

possibility may be realized if the mode expansion for X^{25} is augmented by a term linear in σ: rewrite (1.6a) and (1.6b) as

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \sqrt{2\alpha'}\,\alpha_0^\mu(\tau-\sigma) + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau-\sigma)}$$
$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \sqrt{2\alpha'}\,\tilde{\alpha}_0^\mu(\tau+\sigma) + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-2\mathrm{i}n(\tau+\sigma)},$$

with

$$\tilde{\alpha}_0^\mu + \alpha_0^\mu = \sqrt{2\alpha'}\,p^\mu.$$

For non-compact directions, single-valuedness of X^μ implies $\tilde{\alpha}_0^\mu = \alpha_0^\mu$, but for a compact direction the conditions are slightly more lenient: we may have

$$\tilde{\alpha}_0^{25} - \alpha_0^{25} = \sqrt{\frac{2}{\alpha'}}\,wR,$$

with w integer.

Thus we find

$$\alpha_0^{25} = \sqrt{2\alpha'}\left\{\frac{n}{R} - \frac{wR}{\alpha'}\right\}$$
$$\tilde{\alpha}_0^{25} = \sqrt{2\alpha'}\left\{\frac{n}{R} + \frac{wR}{\alpha'}\right\}. \tag{2.1}$$

The structure of (2.1) suggests a symmetry between winding and momentum modes. Indeed, if we rewrite the theory at radius R in terms of

$$X'(\sigma,\tau) = X_L(\tau+\sigma) - X_R(\tau-\sigma),$$

that is, change the sign of the right-moving modes, the (25-dimensional[1)]) mass spectrum does not change, nor does the energy-momentum tensor $T_{\alpha\beta} = \frac{\delta S}{\delta h^{\alpha\beta}}$. Therefore the interactions do not change. In fact, the only change is in the zero modes: changing the sign of α_0^{25} exchanges n and w, while replacing R by $R' = \frac{\alpha'}{R}$.

Changing the sign of the right-moving modes in a given direction is called a *T-duality transformation*. We have just shown that T-dualizing closed string theory at radius R yields the same theory, but at radius R': closed string theory is said to be self-dual under T-duality.

T-duality implies that at scales much smaller than $\sqrt{\alpha'}$ space-time looks the same to strings as at scales much larger than $\sqrt{\alpha'}$. This has a very important consequence for experiments that attempt to probe space-time at short distances using strings: it simply will not work. If one tries to probe space-time at length scale $r < \sqrt{\alpha'}$, one will get answers pertaining to length scale $r' = \alpha'/r$.

[1)]Viewing the compact dimension as 'internal', we are describing a string in a 25 dimensional space-time, which has some additional degrees of freedom that are associated with the compact direction. The mass is then computed as $M^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu$.

2.2 T-duality and open strings

How does the above duality affect the open string sector? Since open strings do not have winding number, there does not seem to be any quantum number for momentum to mix with. We should clearly look for more subtle effects.

Consider the mode expansion for open strings on a space-time with compact dimensions:

$$X_R^\mu(\sigma-\tau)=\frac{1}{2}x^\mu+\sqrt{2\alpha'}\,\alpha_0^\mu(\tau-\sigma)+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu \mathrm{e}^{-\mathrm{i}n(\tau-\sigma)}$$
$$X_L^\mu(\sigma+\tau)=\frac{1}{2}x^\mu+\sqrt{2\alpha'}\,\tilde{\alpha}_0^\mu(\tau+\sigma)+\mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \mathrm{e}^{-\mathrm{i}n(\tau+\sigma)}.$$

Open string boundary conditions imply that

$$\tilde{\alpha}_n^\mu-\alpha_n^\mu=0, \tag{2.2}$$

while by definition

$$\tilde{\alpha}_0^\mu+\alpha_0^\mu=\sqrt{2\alpha'}\,p^\mu, \tag{2.3}$$

with boundary conditions implying $p^\mu=\frac{n}{R}$ as before.

If we change the sign of α_0^{25}, the boundary condition $\partial_\sigma X^{25}\big|_{\sigma=0,\pi}=0$ no longer holds. Rather:

$$\partial_\tau X^{25}\big|_{\sigma=0,\pi}=0.$$

Neumann boundary conditions are replaced by Dirichlet boundary conditions! This means that the endpoint of an open string cannot move in a T-dualized direction. More precisely, the equations (2.2) and (2.3) turn into

$$\tilde{\alpha}_n^\mu+\alpha_n^\mu=0$$

and

$$\tilde{\alpha}_n^\mu-\alpha_n^\mu=\sqrt{2\alpha'}\,\frac{n}{R}\equiv\sqrt{\frac{2}{\alpha'}}\,wR'$$

respectively, where we have again introduced $R'=\frac{\alpha'}{R}$, and replaced n by w, to emphasize the fact that the quantum number that counted momentum before T-duality, counts winding number afterwards.

From the short calculation

$$\int \mathrm{d}\sigma\,\partial_\sigma X^{25}=\int \mathrm{d}\sigma\,\left[\sqrt{2\alpha'}\left(\tilde{\alpha}_0^{25}-\alpha_0^{25}\right)+\mathrm{osc.}\right]$$
$$=2\pi wR'+0, \tag{2.4}$$

we see that w indeed counts the number of times the string is wound around the compact direction, and that the endpoints lie on the same hyperplane: $X^{25}+2\pi wR'\equiv X^{25}$. Dirichlet conditions enable open strings to carry winding number which ordinary open strings do not, thereby solving the mystery of what quantum number exchanges roles with momentum on T-duality. (Note that an open string that has its endpoints fixed on a hyperplane can not have momentum perpendicular to that plane.)

In connection with the possibility of open string splitting and joining diagrams, (2.4) means that *all* open strings have both endpoints on the same hyperplane[1)], which we may as well take to lie at $X^{25} = 0$. Such a hyperplane, which is characterized by the fact that open strings with Dirichlet boundary conditions may end on it, is called a D-brane.

Although we only considered T-dualizing the 25th direction in the above, there is no reason not to continue. We may T-dualize any number of directions, producing D-k-branes with dimension $k = 24$ all the way down to zero: a D-particle. In fact, it is even possible to T-dualize the time direction, thus producing a D-(-1)-brane, or D-instanton.

2.2.1 D-brane dynamics

As presented above, the D-brane might seem to be perfectly rigid and constant. This however, is not the case, nor would it seem possible to have such a static object in a theory containing gravity. To see how the D-brane becomes a dynamical object, consider what happens to the vertex factor associated with generic vectors coupling to the end of the string, $V_A(\zeta_\mu[X])$[2)], after T-duality.

Before T-duality, this vertex operator is given by

$$V_A(\zeta_\mu[X]) = \oint_{\partial M} \mathrm{d}\tau\, \zeta_\mu(X)\partial_t X^\mu,$$

with the integral tracing the boundary of the world-sheet, and ∂_t denoting the derivative tangential to this boundary. After T-dualitizing the X^{25} direction, this is changed into

$$V_A(\zeta_\mu[X]) = \sum_{\mu=0}^{24} \oint_{\partial M} \mathrm{d}\tau\, \zeta_\mu(X)\partial_t X^\mu + \oint_{\partial M} \mathrm{d}\tau\, \zeta_{25}(X)\partial_n X^{25},$$

with ∂_n denoting the derivative normal to the boundary. In this equation ζ_{25} is coupled to $\partial_n X^{25}$ at the boundary. Since the open string action can be partially integrated to yield a boundary term $S = -\oint_{\partial M} \mathrm{d}\tau\, X_{25}\partial_n X^{25}$, it is clear that acting on a string boundary state with $V_A(\zeta[X])$ translates the boundary by an $X(\tau)$-dependent amount.

One more point that is noteworthy, is the following: since X^{25} is constant along the boundary, the functional ζ_μ cannot depend on this coordinate. Straightforwardly extending this consideration to the general D-k-brane case, we see that after some T-dualities the fields that live on the boundary of the string – or, equivalently, on the D-brane – cannot depend on the transversal coordinates. This will turn out to be very important in the next chapter.

---

1) Taking Γ to be a contour on such a diagram connecting any two endpoints, (2.4) generalizes to

$$\int_\Gamma \mathrm{d}\sigma^\alpha\, \partial_\alpha X^{25} = 2\pi w R',$$

for some $w \in \mathbb{Z}$.

2) ie ζ_μ are a functionals of the limitation of X to the boundary of the world-sheet.

2.2.2 The mass of strings with Dirichlet boundary conditions

As noted in chapter 1, the mass of an open string state is determined by the usual relation $M^2 = -p^2$, and the constraint equation $L_0 = 0$. After T-dualizing, these remain valid, but we should be somewhat careful: after compactification, only the momentum in non-compact directions should be included in the calculation of M^2. Returning to the case where only X^{25} is compact, this entails an increase of M^2 by a term arising from $(p^{25})^2$, which is now an internal excitation. After T-duality, this momentum is replaced by winding number, and we find that L_0 contains a term

$$\frac{1}{2}(\tilde{\alpha}_0^{25} - \alpha_0^{25})^2 = \frac{1}{2}\left[\sqrt{\frac{2}{\alpha'}}wR'\right]^2,$$

which changes the mass formula into

$$M^2 = \frac{1}{(2\pi\alpha')^2}w^2R'^2 + \frac{1}{\alpha'}\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n.$$

Note that in particular the ground state mass becomes

$$M_{\mathrm{GS}} = \frac{1}{2\pi\alpha'}wR', \tag{2.5}$$

which is the string tension $T = \frac{1}{2\pi\alpha'}$ times the length of the wound string.

2.3 D-branes and Chan-Paton factors

Having found that T-dualizing a simple open string yields a string with its endpoints on a D-brane, we may consider whether it is possible to have more than one D-brane. In this section we shall find that T-dualizing an open string theory with $\mathrm{U}(n)$ Chan-Paton group yields a theory of open strings with their endpoints lying on any of n D-branes. To show this, we shall first examine the effect of a constant background gauge field on open strings, and then perform the T-duality to carry the result over to the D-brane picture.

First of all, note that under a gauge transformation the string endpoint states transform as fields in the fundamental ($\boldsymbol{n}$) and anti-fundamental ($\bar{\boldsymbol{n}}$) representation respectively, that is, as

$$\psi \to \psi' = \Lambda\psi,$$

and

$$\bar{\psi} \to \bar{\psi}' = \bar{\psi}\Lambda^{-1}.$$

On the other hand, the massless vector A transforms as

$$A \to A' = \Lambda A\Lambda^{-1} + \mathrm{i}\Lambda\partial\Lambda^{-1}.$$

(For a recap of gauge theory see §4.1, or one of the references therein.)

2.3.1 Open strings with Neumann conditions

Consider a constant gauge background[1)] of the form:

$$A_{25}(X) = \mathrm{diag}(\frac{\theta_1}{2\pi R}, \ldots, \frac{\theta_n}{2\pi R}).$$

In this background consider a string state $\left|\Lambda; a\bar{b}\right\rangle$ that is a momentum eigenstate with $p^{25} = n/R$. Such a state has a wave function that obeys $\Psi(X^{25}+a) = \mathrm{e}^{\mathrm{i}na/R}\Psi(X^{25})$[2)]. At least locally A_{25} is pure gauge, and it may be set to zero globally by performing a gauge transformation with

$$\Lambda = \mathrm{diag}(\mathrm{e}^{-\frac{\mathrm{i}\theta_1 X^{25}}{2\pi R}}, \ldots, \mathrm{e}^{-\frac{\mathrm{i}\theta_n X^{25}}{2\pi R}}).$$

However, this transformation does not leave the string endpoint states untouched:

$$\psi'(X^{25}) = \mathrm{diag}(\mathrm{e}^{-\mathrm{i}\theta_1 \frac{X^{25}}{2\pi R}}, \ldots, \mathrm{e}^{-\mathrm{i}\theta_n \frac{X^{25}}{2\pi R}})\, \psi(X^{25})$$

and

$$\bar{\psi}'(X^{25}) = \bar{\psi}(X^{25})\, \mathrm{diag}(\mathrm{e}^{+\mathrm{i}\theta_1 \frac{X^{25}}{2\pi R}}, \ldots, \mathrm{e}^{+\mathrm{i}\theta_n} \frac{X^{25}}{2\pi R}).$$

This means that after the constant vector potential has been gauged away, the string state $\left|\Lambda; a\bar{b}\right\rangle$ picks up an extra phase factor under translation:

$$\Psi'(X+a) = \mathrm{e}^{\mathrm{i}\frac{(\theta_b-\theta_a)a}{2\pi R}}\Psi(X+a) = \mathrm{e}^{\mathrm{i}na/R+\mathrm{i}\frac{(\theta_b-\theta_a)a}{2\pi R}}\Psi'(X)$$

This identifies the new state as a momentum eigenstate with

$$p = \frac{n + \frac{\theta_b - \theta_a}{2\pi}}{R}.$$

The momentum thus effectively acquires a fractional part.

2.3.2 Open strings with Dirichlet conditions

Having considered the effect of Chan-Paton factors on ordinary open strings, we shall transfer the results to the T-dual theory: after T-dualizing, momentum becomes winding number, and we are left with open strings with fractional winding number. This means that the strings no longer have their endpoints all on the same hyperplane. Rather, a string $\left|\Lambda; a\bar{b}\right\rangle$ has

$$X^{25}(\sigma = \pi) - X^{25}(\sigma = 0) = (\theta_b - \theta_a)\, R',$$

where $R' = \alpha'/R$ is the compactification radius of the dual theory, and integer winding terms $W = 2\pi n R'$ have been suppressed.

---

1) A background in string theory is the result of the presence of a particular set of excited strings that effectively gives rise to a value for the space-time field. In the next chapter we shall present an effective description of the way string dynamics yields an action principle for space-time fields.

2) A momentum eigenstate $|p\rangle$ quite generally has a configuration space wavefunction $\psi(x)$ obeying $\psi(x+a) = \mathrm{e}^{\mathrm{i}pa}\psi(x)$, as can be seen from the Fourier transformation of the momentum space wavefunction $\tilde{\psi}(k) \sim \delta(k-p)$.

Using the same argument as in §2.2, we find that two different open strings – both having an endpoint in state $|a\rangle$ – must have that endpoint lying on the same hyperplane. We may therefore translate the X^{25} coordinate in such a way that a string $\left|\Lambda; a\bar{b}\right\rangle$ has

$$X^{25}(0) = \theta_a R' \qquad \text{and} \qquad X^{25}(\pi) = \theta_b R'.$$

Another way to express this constraint is to say that there are now n D-branes, at the positions $\left\{X^{25} = \theta_a R' :\ a = 1\ldots n\right\}$. In general such a configuration breaks the $\mathrm{U}(n)$ symmetry down to $(\mathrm{U}(1))^n$, (since a more general gauge transformation would move the branes about), but when several D-branes coincide, the symmetry is partially restored[1)].

The mass of strings stretching between D-branes

The formula (2.5) for the mass of an open string with winding number w implies that a string stretching directly[2)] between two branes has ground state mass

$$M_{\mathrm{GS},ab} = \frac{1}{2\pi\alpha'}|\theta_a - \theta_b|R', \tag{2.6}$$

which is the string tension times the minimal length of a string stretching between the two branes. The energy stored in the stretched string in this way, acts like an effective potential pulling the branes together. We shall return to this subject in the last chapter.

As a final remark, note that the above discussion may be extended in an obvious way to D-k-branes with lower values of k. The mass of stretched strings is then found to be proportional to the shortest distance between the branes.

[1)]For each group of k coincident D-branes, k factors of $\mathrm{U}(1)$ are replaced by the larger group $\mathrm{U}(k)$.
[2)]ie not wound around X^{25}.

CHAPTER 3

D-PARTICLE EFFECTIVE ACTION

'Shall we go out?' asked Merlyn. 'I think it is about time we began lessons.'

— T. H. White, 'The once and future king'

Armed with some basic knowledge of string theory, and having seen how D-branes occur in it, we shall formulate an effective action to describes the dynamics of a single D-0-brane, or *D-particle*. In the weak coupling limit, we shall find that a single D-particle behaves like a free relativistic point particle with mass g_c^{-1} times the string mass scale.

In the first section of this chapter, we shall give a general introduction to the concept *effective action*. Then we shall present a calculation made by Fradkin and Tseytlin [16] for the effective action of open strings in an electromagnetic background. In the last section we shall see how the D-particle action is derived as the T-dual of the same calculation.

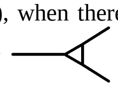
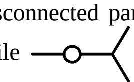
All calculations in this chapter will be performed in the context of bosonic string theory. However, the results can easily be transferred to superstring theory, as will be shown in an appendix.

3.1 Effective actions

An effective action is an object from which amplitudes for *one particle irreducible*[1] diagrams can be found directly by functional differentiation, ie without performing a path integral. An effective action generally looks more complicated than a 'normal' action, since it explicitly contains all the information about possible internal lines, which is normally encoded in simple propagators and vertices. All of this should be clear in a moment.

We will first consider effective actions in the context of quantum field theory before moving on to the string context. The following material can be found in many textbooks on quantum field theory, for instance [20].

We will use a vector-and-mapping notation for the fields and the sources: given two fields $\phi(x)$ and $\chi(x)$ defined on a manifold M, (which may be four dimensional space-time or the

[1] A diagram is called one particle irreducible (OPI), when there are no disconnected parts, and the diagram cannot be split in two by cutting a single particle line, ie  *is* OPI, while  is not.

world-sheet of a string), we define

$$\phi \cdot \chi \equiv \int_M \mathrm{d}^D x \, \phi(x)\chi(x).$$

Furthermore we will be using operators $\mathcal{A}(x,y)$ on the space of fields, for which we define the notation

$$\mathcal{A} \cdot \phi : \; x \mapsto \int_M \mathrm{d}^D y \, \mathcal{A}(x,y)\phi(y).$$

Finally, we must differentiate between $\phi^n : \; x \mapsto (\phi(x))^n$ and $\phi^{\otimes n}$ which is defined by

$$\phi^{\otimes n} : \; (x_1, \ldots, x_n) \mapsto \phi(x_1) \cdots \phi(x_n).$$

3.1.1 Effective actions in QFT

1. General definition in the context of QFT

Given a field theoretical action depending on several fields, $S[\phi]$, we can make a generating functional for scattering amplitudes by introducing

$$W[J] = \int \mathcal{D}\phi \, \mathrm{e}^{\mathrm{i}S[\phi] + \mathrm{i}J \cdot \phi}. \tag{3.1}$$

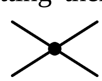
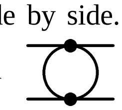
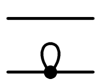
From this functional we can derive the so-called *disconnected Green functions* by functional differentiation:

$$\mathcal{G}^{(n)} = \left. \frac{\delta^n W[J]}{\delta J^{\otimes n}} \right|_{J=0}. \tag{3.2}$$

(Note that $\mathcal{G}^{(n)} : \; (x_1, \ldots, x_n) \mapsto \mathcal{G}^{(n)}(x_1, \ldots, x_n)$.)

n-particle scattering amplitudes are related to these Green functions by

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{W[0]} \mathcal{G}^{(n)}(x_1, \ldots, x_n).$$

$\mathcal{G}^{(n)}$ are called *disconnected*, because they contain contributions from diagrams which are composed from simple diagrams by simply plotting them side by side. For example in ϕ^4-theory, $\mathcal{G}^{(4)}$ will not only contain terms such as  and , but also terms like and .

If we want to construct *connected Green functions*, we can define $X[J]$ by

$$W[J] \equiv \mathrm{e}^{\mathrm{i}X[J]},$$

and write

$$G^{(n)} = \left. \frac{\delta^n X[J]}{\delta J^{\otimes n}} \right|_{J=0}. \tag{3.3}$$

These Green functions contain only contributions from connected diagrams.

At this point we may introduce the *classical field* φ_c by

$$\varphi_c \equiv \frac{\delta X[J]}{\delta J},$$

and finally the effective action $\Gamma[\varphi_c]$ by

$$\Gamma[\varphi_c] \equiv X[J] - J \cdot \varphi_c.$$

(Note that $\Gamma[\varphi_c]$ is indeed independent of J, as the notation suggests, since the functional derivatives with respect to J of the two terms cancel by the definition of φ_c.)

From the effective action, OPI amplitudes can be calculated directly, since it may be expanded as

$$\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int \left(\prod_{i=1}^{n} \mathrm{d}^4 x_i \right) \Gamma^{(n)}(x_1, \ldots, x_n) \varphi(x_1) \cdots \varphi(x_n),$$

with $\Gamma^{(n)}$ the total amplitude for n particle scattering through OPI diagrams only.

2. Effective action of a free relativistic point particle

As an example, consider the following action for a scalar particle:

$$S[\phi] = -\frac{1}{2} \phi \cdot \left(-\partial^2 + m^2\right) \cdot \phi.$$

(Again, integration over space-time is implied.)

Following the steps outlined above, we write

$$W[J] = \int \mathcal{D}\phi \; \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi \cdot \left(-\partial^2 + m^2\right) \cdot \phi + \mathrm{i} J \cdot \phi}. \tag{3.4}$$

To evaluate the path integral we introduce G by $\left(-\partial^2 + m^2\right) G = \mathbb{1}$. This is the normal Feynman propagator, which may be written as

$$G(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \; \mathrm{e}^{\mathrm{i}p \cdot (x-y)} \frac{1}{p^2 + m^2}.$$

We may now write $\left(-\partial^2 + m^2\right) \equiv G^{-1}$, to emphasize the relationship between $\left(-\partial^2 + m^2\right)$ and G.

Following standard procedure we find

$$\begin{aligned} W[J] &= \int \mathcal{D}\phi \; \mathrm{e}^{-\frac{\mathrm{i}}{2}(\phi - G \cdot J) \cdot G^{-1} (\phi - G \cdot J) + \frac{\mathrm{i}}{2} J \cdot G \cdot J} \\ &= \left(\int \mathcal{D}\tilde{\phi} \; \mathrm{e}^{-\frac{\mathrm{i}}{2}\tilde{\phi} \cdot G^{-1} \cdot \tilde{\phi}} \right) \mathrm{e}^{\frac{\mathrm{i}}{2} J \cdot G \cdot J} \\ &\equiv \mathrm{e}^{\frac{\mathrm{i}}{2} J \cdot G \cdot J}. \end{aligned} \tag{3.5}$$

In the last step we have chosen a normalization for the path integral such that $W[0] = 1$. More properly, we should compute $\int \mathcal{D}\phi\, \mathrm{e}^{-\frac{\mathrm{i}}{2}\phi\cdot G^{-1}\cdot\phi}$ by going to Euclidean time $\bar{t} = \mathrm{i}t$ in which we find

$$\int \mathcal{D}\phi\, \mathrm{e}^{-\frac{1}{2}\phi\cdot G_E^{-1}\cdot\phi} = \frac{\mathcal{N}}{\sqrt{\det G_{\mathrm{E}}^{-1}}},$$

where $\mathcal{N}$ is a normalization constant.

Continuing the procedure of §1 we compute $X[J]$, finding

$$X[J] = \frac{1}{2} J\cdot G\cdot J. \tag{3.6}$$

Next we compute $\varphi_c = \frac{\delta X[J]}{\delta J}$. From (3.4) we find that

$$\varphi_c = \frac{\delta X[J]}{\delta J} = \frac{1}{\mathrm{i}}\frac{1}{W[J]}\frac{\delta W[J]}{\delta J} = \frac{1}{\mathrm{i}}\frac{1}{\langle 1\rangle}\mathrm{i}\,\langle\phi\rangle = \frac{\langle\phi\rangle}{\langle 1\rangle},$$

where $\langle\ldots\rangle$ denotes the vacuum to vacuum transition amplitude. The fact that φ_c is equal to the expectation value of ϕ, justifies the term classical field.

On the other hand, (3.6) implies $\varphi_c = G\cdot J$, from which we find $J = G^{-1}\varphi_c$. The effective action is thus given by

$$\Gamma[\varphi_c] = \frac{1}{2}J\cdot G\cdot J - J\cdot\varphi_c = \frac{1}{2}(G^{-1}\cdot\varphi_c)\cdot\varphi_c - (G^{-1}\cdot\varphi_c)\cdot\varphi_c = -\frac{1}{2}\varphi_c\left(-\partial^2 + m^2\right)\varphi_c,$$

which is of exactly the same form as the action we started with. This is as expected, since for a free particle, the only OPI diagram is , since there are no vertices.

3. Effective action for interacting field theory

If the action for a free particle is augmented with interaction terms, for example $S_{\mathrm{int}} = \frac{\lambda}{4!}\phi^4$, it will no longer be possible to find an exact expression for φ_c and the effective action. In such cases we will have to content ourselves by defining the effective action as a series expansion in terms of OPI Green functions, $\Gamma[\varphi] = \sum_n \frac{\mathrm{i}^n}{n!}\Gamma^{(n)}\cdot\varphi^{\otimes n}$, as explained above.

We will not pursue this any further, in order to avoid drifting too far from the main purpose of this chapter.

3.1.2 Effective action for strings

There is a very important distinction between the effective actions we considered in the previous section and the ones we will consider here: our starting point will now be a string world-sheet action: a *first* quantized action. This means that the path integral over the exponentiated action does not contain any disconnected diagrams from the space-time point of view. Therefore, it may be possible to define effective actions direct from this path integral, without taking the logarithm first. This programme was proposed by Fradkin and Tseytlin in [13], and elaborated upon in [14] and [15]. They wrote down an effective action expressed in sources[1] for the

[1] The terms *field* and *source* have slightly subtle meanings in string theory: sources in world-sheet language, behave like fields in the space-time sense, while fields on the world-sheet encode the coordinates of space-time.

closed string excitations:

$$\Gamma[\Phi, C, G_{\mu\nu}, B_{\mu\nu}, \ldots] = \sum_{\chi=2,0,-2,\ldots} g_c^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu\, \mathrm{e}^{-S_1[X]}, \tag{3.7}$$

where $S_1[X]$ is defined by

$$\begin{aligned} S_1[X] = & \int_{M_\chi} \mathrm{d}^2\sigma\, \sqrt{h}\, \Phi(X) + \frac{1}{4\pi\alpha'} \int_{M_\chi} \mathrm{d}^2\sigma\, \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, G_{\mu\nu}(X) \\ & + \frac{1}{4\pi} \int_{M_\chi} \mathrm{d}^2\sigma\, \sqrt{h}\, R^{(2)} C(X) + \mathrm{i} \int_{M_\chi} \mathrm{d}^2\sigma\, \varepsilon^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, B_{\mu\nu}(X) \\ & + \cdots \end{aligned} \tag{3.8}$$

In these formulas g_c is the closed string coupling constant and χ is the Euler characteristic of the world-sheet. χ counts the number of handles k: $\chi = 2 - 2k$. In particular, for sphere diagrams (the tree level of closed string theory) $\chi = 2$. The fields occurring in the action are: $R^{(2)}$: the curvature of the world-sheet, and Φ, C, $G_{\mu\nu}$ and $B_{\mu\nu}$: the sources for the string excitations: Φ is the scalar tachyon, $G_{\mu\nu}$ is the graviton, $B_{\mu\nu}$ is the antisymmetric tensor and C is the dilaton.

Functional differentiation with respect to these sources (and multiplying by g_c) can be used to find scattering amplitudes. As an example (still in closed string theory), consider how differentiating with respect to $G_{\mu\nu}$ brings down the vertex factor for graviton emission: expanding $G_{\mu\nu}(X)$ in Fourier modes: $G_{\mu\nu}(X) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\, \mathrm{e}^{\mathrm{i}pX} G_{\mu\nu}(p)$, we have

$$\zeta_{\mu\nu} \frac{\delta\Gamma}{\delta G_{\mu\nu}(p)} = \sum_{\chi=2,0,-2,\ldots} g_c^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu\, \mathrm{e}^{-S_1[X]}\, \zeta_{\mu\nu} \int \mathrm{d}^2\sigma\, \mathrm{e}^{\mathrm{i}pX} \partial_\alpha X^\mu \partial^\alpha X^\nu.$$

The integrand $\zeta_{\mu\nu} \mathrm{e}^{\mathrm{i}pX} \partial_\alpha X^\mu \partial^\alpha X^\nu$ is recognized as the vertex factor $V_G(p_\mu, \zeta_{\mu\nu})$ for emission of a graviton with momentum p_μ and polarization $\zeta_{\mu\nu}$.

1. Adding open string fields

The effective action may be augmented with open string contributions, by replacing (3.7) with

$$\Gamma[\Phi, C, G_{\mu\nu}, B_{\mu\nu}, \ldots; \phi, A_\mu, \ldots] = \sum_\chi g_c^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu\, \mathrm{e}^{-S_1[X] - S_2[X]}, \tag{3.9}$$

where χ can also take odd values, since a more general Riemann surface with n holes as well as k handles has $\chi = 2 - 2k - n$. Tree level in open string theory corresponds to a world-sheet with the topology of a disk, so the highest allowable value for χ is $\chi = 1$. In addition to the closed string action, the string endpoints may also interact with the background. This fact is expressed by including

$$S_2[X] = \int_{\partial M_\chi} \mathrm{d}t\, e\phi(X) + \mathrm{i} \int_{\partial M_\chi} \mathrm{d}t\, \dot{X}^\mu A_\mu(X) + \cdots$$

in the exponent. Here, ϕ is the open-string tachyon and A_μ the massless vector field. Note that e is the *einbein* along the boundary: $e = \sqrt{h_{\alpha\beta} \dot{\sigma}^\alpha(t) \dot{\sigma}^\beta(t)}$.

2. Interpretation
Having written down these formulas, we take one step back and consider what they actually *mean*. The actions S_1 and S_2 can be viewed as a description of the dynamics of a string that interacts with background fields (or 'sources') living on space-time. This point of view isn't fully satisfactory though: the sources do not exist independently of the string, rather they are carried by the string. The picture is improved slightly, if we view S_1 and S_2 as actions for a string moving in a background set up by an ensemble of other strings. Better still, these actions describe the interaction between fields living on the string world-sheet (X^μ and $h_{\alpha\beta}$) and fields living in space-time (Φ, $G_{\mu\nu}$, A_μ, etc). Exponentiating and integrating the string fields away leaves us with an effective description for the dynamics of the metric of space-time and a set of fields living in that space-time, which is a direct result of the way these fields couple to the various excitation modes of the string.

This then is our interpretation of (3.9): Γ describes the (low-energy)[1)] behaviour of a set of interacting particles. This set includes the graviton, so space-time actively participates in the dynamics, rather then being a static background. The properties of the particles are completely determined by their previous interpretation as sources for excitations on the string world-sheet.

3. Low energy effective string field theory
As stated above, the effective action (3.9) contains only contributions from connected diagrams. It is in fact possible to obtain *all* diagrams from this action by exponentiation. We could write

$$Z = \int \mathcal{D}\Phi\, \mathcal{D}G_{\mu\nu} \cdots \mathrm{e}^{\mathrm{i}\Gamma[\Phi, G_{\mu\nu}, \ldots]},$$

and couple sources to the objects which have now become fields in the space-time sense, Φ, G, etc, to obtain a generating functional for all possible amplitudes. This scheme, which could in principle be regarded as a sort of string field theory, is not useful in practice, since there are an infinite number of (massive) fields to integrate over, which is not a very pretty feature for a Theory Of Everything.

We will therefore limit our attention in this chapter to the massless sector, and consider only special cases in detail.

3.2 Open strings and the electromagnetic vector field

In this section we shall apply the procedures introduced above to calculate an effective action for the electromagnetic vector field A_μ in open string theory. We shall find that in the weak coupling limit, that is ignoring loop corrections and higher derivatives of A_μ, this action equals the Born-Infeld action $S = \int \mathrm{d}^D x \sqrt{\det\left[\eta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}\right]}$ [21]. This result was first described by Fradkin and Tseytlin in [16]. The reason why we repeat their calculation in detail, is that we wish to transfer the results to the T-dual theory: open strings coupling to a D-particle. Before attempting this new subject, we wish to test our techniques on solid ground.

1) Ignoring fields that couple to higher string excitations limits the energy range for which the description is valid: increasing the energy scale, the interaction with the more massive fields becomes a more and more important factor in the dynamics of the low-mass fields.

We shall work with Euclidean metric on the world-sheet ($h_{\alpha\beta} = \mathrm{diag}(+1,+1)$), and in space-time ($\eta_{\mu\nu} = \mathrm{diag}(+1,+1,\ldots,+1)$), which makes some properties of integrals and the field strength more transparent.

3.2.1 The setting

Consider the following expression for the electromagnetic field coupling to the ends of an open string:

$$\Gamma[A_\mu] = \sum_{\chi=1,0,-1,\ldots} g^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, \mathcal{D}X^\mu\, \mathrm{e}^{-S[X]}, \tag{3.10}$$

where $S[X]$ is defined by:

$$S[X] = \frac{1}{4\pi\alpha'} \int\limits_{M_\chi} \mathrm{d}^2\sigma\, \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X_\mu + \mathrm{i} \oint\limits_{\partial M_\chi} \mathrm{d}t\, \dot{X}^\mu A_\mu(X), \tag{3.11}$$

and we have Wick rotated to Euclidean time in order to facilitate the calculations.

In writing the expressions in this simple form, we have implicitly set the sources for closed string excitations to trivial values ($\Phi = 0$, $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$, …. The sources for the open string scalar tachyon and for higher spin excitations have also been set to zero. Furthermore, we will limit ourselves to $\mathrm{U}(1)$ as the choice for the gauge group for A, ie we shall not consider Chan-Paton charges.

We write

$$\Gamma[A_\mu] \equiv \sum_{\chi=1,0,-1,\ldots} g^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, Z[A_\mu], \tag{3.12}$$

and postpone the integration and summation over the metric. At this point we split $X^\mu(\sigma,\tau)$ into a term $\xi^\mu(\sigma,\tau)$ which is non-constant[1)], and a constant term x^μ encoding the centre of mass coordinate: $X^\mu(\sigma,\tau) \equiv x^\mu(\sigma,\tau) + \xi^\mu$. This gives us:

$$Z[A_\mu] = \int \frac{\mathrm{d}^D x}{(2\pi\alpha')^{D/2}} \int \mathcal{D}\xi^\mu\, \mathrm{e}^{-\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\, \sqrt{h}\, h^{\alpha\beta}\, \partial_\alpha \xi^\mu \partial_\beta \xi_\mu - \mathrm{i} \oint \mathrm{d}t\, \dot{\xi}^\mu A_\mu(x+\xi)}. \tag{3.13}$$

The x-integral will be postponed and suppressed in the following. Furthermore, we will partially integrate the first term, yielding

$$\frac{1}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \sqrt{h}\, h^{\alpha\beta} \partial_\alpha \xi^\mu\, \partial_\beta \xi_\mu = -\frac{1}{2\pi\alpha'} \int\limits_M \mathrm{d}^2\sigma\, \xi^\mu \partial_\alpha \left(\sqrt{h}\, h^{\alpha\beta}\, \partial_\beta \xi_\mu\right) \equiv \xi^\mu \cdot \mathcal{A} \cdot \xi_\mu,$$

where we used the fact that ξ obeys Neumann boundary conditions, and we extended the notation defined in §3.1 to integration over world-sheets.

We introduce the notation[2)]

$$\phi \circ \chi \equiv \oint\limits_{\partial M} \mathrm{d}t\, \phi(t)\chi(t),$$

[1)] A function without a zeroth Fourier component is called non-constant: eg $f : \mathbb{R} \to \mathbb{R}$ is called non-constant if $\int \mathrm{d}x\, f(x) = 0$.

[2)] Parametrizing the boundary ∂M by $t \mapsto \gamma^\mu(t)$, we write $\phi(t) \equiv \phi(\gamma(t))$ and similarly for operators $A(t,t')$.

and

$$A\circ\phi:\ x\mapsto\oint_{\partial M}\mathrm{d}t\,A(x,t)\phi(t),\quad\text{or}\quad A\circ\phi:\ t'\mapsto\oint_{\partial M}\mathrm{d}t\,A(t',t)\phi(t),$$

depending on context. With the centre of mass-integral suppressed as promised, we may then write (3.13) as:

$$Z[A^\mu]=\int\mathcal{D}\xi^\mu\ \mathrm{e}^{-\frac{1}{2}\xi^\mu\cdot\mathcal{A}\cdot\xi_\mu-\mathrm{i}\dot{\xi}^\mu\circ A_\mu(x+\xi)}.\tag{3.14}$$

3.2.2 The calculation

Integrals such as (3.14) are usually performed by completing squares. In the present case, a boundary current coupled to ξ, it is appropriate to try to start by integrating out the parts of ξ pertaining to the interior. In order to do so, we introduce a new field $\eta^\mu(t)$ defined on the boundary[1)], and insert a factor

$$1=\int\mathcal{D}\eta^\mu\ \delta(\xi-\eta)\tag{3.15}$$

into (3.14)[2)] . We may now use the presence of the δ-functional to replace $A_\mu(x+\xi)$ in (3.14) by $A_\mu(x+\eta)$, removing the ξ-dependence of the current. The ξ-integrals can now be performed if we write

$$\delta(\xi-\eta)\equiv\int\mathcal{D}\nu^\mu\ \mathrm{e}^{\mathrm{i}\nu_\mu\circ[\xi^\mu-\eta^\mu]},$$

extending the Fourier transform of the ordinary δ-function (and absorbing the factors 2π in the measure)[3)]. With the results (3.34) and (3.35) from appendix 3.A, the square in (3.14) may be completed:

$$\begin{aligned}Z[A_\mu]&=\int_M\mathcal{D}\xi^\mu\int_{\partial M}\mathcal{D}\eta^\mu\int_{\partial M}\mathcal{D}\nu^\mu\ \mathrm{e}^{-\frac{1}{2}\xi^\mu\cdot\mathcal{A}\cdot\xi_\mu+\mathrm{i}\xi^\mu\circ\nu_\mu-\mathrm{i}\eta^\mu\circ\nu_\mu-\mathrm{i}\dot{\xi}^\mu\circ A_\mu(x+\eta)} \qquad (3.16)\\&=\int_M\mathcal{D}\xi^\mu\int_{\partial M}\mathcal{D}\eta^\mu\int_{\partial M}\mathcal{D}\nu^\mu\ \mathrm{e}^{-\frac{1}{2}\tilde{\xi}^\mu\cdot\mathcal{A}\cdot\tilde{\xi}_\mu-\frac{1}{2}\nu^\mu\circ G\circ\nu_\mu+\nu^\mu\circ\dot{G}\circ A_\mu-\frac{1}{2}A^\mu\circ\ddot{G}\circ A_\mu-\mathrm{i}\eta^\mu\circ\nu_\mu},\end{aligned}$$

where $\tilde{\xi}=\xi-\mathrm{i}G\circ\nu+\mathrm{i}\dot{G}\circ A$. For the purpose of the ξ-integrals, the difference between ξ and $\tilde{\xi}$ is just a constant, which doesn't change the result of the integration. Defining

1) More properly, if the boundary consists of a number of parts, $\partial M=\bigcup_i C_i$, introduce fields $\eta_i^\mu(t)$ on each of these parts.

2) This equation effectively defines the δ-functional: it can be made exact by timeslicing: split the length T of the boundary into N pieces, and define $t_i=\frac{i}{N}T$. Eq. (3.15) then becomes

$$1=\lim_{N\to\infty}\int\left[\prod_{i=1}^N\frac{\mathrm{d}^D\eta(t_i)}{(2\pi\alpha')^{D/2}}\right]\prod_{i=1}^N(2\pi\alpha')^{D/2}\delta^{(D)}\left(\xi(t_i)-\eta(t_i)\right).$$

3) That is:

$$\prod_{i=1}^N(2\pi\alpha')^{D/2}\delta^{(D)}\left(\xi(t_i)-\eta(t_i)\right)=\int\left[\prod_{i=1}^N\left(\frac{\alpha'}{2\pi}\right)^{D/2}\mathrm{d}^D\nu(t_i)\right]\mathrm{e}^{\mathrm{i}\sum_{i=1}^N\nu_\mu(t_i)[\xi^\mu(t_i)-\eta^\mu(t_i)]}.$$

$Z_0 = \int_M \mathcal{D}\xi^\mu \, e^{-\frac{1}{2}\xi^\mu \cdot \mathcal{A} \cdot \xi_\mu} = (\det \mathcal{A})^{-D/2}$, we may next focus on the ν-integrals:

$$\int_{\partial M} \mathcal{D}\nu^\mu \quad e^{-\frac{1}{2}\nu^\mu \circ G \circ \nu_\mu + \nu^\mu \circ \left(\dot{G} \circ A_\mu - i\eta_\mu\right)}$$
$$= \int_{\partial M} \mathcal{D}\nu^\mu \, e^{-\frac{1}{2}\tilde{\nu}^\mu \circ G \circ \tilde{\nu}_\mu - \frac{1}{2}\eta^\mu \circ G^{-1}\eta_\mu - i\dot{\eta}^\mu \circ A_\mu + \frac{1}{2}A^\mu \circ \ddot{G} \circ A_\mu},$$

where $\tilde{\nu} = \nu + G^{-1} \circ \left(i\eta - \dot{G} \circ A\right)$, and G^{-1} is defined by $G \circ G^{-1} = \mathbb{1}_{\partial M}$, in other words, G^{-1} is $\mathcal{A}$ limited to the boundary.

The ν-integrals can now be performed and the $A \circ \ddot{G} \circ A$-terms cancel, so that we are left with[1]:

$$Z[A_\mu] = Z_0 \mathcal{N}^D \int_{\partial M} \mathcal{D}\eta^\mu \, e^{-\frac{1}{2}\eta^\mu \circ G^{-1} \circ \eta_\mu - i\dot{\eta}^\mu \circ A_\mu(x+\eta)}, \tag{3.17}$$

with

$$\mathcal{N} = \int_{\partial M} \mathcal{D}\nu \, e^{-\frac{1}{2}\nu \circ G \circ \nu} = (\det G)^{-1/2}.$$

We cannot hope to evaluate the η-integral exactly. Therefore, we will expand A_μ in powers of η^μ:

$$A_\mu(x+\eta) = A_\mu(x) + \eta^\nu \partial_\nu A_\mu(x) + \frac{1}{2}\eta^\nu \eta^\lambda \partial_\nu \partial_\lambda A_\mu(x) + \ldots.$$

Properly anti-symmetrizing and noting that $\oint dt \;\dot{\eta}^\mu = 0$, yields:

$$\dot{\eta}^\mu \circ A_\mu(y+\eta) = \frac{1}{2}F_{\mu\nu} \oint dt \;\dot{\eta}^\nu \eta^\mu + \frac{1}{3}\partial_\lambda F_{\mu\nu} \oint dt \;\dot{\eta}^\nu \eta^\mu \eta^\lambda + \ldots.$$

Following in the footsteps of Fradkin and Tseytlin, we consider only constant fields $F_{\mu\nu}$[2]. In this sector (3.17) may be computed exactly.

At this point $F_{\mu\nu}$ is just a constant anti-symmetric matrix. As such, it is possible to find an element $U \in \mathrm{SO}(D)$[3] for which $U^T F_{\mu\nu} U \equiv \bar{F}$ has a standard form:

$$\bar{F} = \begin{pmatrix} 0 & f_1 & & & \\ -f_1 & 0 & & \emptyset & \\ & & \ddots & & \\ & \emptyset & & 0 & f_{D/2} \\ & & & -f_{D/2} & 0 \end{pmatrix}.$$

In the sector where $F_{\mu\nu}$ is constant, we may thus write (3.17) as

$$Z[F_{\mu\nu}] = Z_0 \mathcal{N}^D \int_{\partial M} \left[\prod_{i=1}^{D/2} \mathcal{D}\eta_i\right] e^{-\frac{1}{2}\eta^\mu \circ G^{-1} \circ \eta_\mu - \frac{i}{2}\sum_{j=1}^{D/2} f_j \left[\dot{\eta}_{2j-1} \circ \eta^{2j} - \eta_{2j-1} \circ \dot{\eta}^{2j}\right]}.$$

---

[1] This result could have been found far more easily by replacing $\dot{\xi}$ by $\dot{\eta}$ in the final term of (3.16), but the method employed here can be generalized more easily, as will be apparent in the next section.

[2] In fact, taking $F_{\mu\nu}$ to be constant is equivalent to assuming that $F_{\mu\nu}$ is slowly changing on the string scale.

[3] Recall that we are working in Euclidean space-time.

Integrating the very last term by parts yields a form without derivatives of any of the η^{2j}. Therefore, the functional integral over half of the η's may be performed, yielding

$$Z[F_{\mu\nu}] = Z_0 \mathcal{N}^{D/2} \int_{\partial M} \left[\prod_{i=1}^{D/2} \mathcal{D}\eta_{2i-1} \right] \mathrm{e}^{-\frac{1}{2}\sum_j \eta^{2j-1}\circ G^{-1}\circ\eta^{2j-1} - \frac{1}{2}\sum_j f_j \dot\eta^{2j-1}\circ G\circ\dot\eta^{2j-1}},$$

since

$$\int_{\partial M} \mathcal{D}\eta\, \mathrm{e}^{-\frac{1}{2}\eta\circ G^{-1}\circ\eta} = \frac{1}{\sqrt{\det G^{-1}}} = \mathcal{N}^{-1}$$

In the functional integrals the η's are dummies, so, there being no cross-terms left, we may separate the integrals to find:

$$Z[F_{\mu\nu}] = Z_0 \mathcal{N}^{D/2} \prod_{i=1}^{D/2} \int_{\partial M} \mathcal{D}\eta\, \mathrm{e}^{-\frac{1}{2}\eta\circ G^{-1}\eta - \frac{1}{2} f_i^2 \dot\eta\circ G\circ\dot\eta}.$$

Taking out another $D/2$ factors of $\mathcal{N}^{-1}$ this may be written as:

$$Z[F_{\mu\nu}] = Z_0 \prod_{i=1}^{D/2} \int_{\partial M} \mathcal{D}\eta\, \mathrm{e}^{-\frac{1}{2}\eta\circ\Delta_i\circ\eta} = Z_0 \int \frac{\mathrm{d}^D x}{(2\pi\alpha')^{D/2}} \prod_{i=1}^{D/2} \frac{1}{\sqrt{\det \Delta_i}}, \tag{3.18}$$

where $\Delta_i = \mathbb{1} + f_i^2 \ddot{G}\circ G$, and we have re-instated the integral over the centre of mass position. (We have partially integrated the $\dot\eta\circ G\circ\dot\eta$ term twice to yield $\eta\circ\ddot{G}\circ\eta$.)

This is the final expression to be inserted into (3.12).

3.2.3 Tree level approximation

At loop level, the integration over the metric is quite complicated, but at tree level it can be done easily: by conformal invariance we can gauge fix $h_{\alpha\beta}$ to any convenient value. We will set $h_{\alpha\beta} = \eta_{\alpha\beta}$, and take M to be the unit disk. Re-instating the integral over space-time, the tree level term ($\chi = 1$) in (3.12) reads:

$$\Gamma_{\text{tree}}[F_{\mu\nu}] = \frac{1}{g_c} Z[F_{\mu\nu}]. \tag{3.19}$$

Introducing $z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\phi}$, the basic form of the propagator is $\Box_0^{-1}(z,z') = \frac{1}{2\pi}\ln|z-z'|$, since $\Box = 4\partial\bar\partial$[1]. (Acting with $\Box$ on $\Box_0^{-1}$ yields $\frac{1}{\pi}\bar\partial\frac{1}{z-z'}$. This is zero everywhere except where $z = z'$. Noting that $\int_M \mathrm{d}^2 z\, \bar\partial f = -\oint_{\partial M} \mathrm{d}z\, f$[2], and using the fact that $\oint_C \mathrm{d}z\, z^{n-1} = 2\pi\mathrm{i}\,\delta_n$ when C circles the point $z = 0$ exactly once in the positive direction, we see that $\Box\Box_0^{-1}(z,z') = \delta^{(2)}(z,z')$ as claimed.)

As presented, $\Box_0^{-1}$ does not obey Neumann boundary conditions, but this can easily be mended by adding a mirror charge:

$$\Box^{-1}(z,z') = \frac{1}{2\pi}\ln|z-z'||z-(\bar{z}')^{-1}|$$

[1] Note that $\partial = \frac{1}{2}(\partial_x - \mathrm{i}\partial_y)$ and $\bar\partial = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$.
[2] In our conventions, $\mathrm{d}^2 z = \mathrm{d}z\,\mathrm{d}\bar{z} = -2\mathrm{i}\,\mathrm{d}x\,\mathrm{d}y$.

is the same as the basic form inside the unit disk, but *does* obey Neumann boundary conditions on the boundary[1)].

To continue our calculation, we need to limit the propagator to the boundary. Parametrizing the boundary by $z(t) = \mathrm{e}^{it}$, we find

$$G(t,t') = -2\pi\alpha'\Box^{-1} = -\alpha' \ln\left(2 - 2\cos(t-t')\right) = 2\alpha' \sum_{n=1}^{\infty} \frac{1}{n} \cos n(t-t').$$

(The equality used in the last step can be obtained from any sufficiently large table of mathematical facts.)

This can easily by differentiated to yield

$$\ddot{G}(t,t') \equiv \partial_t \partial_{t'} G(t,t') = 2\alpha' \sum_n n \cos n(t-t').$$

This gives us

$$\begin{aligned} \ddot{G} \circ G : (t,t') \mapsto \int_0^{2\pi} \mathrm{d}t'' \, \left(2\alpha'\right)^2 \sum_{n,m>0} \frac{n}{m} \cos n(t-t'') \cos m(t''-t') \\ = \left(2\pi\alpha'\right)^2 \sum_{n>0} \frac{1}{\pi} \cos n(t-t'), \end{aligned}$$

which differs from the δ-function by a constant term only:

$$\delta(t) = \frac{1}{2\pi} + \sum_{n>0} \frac{1}{\pi} \cos nt.$$

For non-constant functions such as η, for which $\oint \mathrm{d}t \, \eta(t) = 0$, this first term is unimportant, so we have

$$(\ddot{G} \circ G) \circ \eta = \left(2\pi\alpha'\right)^2 \mathbb{1} \circ \eta = \left(2\pi\alpha'\right)^2 \eta.$$

In order to compute the determinant of Δ_i, we must expand η on a basis for non-constant functions $[0, 2\pi] \to \mathbb{R}$. The normalization is immaterial, since the measure of the path integral hides an arbitrary constant. Taking $\eta = \sum_{n\neq 0} \alpha_n a_n$, with

$$\begin{aligned} a_n(t) &= \frac{1}{\sqrt{\pi}} \cos(nt) \qquad (n > 0), \\ a_n(t) &= \frac{1}{\sqrt{\pi}} \sin(-nt) \qquad (n < 0), \end{aligned}$$

and $\{\alpha_n\}$ a set of constant coefficients, we find

$$\begin{aligned} (\det \Delta_i)^{-\frac{1}{2}} &= \int \mathcal{D}\eta \, \mathrm{e}^{-\frac{1}{2}\eta\circ\Delta_i\circ\eta} \\ &= \int \prod_{n\neq 0} \mathrm{d}\alpha_n \, \mathrm{e}^{-\frac{1}{2}\sum_{n,m\neq 0} \alpha_n \alpha_m \, a_n \circ \left[\mathbb{1} + (2\pi\alpha' f_i)^2 \ddot{G}\circ G\right] \circ a_m} \end{aligned}$$

[1)] Replacing z by $r\mathrm{e}^{i\phi}$, it is quite trivial to show that $\partial_r \Box^{-1} = 0$ when $r = 1$.

$$
\begin{aligned}
&= \int \prod_{n\neq 0} \mathrm{d}\alpha_n \, \mathrm{e}^{-\frac{1}{2}\sum_{n,m\neq 0}\left(1+(2\pi\alpha' f_i)^2\right)\alpha_n\alpha_m a_n\circ a_m} \\
&= \int \prod_{n\neq 0} \mathrm{d}\alpha_n \, \mathrm{e}^{-\frac{1}{2}\sum_{n\neq 0}\left(1+(2\pi\alpha' f_i)^2\right)\alpha_n^2} \\
&= \prod_{n\neq 0}\left(1+\left(2\pi\alpha' f_i\right)^2\right)^{-\frac{1}{2}}.
\end{aligned}
$$

Inserting this into (3.18) yields

$$
Z[F_{\mu\nu}] = \prod_{i=1}^{D/2}\prod_{n>0}\frac{1}{1+(2\pi\alpha' f_i)^2}. \tag{3.20}
$$

Using Riemann ζ-functions[1)], it is possible to find a prescription for such infinite products. In this way (3.20) may be taken to mean

$$
Z[F_{\mu\nu}] = \prod_{i=1}^{D/2}\sqrt{1+(2\pi\alpha' f_i)^2}.
$$

To obtain a nice expression in terms of $F_{\mu\nu}$ again, note that

$$
\begin{aligned}
\prod_{i=1}^{D/2}\left(1+\left(2\pi\alpha' f_i\right)^2\right) &= \prod_{i=1}^{D/2}\det\begin{pmatrix} 1 & 2\pi\alpha' f_i \\ -2\pi\alpha' f_i & 1\end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 2\pi\alpha' f_1 & & & \\ -2\pi\alpha' f_1 & 1 & & \emptyset & \\ & & \ddots & & \\ & \emptyset & & 1 & 2\pi\alpha' f_{D/2} \\ & & & -2\pi\alpha' f_{D/2} & 1 \end{pmatrix} \\
&= \det\left(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right).
\end{aligned} \tag{3.21}
$$

The final result is obtained by inserting this into (3.19):

$$
\Gamma_{\text{tree}}[F_{\mu\nu}] = Z_0 g_c^{-1}\int\frac{\mathrm{d}^D x}{(2\pi\alpha')^{D/2}}\sqrt{\det\left(\delta_{\mu\nu}+2\pi\alpha' F_{\mu\nu}\right)}. \tag{3.22}
$$

Rotating back to ordinary Minkowski metric simply means that all terms in the expansion of (3.22) should be written in a Lorentz covariant way, eg the term $\int F_{\mu\nu}F_{\mu\nu}$ should be replaced by $\int F_{\mu\nu}F^{\mu\nu}$.

We have thus shown that the effective action for a vector potential coupling to the end of an open string is in fact equal to the Born-Infeld action for non-linear electrodynamics. This surprising result is exact in powers of $F_{\mu\nu}$, but it is only the first order approximation to the full derivative expansion (terms containing ∂F, $\partial\partial F$, $\ldots$), although it has been shown [24], that for the supersymmetric equivalent to this calculation (which we present in appendix 3.B) the $F^n\partial F\partial F$-terms vanish, and the first corrections to (3.22) are $FF\,\partial\partial F\,\partial\partial F$-terms.

1) In particular, $\zeta(0) = \sum_{n>0} 1 \equiv -\frac{1}{2}$, so $\prod_{n>0}\mathcal{A} = \mathrm{e}^{\sum\log\mathcal{A}} = \mathcal{A}^{\sum 1} = \mathcal{A}^{-\frac{1}{2}}$. See [22] for further details, and [23] for a justification of the use of this prescription in the present context.

3.3 D-particles

In the previous section we considered open strings with Neumann boundary conditions. Now we will take Dirichlet boundary conditions for all spatial coordinates. Following basically the same procedure as before, this will enable us to derive an effective action for a single D-particle in bosonic open string theory.

We start of from (3.12) again, but this time we take[1]:

$$S[X] = \frac{1}{4\pi\alpha'}\int_{M_\chi} d^2\sigma\,\sqrt{h}\,h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + \frac{1}{2\pi\alpha'}\int_{\partial M_\chi} dt\, X'^m Y_m(X). \tag{3.23}$$

m runs over the coordinates transversal to the particle, ie the spatial coordinates: $m = 1,\ldots,D-1$. We have not included a term $\int dt\,\dot{X}^0 A_0(X)$, since a (0+1)-dimensional gauge field does not carry any physical degrees of freedom. In a more general setting, with Neumann boundary conditions for some of the coordinates and Dirichlet for others, we would have to include both $\dot{X}^i A_i(X)$ and $X'^m Y_m(X)$ terms. Also note that we again use Euclidean metric on the world-sheet and in space-time.

As above we set all excitations to zero, except for the massless vector, and split $X^\mu \equiv x^\mu + \xi^\mu$, with ξ^μ non-constant on the boundary of the world-sheet and x^μ constant. We thus have

$$\Gamma[Y^m] \equiv \sum_{\chi=1,0,-1,\ldots} g^{-\chi} \int \mathcal{D}h_{\alpha\beta}\, Z[Y^m], \tag{3.24}$$

with

$$Z[Y^m] = \int \frac{d^D x}{(2\pi\alpha')^{D/2}} \int \mathcal{D}\xi^\mu\, e^{-\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{h}\,h^{\alpha\beta}\,\partial_\alpha\xi^\mu\partial_\beta\xi_\mu - \frac{1}{2\pi\alpha'}\oint dt\,\xi'^m Y_m(x+\xi)}. \tag{3.25}$$

Again we postpone the x-integration and suppress it.

Analogously to the first steps in §3.2, we want to change (3.25) into a Gaussian integral. In order to do so, the first term must be partially integrated. Note that the boundary term $\xi'\circ\xi$ vanishes: ξ'^0 is zero because of the Neumann condition on the time-direction, and the ξ^m's are zero, because the X^m's are constant on the boundary due to the Dirichlet condition. ξ^m, being the non-constant part of X^m, therefore vanishes. Thus we find:

$$Z[Y^m] = \int \mathcal{D}\xi^\mu\, e^{-\frac{1}{2}\xi^\mu\cdot\mathcal{A}\cdot\xi_\mu - \frac{1}{2\pi\alpha'}\xi'^m\circ Y_m(x+\xi)}, \tag{3.26}$$

with $\mathcal{A} = \frac{1}{2\pi\alpha'}\partial_\alpha\sqrt{h}\,h^{\alpha\beta}\partial_\beta$, as before. Although it is not explicit in the equation, one should note that Y_m depends only on ξ^0, since the other components of ξ vanish.

The ξ^0-dependence of the current Y must be removed before any progress can be made. To this end, we again introduce auxiliary fields η defined on the boundary, replace $Y(\xi)$ by $Y(\eta)$ in the last term of (3.26), and complete the square:

$$Z[Y^m] = \int_M \mathcal{D}\xi^\mu \int_{\partial M} \mathcal{D}\eta^0 \int_{\partial M} \mathcal{D}\nu^0\, e^{-\frac{1}{2}\xi^\mu\cdot\mathcal{A}\cdot\xi_\mu + i\xi^0\circ\nu_0 - i\eta^0\circ\nu_0 - \frac{1}{2\pi\alpha'}\xi'^m\circ Y_m(x+\eta)}$$

[1] Note that the factor i which seems to be missing from the second term when compared to (3.11) is due to the fact that we are working in Euclidean time: unlike $\dot{X}$, X' does not acquire a factor i on Wick rotation.

$$= \int_M \mathcal{D}\xi^\mu \int_{\partial M} \mathcal{D}\eta^0 \int_{\partial M} \mathcal{D}\nu^0 \, \mathrm{e}^{-\frac{1}{2}\tilde{\xi}^\mu \cdot \mathcal{A} \cdot \tilde{\xi}_\mu - \frac{1}{2}\nu^0 \circ G_N \circ \nu_0 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2} Y^m \circ G_D'' \circ Y_m - \mathrm{i}\eta^0 \circ \nu_0},$$

where $\tilde{\xi}^\mu = \xi^\mu - \mathrm{i}G_N \circ \nu_0 \delta^{\mu 0} + \frac{1}{2\pi\alpha'} G_D' \circ Y_m \delta^{\mu m}$. Note that we have not bothered to introduce auxiliary fields for the spatial coordinates, since the current does not depend on these, even in (3.25). Also note that, while in the interior the Green function for Neumann and Dirichlet conditions is the same, on the boundary we must be careful about the difference[1)].

The ξ-integrals can now be performed, and produces the same factor Z_0 as in the EM-field case. The ν^0-integrals can be performed totally analogously as well, yielding

$$Z[Y^m] = Z_0 \mathcal{N} \int_{\partial M} \mathcal{D}\eta^0 \, \mathrm{e}^{-\frac{1}{2}\eta^0 \circ G_N^{-1} \circ \eta_0 - \frac{1}{2}\frac{1}{(2\pi\alpha')^2} Y^m \circ G_D'' \circ Y_m}. \tag{3.27}$$

Noting again that Y^m only depends on η^0, we may expand:

$$Y^m(x+\eta) = Y^m(x) + \eta^0 \partial_0 Y^m(x) + \frac{1}{2}\left(\eta^0\right)^2 \partial_0^2 Y^m(x) + \cdots.$$

Keeping only the first two terms and writing $\partial_0 Y^m(x) \equiv v^m$, (3.27) becomes

$$Z[v^m] = Z_0 \mathcal{N} \int \mathcal{D}\eta^0 \, \mathrm{e}^{-\frac{1}{2}\eta^0 \circ G_N^{-1} \circ \eta_0 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2} v^m \eta^0 \circ G_D'' \circ \eta^0 v_m},$$

At this point it is suitable to absord the integral over the spatial x^m's – which is just the volume of space – into Z_0[2)].

The factor of $\mathcal{N}$ may be absorbed in the exponent, yielding

$$Z[v^m] = Z_0 \int \mathcal{D}\eta^0 \, \mathrm{e}^{-\frac{1}{2}\eta^0 \Delta \eta^0}, \tag{3.28}$$

where $\Delta = \mathbb{1} - \frac{1}{(2\pi\alpha')^2} v^2 G_N \circ G_D''$.

The Green functions obey $G_N \circ G_D'' = -(2\pi\alpha')^2 \mathbb{1}$ when used on non-constant functions η[3)]. The path integral may therefore be performed:

$$Z[v^m] = Z_0 \int \mathcal{D}\eta^0 \, \mathrm{e}^{-\frac{1}{2}(1+v^2)\eta^0 \circ \eta^0} = \prod_{n \neq 0} \frac{1}{\sqrt{1+v^2}}. \tag{3.29}$$

With the same Riemann function trick as before this may be written as

$$Z[v^m] = Z_0 \int \frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}} \sqrt{1+v^2}, \tag{3.30}$$

---

1) Explicitly, the propagator that obeys Dirichlet boundary conditions on the unit disk, is

$$\Box^{-1}(z,z') = \frac{1}{2\pi} \ln |z - z'| |z - (\bar{z}')^{-1}|^{-1}.$$

2) This is reasonable, since even if we wouldn't take v^m to be constant, we would still have Y^m independent of the x^m, the latter being constant on the boundary.

3) This follows from the calculation of $G_N \circ \ddot{G}_N$ in the previous section, if we note that on the boundary $G_D'' = -\ddot{G}_N$.

where we have re-instated the integral over x^0, which is the only part that remains of the x^μ-integrals after the spatial coordinates have been integrated out. Inserting this result into (3.24), we find:

$$\Gamma_{\text{tree}}[v^m] = Z_0 g_c^{-1} \int \frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\, \sqrt{1+v^2}. \tag{3.31}$$

After Wick rotating back to Minkowski time, the final result is

$$\Gamma_{\text{tree}}[v^m] = Z_0 g_c^{-1} \int \frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\, \sqrt{1-v^2}. \tag{3.32}$$

The D-particle effective action is thus found to equal the action for a relativistic point particle with mass $m \sim \frac{1}{g_c\sqrt{\alpha'}}$ [1)], provided that the velocity is slowly changing on the world-line: as before, the result is exact in powers of v, but we have ignored acceleration terms and loop corrections. Both of these correspond to corrections of order g_c^0.

3.A Appendix: Completing squares with boundary currents

In this appendix we shall prove some of the equalities used to complete squares in the path integrals of section §3.2. Consider the following integral:

$$[\phi - \mathrm{i}G \circ J] \cdot \mathcal{A} \cdot [\phi - \mathrm{i}G \circ J]\,. \tag{3.33}$$

The operators G and $\mathcal{A}$ are related by $\mathcal{A} \cdot G = G \cdot \mathcal{A} = \mathbb{1}$, with $\mathbb{1}$ the unit operator defined by $\mathbb{1} \cdot f = f$, (which means $\int_M \mathrm{d}^2y\, \delta(x-y)\, f(y) = f(x)$).

We may evaluate (3.33) as follows:

$$\begin{aligned}(3.33) &= \phi \cdot \mathcal{A} \cdot \phi - \mathrm{i}\,(G \circ J) \cdot \mathcal{A} \cdot \phi - \mathrm{i}\phi \cdot \mathcal{A} \cdot (G \circ J) - (G \circ J) \cdot \mathcal{A} \cdot (G \circ J)\\ &= \phi \cdot \mathcal{A} \cdot \phi - \mathrm{i}J \circ G \cdot \mathcal{A} \cdot \phi - \mathrm{i}\phi \cdot \mathcal{A} \cdot G \circ J - J \circ G \cdot \mathcal{A} \cdot G \circ J\\ &= \phi \cdot \mathcal{A} \cdot \phi - 2\mathrm{i}J \circ \phi - J \circ G \circ J.\end{aligned}$$

Comparing the left- and right-hand sides, we find:

$$-\frac{1}{2}\phi \cdot \mathcal{A} \cdot \phi + \mathrm{i}J \circ \phi = -\frac{1}{2}[\phi - \mathrm{i}G \circ J] \cdot \mathcal{A} \cdot [\phi - \mathrm{i}G \circ J] - \frac{1}{2}J \circ G \circ J. \tag{3.34}$$

Completing squares is thus possible for boundary currents. Note that $G \circ J$ is defined on M, *not* just on ∂M, so when using (3.34) inside the exponent of a path integral, there will be no problem in changing variables from ϕ to $\tilde{\phi} = \phi + \mathrm{i}G \circ J$ and performing the integral.

A similar result may be found when the current is not coupled to ϕ itself, but to some derivative (tangential or normal) $\partial\phi$. We will write $\partial_{1(2)}G$ to denote the derivative with respect to the first (second) entry of an operator G. We may then write out

$$[\phi - \mathrm{i}\partial_2 G \circ J] \cdot \mathcal{A} \cdot [\phi - \mathrm{i}\partial_2 G \circ J]\,.$$

[1)] The exact numerical prefactor (which is dimensionless) can be discovered by a loop level calculation, see eg [19]. In the present formulation, it is less clear which numerical factors should properly be absorbed in Z_0.

The first term is as before. The second and third terms can be simplified by using

$$\begin{aligned}\phi\cdot\mathcal{A}\cdot\partial_2 G\circ J &= \phi\cdot\partial_2(\mathcal{A}\cdot G)\circ J = \phi\cdot\partial_2\mathbb{1}\circ J\\ &= \partial(\phi\cdot\mathbb{1})\circ J = \partial\phi\circ J.\end{aligned}$$

In this derivation, we have repeatedly used the fact that $g(y)\partial_x f(y,x) = \partial_x(g(y)f(y,x))$.

The final term may be simplified because

$$\begin{aligned}(\partial_2 G\circ J)\cdot\mathcal{A}(\partial_2 G\circ J) &= (J\circ\partial_1 G)\cdot\mathcal{A}\cdot(\partial_2 G\circ J) = (J\circ\partial_1 G)\cdot(\partial_2(\mathcal{A}\cdot G)\circ J)\\ &= (J\circ\partial_1 G)\cdot(\partial_2\mathbb{1}\circ J) = \partial\left[(J\circ\partial_1 G)\cdot\mathbb{1}\right]\circ J\\ &= \partial\left[(J\circ\partial_1 G)\right]\circ J = J\circ\partial_1\partial_2 G\circ J.\end{aligned}$$

Taken together, this means that

$$-\frac{1}{2}\phi\cdot\mathcal{A}\cdot\phi + \mathrm{i}\partial\phi\circ J = -\frac{1}{2}(\phi-\mathrm{i}\partial_2 G\circ J)\cdot\mathcal{A}\cdot(\phi-\mathrm{i}\partial_2 G\circ J) - \frac{1}{2}J\circ\partial_1\partial_2 G\circ J. \tag{3.35}$$

It is important to note that both (3.34) and (3.35) are completely independent of the form of J. In particular, J may depend functionally on ϕ.

3.B Appendix: Supersymmetric extension

All of the above calculations were performed in the context of bosonic string theory. To be relevant to the physics of the real world, they should be ported to superstring theory. In this appendix we shall extend the calculation of §3.2 to the supersymmetric theory.

3.B.1 Supersymmetry in string theory

Recall that the bosonic string may be supersymmetrized by adding a fermionic term to the action. On a world-sheet with Euclidean metric, the supersymmetric open string action reads

$$S_{\mathrm{susy}} = \frac{1}{4\pi\alpha'}\int \mathrm{d}\tau\,\mathrm{d}\sigma\,\left\{\partial_\alpha X^\mu\partial^\alpha X_\mu - \bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu\right\}. \tag{3.36}$$

The Dirac matrices that obey $\{\rho^\alpha,\rho^\beta\} = -2\delta^{\alpha\beta}$ are

$$\rho^0 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}, \qquad \text{and} \qquad \rho^1 = \begin{pmatrix} 0 & \mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},$$

and the infinitesimal supersymmetry transformation is given by

$$\begin{aligned}\delta X^\mu &= -\mathrm{i}\bar{\varepsilon}\psi^\mu\\ \delta\psi^\mu &= -\mathrm{i}\rho^\alpha\varepsilon\partial_\alpha X^\mu.\end{aligned} \tag{3.37}$$

In the following we will often use that for Majorana spinors (which are real), $\psi^\dagger = \psi^T$, so $\bar{\psi}\chi = \psi^A\rho^0_{AB}\chi^B = \bar{\chi}\psi$, since ρ^0 is antisymmetric and the components of ψ and χ are (anticommuting!) Grassmann numbers. Similarly, we may show that $\bar{\psi}\rho^\alpha\chi = -\bar{\chi}\rho^\alpha\psi$, $\bar{\psi}\rho^\alpha\rho^\beta\chi = \bar{\chi}\rho^\beta\rho^\alpha\psi$, etcetera.

3.B.2 Supersymmetry in open strings with electromagnetic boundary charge

We return now to the calculation of §3.2. To supersymmetrize (3.11), we must add a fermionic boundary term as well as a fermionic bulk term. The resulting action is

$$S_{\text{susy}} = \frac{1}{4\pi\alpha'} \int_{M_\chi} \mathrm{d}^2\sigma \, \left\{ \partial_\alpha X^\mu \partial^\alpha X_\mu - \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu \right\} + \mathrm{i} \oint_{\partial M_\chi} \mathrm{d}\sigma^\alpha \, \left\{ \partial_\alpha X^\mu A_\mu + \frac{1}{2} \overline{\psi}^\mu \rho_\alpha \psi^\nu F_{\mu\nu} \right\}. \tag{3.38}$$

(Note that in Euclidean metric $\rho^\alpha = \rho_\alpha$.)

Choosing a gauge in which $A_\mu = -\frac{1}{2} F_{\mu\nu} X^\nu$ – which is possible globally if $F_{\mu\nu} = cst$ as in §3.2 – and noting that on the boundary $\partial_n X = 0$, it is a trivial exercise to insert (3.37) into (3.38) and show that it is indeed supersymmetric.

Having limited ourselves to the case $F_{\mu\nu} = cst$, there is no interaction between bosonic and fermionic variables left in the action. Therefore the path integral may be split, and the bosonic part of the calculation proceeds as before, upto (3.20).

The fermionic path integral looks as follows:

$$\int_M \mathcal{D}\psi \; \mathrm{e}^{\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \overline{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu + \frac{\mathrm{i}}{2} \oint \mathrm{d}t \, \overline{\psi}^\mu \rho^\parallel \psi^\nu F_{\mu\nu}},$$

in which $\rho^\parallel$ denotes the Dirac matrix tangential to the boundary: $\mathrm{d}t \, \rho^\parallel \equiv \mathrm{d}\sigma^\alpha \, \rho_\alpha$.

Just as for the bosonic case, the interior parts of the integration may be performed, leaving just the path integral over the boundary parts of ψ. Introducing

$$G_F^{-1} \equiv \left. \frac{-1}{4\pi\alpha'} \rho^\alpha \partial_\alpha \right|_{\partial M},$$

the result may be written as

$$Z_{\text{fermion}} \sim \int_{\partial M} \mathcal{D}\psi \; \mathrm{e}^{-\overline{\psi}^\mu \circ G_F^{-1} \circ \psi_\mu + \frac{\mathrm{i}}{2} F_{\mu\nu} \overline{\psi}^\mu \rho^\parallel \circ \psi^\nu}.$$

Writing $F_{\mu\nu}$ in standard form as before, half of the ψ's may be integrated away, leaving

$$Z_{\text{fermion}} \sim \prod_{k=1}^{D/2} \int_{\partial M} \mathcal{D}\psi_k \; \mathrm{e}^{-\overline{\psi}_k \circ G_F^{-1} \circ \psi_k + \frac{1}{4} f_k^2 \overline{\psi}_k \circ G_F^T \circ \psi_k}$$

Taking out some factors of $\sqrt{\det G_F^{-1}}$ this may be rewritten as

$$Z_{\text{fermion}} = \prod_{k=1}^{D/2} \int_{\partial M} \mathcal{D}\psi_k \; \mathrm{e}^{-\overline{\psi}_k \circ \psi_k + \frac{1}{4} f_k^2 \overline{\psi}_k \circ (G_F^T \circ G_F) \circ \psi_k} \tag{3.39}$$

where $(Z_0)_{\text{fermion}}$, the square root of the determinant of the propagator in the bulk has been suppressed.

It is possible to show that

$$(G_F^T \circ G_F) : (t,t') \mapsto -4\,(2\pi\alpha')^2 \delta(t-t').$$

Therefore (3.39) reduces to

$$Z_{\text{fermion}} = \prod_{k=1}^{D/2} \int_{\partial M} \mathcal{D}\psi_k \; \mathrm{e}^{-\left[1+(2\pi\alpha' f_k)^2\right] \bar{\psi}_k \circ \psi_k}. \tag{3.40}$$

To conclude the calculation, note that for Majorana fermions

$$\int \mathcal{D}\psi \; \mathrm{e}^{-\bar{\psi}\cdot\mathcal{A}\cdot\psi} = \sqrt{\det \mathcal{A}}.$$

Expanding (3.40) in Fourier modes, and noting that the zero mode *is* present, we find that

$$Z_{\text{fermion}} = \prod_{k=1}^{D/2} \prod_{n} \sqrt{1+(2\pi\alpha' f_k)^2}.$$

This should be multiplied with (3.20) to produce

$$Z_{\text{susy}} = \prod_{k=1}^{D/2} \sqrt{1+(2\pi\alpha' f_k)^2}.$$

As before (cf (3.21)), this can be rewritten in terms of $F_{\mu\nu}$, yielding exactly (3.22), but without any regularization. It is important to note that this happens because there is a one-to-one mapping between fermionic and bosonic modes.

3.B.3 D-branes and supersymmetry

In essentially the same way the calculation of §3.3 may be repeated in a superstring context. Again, no regularization is needed to obtain the finite result (3.30).

CHAPTER 4

N D-PARTICLES

Points
Have no parts or joints.
How then can they combine
to form a line?

— J. A. Lindon

In this chapter we shall consider a system of N D-particles. It will be found that a low-energy effective action for such a configuration is given by the Yang-Mills action reduced to $0+1$ dimensions. After a short introduction about Yang-Mills theory, we shall give an argument for the above statement using T-duality. Having arrived at the effective action using mathematical techniques, we shall then proceed to try to understand the results from a more physical point of view.

4.1 Yang-Mills theory – recap

This section gives a minimal introduction to Yang-Mills theory, mainly intended to lay down conventions. The material presented here can be found in many books on quantum field theory, eg [20]. For a more in-depth treatment, see [25].

4.1.1 General relations

Recall that $(D_\mu\phi)^* D^\mu\phi$ (with ϕ a (complex or real) scalar function) is invariant under $\phi \to \mathrm{e}^{-\mathrm{i}\alpha(x)}\phi$ if D_μ is defined by

$$D_\mu = \partial_\mu - \mathrm{i}A_\mu,$$

and A_μ transforms as

$$A_\mu \to A_\mu - \partial_\mu\alpha.$$

This simple picture can be extended if we replace the scalar ϕ by a vector $\phi \equiv (\phi^i)$, with $i = 1 \ldots r$, and replace α by $\alpha \equiv \alpha^a T_a$, with $a = 1 \ldots n$, and T_a a set of $(r \times r)$-matrices, that represent the generators of a Lie algebra on the space of ϕ's. Note that this means that the T_a's obey

$$[T_a, T_b] = \mathrm{i} f_{abc} T_c,$$

with f_{abc} the structure constants of the algebra.

Ensuring that

$$D_\mu\phi \to e^{-i\alpha} D_\mu\phi \tag{4.1}$$

under $\phi \to e^{-i\alpha}\phi$ now presents a slightly more demanding problem than in the Abelian case.

Write $D_\mu = \partial_\mu - iA_\mu$, which now means

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - iA_\mu^a\,(T_a)_{ij}\,.$$

With this definition of the covariant derivative, (4.1) holds if A_μ transforms as

$$A_\mu \to e^{-i\alpha} A_\mu e^{i\alpha} + ie^{-i\alpha}\partial_\mu e^{i\alpha}. \tag{4.2}$$

The covariant[1)] field strength $F_{\mu\nu}$ is now given by

$$F_{\mu\nu} = i\,[D_\mu, D_\nu] = \partial_{[\mu}A_{\nu]} - i[A_\mu, A_\nu].$$

The extension of the Maxwell Lagrangian ($\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$) to the non-abelian case is

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Tr} F_{\mu\nu}F^{\mu\nu}, \tag{4.3}$$

with the trace acting on the matrices T_a. (4.3) is called the Yang-Mills Lagrangian.

4.1.2 Dimensional reduction

For the purpose of describing Yang-Mills theory living on a D-brane instead of the entire space-time, (4.3) should be dimensionally reduced. This means that the field $A_\mu(X^\nu)$ is split into $A_i(X^j)$, with i and j running from 0 to p, the dimension of the brane, and a set of scalars $\Phi_m(X^i)$, with $m = p{+}1\ldots D{-}1$ (D being the dimension of space-time). Splitting A_μ obviously affects $F_{\mu\nu}$ as well, and in fact $F_{\mu\nu}$ is split into

$$\begin{aligned} F_{ij} &= \partial_{[i}A_{j]} - i[A_i, A_j], \\ F_{im} &= \partial_i\Phi_m - i[A_i, \Phi_m], \\ F_{mn} &= -i[\Phi_m, \Phi_n]. \end{aligned}$$

4.1.3 Dimensional reduction to 0+1 dimensions

In the following, the above will be applied in the context of D-particles, where $p = 0$, and the only component of the vector potential that survives is A_0. Since it is possible to gauge the latter away, it will be suppressed in the discussion. We are then left with

$$\begin{aligned} F_{0m} &= \frac{1}{2\pi\alpha'}\partial_0 Y_m, \\ F_{mn} &= \frac{-i}{(2\pi\alpha')^2}\,[Y_m, Y_n]\,, \end{aligned}$$

with m and n now ranging from 1 to $D-1$. The scalars have been renamed and rescaled to emphasize their later interpretation as spatial coordinates.

1) meaning transformation according to $F \to e^{-i\alpha} F e^{i\alpha}$.

The Lagrangian becomes

$$\mathcal{L} = \mathrm{Tr}\left[\frac{1}{(2\pi\alpha')^2}(\partial_0 Y_m)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^4}[Y_m, Y_n]^2\right]. \tag{4.4}$$

This ends our short recap of Yang-Mills theory. In the next section we shall see how a dimensionally reduced Yang-Mills action arises from open strings coupled to D-0-branes.

4.2 The open string and N D-particles

We would like to extend the calculation of §3.3 to N branes, but unfortunately the techniques used in the calculation cannot be straightforwardly generalized to the non-abelian case, because it is not immediately obvious how the Wilson line should be generalized. However, we may proceed by invoking T-duality: by investigating the properties of the generalization of (3.10) and (3.11) to a $\mathrm{U}(N)$ vector potential, and T-dualizing the result, we arrive at an action which can be interpreted as a low-energy effective action for a set of N D-particles.

In this section, we shall take all radii to infinity immediately after T-dualizing, thus avoiding the intricacies connected with quantization with periodic boundary conditions[1)]. In §4.4 we shall return to compact space and find that the effective dynamics are very different from what we shall find here for the non-compact situation.

4.2.1 Open strings in a U(N) gauge field background

As in chapter 3, we shall limit our attention to the vector potential A_μ. With A_μ a $\mathrm{U}(N)$ field, (3.13) should be replaced by

$$Z[A_\mu] = \int \mathcal{D}\xi\, \mathrm{e}^{-\frac{1}{2}\xi^\mu \cdot \mathcal{A} \cdot \xi_\mu}\, \mathrm{Tr}\,\mathrm{P}\,\mathrm{e}^{-\mathrm{i}\dot{\xi}^\mu \circ A_\mu(x+\xi)}. \tag{4.5}$$

The trace is over the $\mathrm{U}(N)$ Lie algebra indices, and P denotes *path ordening*: for non-commuting $\mathcal{A}$, the expression $\mathrm{e}^{\mathcal{A}}$ is ill-defined unless some ordering convention is specified. The path ordered exponential is defined by

$$\mathrm{P}\,\mathrm{e}^{\int \mathrm{d}t\, \mathcal{A}(t)} = \sum_{n=0}^{\infty} \int_0^T \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n\, \mathcal{A}(t_1)\cdots\mathcal{A}(t_n). \tag{4.6}$$

(Note that for Abelian $\mathcal{A}$ the ordering becomes immaterial, and (4.6) reduces to the well known expansion for $\mathrm{e}^{\int \mathrm{d}t\, \mathcal{A}}$. In chapter 3 we took $\mathrm{U}(1)$ as the gauge group, so there was no need to write Tr P explicitly.)

Tseytlin has shown in some detail [26] that (4.5) yields a Yang-Mills action. T-dualizing this result yields the dimensionally reduced Yang-Mills action that was derived in §4.1. We shall check on this by T-dualizing (4.5) and expanding the result.

---

1) Periodic boundary conditions entail taking the possibility of (multiply) wound strings into account, which complicates matters considerably.

4.2.2 Open strings in the T-dual of U(N) gauge theory

T-dualizing all spatial directions transforms (4.5) into

$$Z[Y^m] = \int \mathcal{D}\xi \, \mathrm{e}^{-\frac{1}{2}\xi^\mu \cdot \mathcal{A} \cdot \xi_\mu} \, \mathrm{Tr}\, \mathrm{P}\, \mathrm{e}^{-\frac{1}{2\pi\alpha'}\xi'^m \circ Y_m(x+\xi)}. \tag{4.7}$$

In this expression the Y_m's are in the adjoint representation of $\mathrm{U}(N)$, that is $Y_m = Y_m^a T_a$, with T_a the generators of the Lie algebra of $\mathrm{U}(N)$.

In the above expression, the term $\dot{\xi}^0 \circ A_0$ has been suppressed, since the equation of motion for A_0 only yields a constraint equation (Gauss's law), and we may choose a gauge in which all of the A_0^a's are zero.

In order to show that (4.7) yields the dimensionally reduced Yang-Mills action of the previous section, we shall first compute the lowest order term in the expansion of (4.7), and then invoke gauge symmetry to complete the calculation.

1. The lowest order term of the effective action

The proposed computation can be facilitated considerably if we first split Y_m into a $\mathrm{U}(1)$ part and a $\mathrm{SU}(N)$ part:

$$Y_m \equiv Y_m^{\mathrm{cm}} \mathbb{1} + Y_m^a T_a,$$

where from now on the T_a are the generators of $\mathrm{SU}(N)$. (4.7) can then be rewritten as

$$Z[Y^m] = \left\langle \mathrm{e}^{-\frac{1}{2\pi\alpha'}\xi'^m \circ Y_m^{\mathrm{cm}}} \, \mathrm{Tr}\, \mathrm{P}\, \mathrm{e}^{-\frac{1}{2\pi\alpha'}\xi'^m \circ Y_m^a T_a} \right\rangle.$$

The expansion becomes

$$\begin{aligned} Z[Y^m] = & \left\langle \left(1 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}\int \mathrm{d}t\, \mathrm{d}t'\ \xi'^m(t)\xi'^n(t') Y_m^{\mathrm{cm}}(t) Y_n^{\mathrm{cm}}(t') \right) \mathrm{Tr}\,\mathbb{1} \right\rangle \\ & + \left\langle \frac{1}{(2\pi\alpha')^2} \int_{t \geq t'} \mathrm{d}t\, \mathrm{d}t'\ \xi'^m(t)\xi'^n(t') Y_m^a(t) Y_n^b(t')\, \mathrm{Tr}\, T_a T_b \right\rangle + \cdots . \end{aligned} \tag{4.8}$$

There are now two ways to proceed. The most obvious one is simply to expand the Y's as $Y(t) = y + \xi^0 v + \frac{1}{2}(\xi^0)^a + \cdots$, and compute (4.8) from scratch. Some infinities occur in the calculation of $\langle \int \mathrm{d}t\, \mathrm{d}t'\ \xi'(t)\xi'(t') \rangle$ and $\left\langle \int \mathrm{d}t\, \mathrm{d}t'\ \xi'^m(t)\xi'^m(t')\xi^0(t)\xi^0(t') \right\rangle$, but these can be dealt with in a straightforward manner. Even so, it is much easier to note that for the $N = 1$ case the second term in (4.8) disappears, and we are left with the first term, which should match the expansion of the previous $\mathrm{U}(1)$ result, that is (3.30).

The first thing to do is to rewrite (4.8) in a more transparent form. To this end we investigate

$$\left\langle \xi'^m(t)\xi'^n(t) Y_m(t) Y_n(t') \right\rangle, \tag{4.9}$$

where Y_m could be either Y_m^{cm} or one of the Y_m^a's. Expand $Y^m(t) = y^m + \xi^0 v^m + \cdots$, and note that

$$\begin{aligned} &\left\langle \xi^0(t)\xi^0(t') \right\rangle = G_N(t,t'), \\ &\left\langle \xi'^m(t)\xi'^n(t') \right\rangle = G''_D(t,t')\delta^{mn}, \\ &\left\langle \xi'^m(t)\xi^0(t') \right\rangle = 0. \end{aligned}$$

With these facts (4.9) becomes

$$G_N(t,t')y^m y_m + G_N(t,t')G''_D(t,t')v^m v_m.$$

In the $\mathrm{U}(1)$ case (4.8) is thus found to equal

$$Z_{\mathrm{U}(1)}[y^m_{\mathrm{cm}};v^m_{\mathrm{cm}}] = \left(\langle 1\rangle + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}\int \mathrm{d}t\,\mathrm{d}t' \;\left[G_N(t,t')(y_{\mathrm{cm}})^2 + G_N(t,t')G''_D(t,t')(v_{\mathrm{cm}})^2\right]\right)\mathrm{Tr}\,\mathbb{1}. \tag{4.10}$$

We shall use a trick to compute the Green functions: comparing (4.10) with (3.30), which expands to

$$Z[v^m] = Z_0\left[1 - \frac{1}{2}v^2 + \ldots\right],$$

we find that

$$\int \mathrm{d}t\,\mathrm{d}t'\; G_N(t,t') = 0,$$

and

$$\int \mathrm{d}t\,\mathrm{d}t'\; G_N(t,t')G''_D(t,t') = -(2\pi\alpha')^2\,\langle 1\rangle\,.$$

As noted before, both of these results could also have been found by direct calculation. (The latter has actually been found in chapter 3.)

Armed with this knowledge, tackling the second term in (4.8) is easy: normalizing the generators of $\mathrm{SU}(N)$ in such a manner that

$$\mathrm{Tr}\,T_a T_b = \frac{1}{2}\delta_{ab}$$

it can be written as

$$\left\langle \sum_a \frac{1}{2}\frac{1}{(2\pi\alpha')^2}\int_{t\geq t'} \mathrm{d}t\,\mathrm{d}t'\; \xi'^m(t)\xi'^n(t')Y^a_m(t)Y^a_n(t')\right\rangle.$$

The integrand is now symmetric in $t \leftrightarrow t'$, so the integration can also be symmetrized, yielding

$$\left\langle \sum_a \frac{1}{4}\frac{1}{(2\pi\alpha')^2}\int \mathrm{d}t\,\mathrm{d}t'\; \xi'^m(t)\xi'^n(t')Y^a_m(t)Y^a_n(t')\right\rangle.$$

Each of the terms in the sum can individually be treated in exactly the same way as the $\mathrm{U}(1)$ part, resulting in

$$\sum_a \frac{-1}{4}(v_a)^2\,\langle 1\rangle\,.$$

Taking it all together again, we find that

$$Z[Y^m] = \left(N\left[1 - \frac{1}{2}(v_{\mathrm{cm}})^2\right] - \frac{1}{4}\sum_a (v_a)^2\right)\langle 1\rangle\,, \tag{4.11}$$

where we have used the fact that $\mathrm{Tr}\,\mathbb{1} = N$.

This expression treats the $\mathrm{U}(1)$ and the $\mathrm{SU}(N)$ parts on unequal footing. It is not necessary to do so. In fact, $\mathrm{SU}(N)$ can be extended to $\mathrm{U}(N)$ by adding a generator $T_0 = \frac{1}{\sqrt{2N}}\mathbb{1}$, and replacing $Y_m^{cm}\mathbb{1}$ by $Y_m^0 T_0$ in the splitting of Y_m. (4.11) is then replaced by

$$Z[Y^m] = \left(N - \frac{1}{4}\sum_a (v_a)^2\right)\langle 1\rangle\,,$$

where a sums over $\mathrm{U}(N)$ indices again.

The full tree-level effective action – upto lowest order, that is – is then given by (cf (3.32))

$$\Gamma_{\text{tree}}[v^m] = Z_0 g_c^{-1}\int\frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\,\left(N - \frac{1}{2}\,\mathrm{Tr}(\dot{Y}_m)^2\right), \tag{4.12}$$

where v_a^m has been replaced by its origin $\dot{Y}_a^m$, and the sum has been replaced by the $\mathrm{U}(N)$-trace that yielded it. Note that the above expression is valid only for slowly changing $\dot{Y}$.

2. Making the effective action gauge invariant

Upto order Y^2 (and $(\partial Y)^2$), (4.12) has the same structure as (4.4). (Note that the constant term $\int \mathrm{d}t\, N$ may be discarded as irrelevant.) Since (4.12) is derived from (4.7), which is a gauge invariant expression, it makes sense to add higher order terms to (4.12) to render it gauge invariant as well. $\mathrm{Tr}(\dot{Y}_m)^2$ is not gauge invariant on its own, but comparison with (4.4) shows that adding a $[Y,Y]^2$ term produces a gauge invariant combination. We therefore replace (4.12) by

$$\Gamma_{\text{tree}}[Y_m] = Z_0 g_c^{-1}\int\frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\,\left(N - \frac{1}{2}\,\mathrm{Tr}(\dot{Y}_m)^2 - \frac{1}{4}\,\frac{1}{(2\pi\alpha')^2}\,\mathrm{Tr}[Y_m,Y_n]^2\right). \tag{4.13}$$

This operation is valid, as long as we keep in mind the possibility that there may be other Y^4 terms in the expansion of (4.7) – which would probably require even higher order terms to restore gauge invariance.

Apart from the normalization and the irrelevant volume term $\int \mathrm{d}x^0\, N$, (4.13) matches (4.4).

3. A word on higher order terms

We shall not investigate the higher order terms that may appear as corrections to (4.13) in any detail. Let it suffice to note that a Y^n term is always expressed as some tensor times $\mathrm{Tr}\, T_{a_1}\cdots T_{a_n}$. Such a trace is proportional to $(\frac{1}{N})^{n/2-1}$. This means that for large N the higher order terms become small.

4. Proper normalization

The normalization and the sign of the D-particle action is of some importance for what follows. We shall therefore not accept (4.13) without further questioning. (4.13) is the first term in an exponential expansion: it contains contributions from connected diagrams only, while in a higher order approximation one would also have to include contributions from disconnected diagrams, which correspond to D-particles which meet several disconnected open strings along their world-lines. This reasoning shows that although a part of the prefactor in (4.13) properly belongs in the exponent, another part actually belongs in front of the entire expansion. The latter factors should not be included in the final expression for the action.

We do not know the full expression for the series expansion, so we must resort to general reasoning to find the proper normalization: Polchinski argues [19] that the physical definition of the string coupling constant is the mass ratio between the fundamental string and the D-string. One might also argue that the most elementary tree level term in the effective action expansion should have one factor of g_c^{-1} in front of it, and no other dimensionless factors. These arguments lead us to replace (4.13) with

$$S[Y_m] = g_c^{-1} \int \frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\ \mathrm{Tr}\left[(\dot{Y}_m)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}[Y_m, Y_n]^2\right], \tag{4.14}$$

where the volume term has been discarded, and the sign has been flipped to ensure that the Hamiltonian obtained from the action by canonical conjugation has positive kinetic energy. The normalization of (4.14) matches (4.4) while preserving the natural g_c and α' dependence.

4.3 Interpretation of the dimensionally reduced Yang-Mills action

In the previous section we have derived the action for a system of D-particles in a formal way. Here, we shall try to gain some insight in the physical reasons leading to (4.14).

Consider a system of a number of D-particles with a large spatial separation. To be specific, let's take the simplest example: two D-particles at a distance a from each other. In the low energy limit, and for large a, the particles do not feel one another's presence, and may be described separately by the action of §3.3. The combined action has a $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry [27]. For finite a, however, there may also be strings stretching from the one brane to the other. Such strings have $\mathrm{U}(1) \times \mathrm{U}(1)$ charges $(1, -1)$ or $(-1, 1)$, and according to (2.6), have ground state mass equal to a times the string tension. Note that as the branes approach one another, the strings stretched between them become massless, the difference between the stretched and non-stretched string vanishes, and the full $\mathrm{U}(2)$ symmetry is restored.

In a more general setting, we may have N D-particles at positions $\{y^m_{(1)}, \dots, y^m_{(N)}\}$. Such a configuration has a $\mathrm{U}(1)^N$ symmetry. Strings stretching from brane i to brane j have mass $M_{ij} = T|\vec{y}_{(i)} - \vec{y}_{(j)}|$. Introducing $\vec{\phi}_{ij}$ to represent these states, we could try to write down a potential term for them:

$$V \stackrel{?}{=} \sum_{i \neq j} \frac{1}{2} M_{ij}^2 |\phi_{ij}|^2. \tag{4.15}$$

(4.15) is slightly inaccurate, since excitations parallel to the stretched string only correspond to reparametrization of the world-sheet, and hence should not have a mass term associated with them. To account for this fact, we replace (4.15) by

$$V = \sum_{i \neq j} \frac{1}{2} M_{ij}^2 \vec{\phi}_{ij} \cdot \left[\mathbb{1} - \frac{(\vec{y}_{(i)} - \vec{y}_{(j)})(\vec{y}_{(i)} - \vec{y}_{(j)})}{(y_{(i)} - y_{(j)})^2}\right] \cdot \vec{\phi}_{ij}. \tag{4.16}$$

Actually, precisely such a potential term is found when we expand (4.14) around a classical solution, ie around some point in configuration space where the Y_m's commute. In such a point the matrices Y_m may be simultaneously diagonalized, and we may write them as

$$(Y^m)_{ij} = y^m_{(i)}\delta_{ij} + \sqrt{g_c}\,\varphi^m_{ij} \qquad \text{(no summation)}. \tag{4.17}$$

The $y^m_{(i)}$'s can then be interpreted as the positions of the branes. (Obviously, after including the correction terms, the Y^m's cannot be interpreted in this way, since they have become matrices.)

Expanding the potential term in (4.14) as suggested in (4.17) to order φ^2 yields (4.16), with $\vec{\phi}_{ij} = \vec{\varphi}^{\mathrm{S}}_{ij} + i\vec{\varphi}^{\mathrm{A}}_{ij}$; $\vec{\varphi}^{\mathrm{S}}$ and $\vec{\varphi}^{\mathrm{A}}$ being the symmetric and antisymmetric parts of the matrices $\vec{\varphi}$. (Note that the expansion in power of ϕ is equivalent to an expansion in powers of g_c: the factor $\sqrt{g_c}$ in (4.17) – which has been chosen to get the correct normalization for the mass of the stretched strings – shows how the open strings are corrections to the D-particle action which become small in the weak coupling limit.

There are also cubic and quartic terms in the expansion. Most of these should probably be interpreted as interaction terms. However, a subset of the cubic terms can be identified as follows: when a string is stretched between D-branes i and j, some change in energy is involved in moving these branes. One may argue that this comes down to a correction of the mass of the ϕ_{ij}-states proportional to the amount by which the length of the string is changed. In fact the expansion of (4.14) contains the required terms:

$$\sum_{i\neq j}\sqrt{g_c}\,T^2\big(\vec{y}_{(i)} - \vec{y}_{(j)}\big)\cdot\big(\vec{\varphi}_{ii} - \vec{\varphi}_{jj}\big)\,\vec{\varphi}_{ij}\cdot\vec{\varphi}_{ji}.$$

Obviously all of this does not prove conclusively that (4.14) is the proper action to describe a system of N D-particles. It might be possible to find a full physical proof of (4.14) by carefully investigating the stringy interactions, but we shall not pursue this track. The evidence that we have accumulated here should be enough to give us confidence in the relevance of the mathematical proof presented in §4.2.

4.4 D-particles on a compact space

As promissed in §4.2, we shall now consider a configuration of N D-particles on a partially compactified space-time. According to T-duality, a quantum description of N D-particles on a space-time with a compact direction[1)] with radius R should also describe the $(1+1)$-dimensional Yang-Mills theory living on a D-string, in a space-time with a compact direction with radius $R' = \frac{\alpha'}{R}$. That is, there should be a way to rewrite the D-particle action in such a way, that it acquires the form of the D-string action. The following calculation was first presented by W. Taylor [28]. Before we plunge into it, I'd like to draw attention to another article in which the large N limit of D-particles is considered: in [12] the authors argue for an equilance between the large N limit of D-particle quantum mechanics and M-theory.

Consider a system of N D-particles on $\mathrm{S}^1 \times \mathbb{R}^8 \times \mathbb{R}$, where S^1 has radius R, that is, $X^1 \equiv X^1 + 2\pi R$. The dynamics of this system may be captured by placing an infinite number of copies

[1)]There may or may not be additional compact dimensions; only one is needed for the argument.

of $[0,2\pi]\times\mathbb{R}^8$ adjacent to one another and imposing periodicity. Each of the N D-particles then acquires an infinite number of copies, and strings wound around the S^1 are represented by strings stretching between different copies of the particles. All of the dynamical information of the particles and the strings may be conveniently collected in a set of matrices $(Y^m_{IJ})_{ij}$, where i and j count from 1 to N, corresponding to the original labelling of particles, while I and J take values in $\mathbb{Z}$, and enumerate the copies of the space-time.

In the following we shall often need to treat the compact direction (X^1) differently than the other ones. We shall therefore introduce α and β to sum over the transverse directions: $\alpha,\beta=2\dots9$. (m and n run from 1 through 9 as before.) As always, summation over repeated labels is implied. The $U(N)$ labels (i and j) will mostly be suppressed, and Tr is written to represent the trace over these suppresed labels.

The requirement of periodicity[1)] is summed up by the following relations:

$$Y^m_{IJ}=Y^m_{(I-1)(J-1)}+2\pi R\delta_{m1}\delta_{IJ},$$

or

$$Y^m_{IJ}=Y^m_{0,(J-I)}+2\pi RI\delta_{m1}\delta_{IJ}. \tag{4.18}$$

The additional term is needed to ensure that Y^1_{II} actually take values in the I'th copy of the space-time.

Note that the Y^m's are Hermitian when considered as one big matrix. With the block labels separated however, we have

$$(Y^m_{IJ})^\dagger=Y^m_{JI}.$$

Having spent some time introducing notation, we may now attempt to tackle the action describing our system. We shall start of from the action (4.14), which extends to

$$S=\int \mathrm{d}t\,\left(\mathrm{Tr}\,\dot{Y}^m_{IJ}\dot{Y}^m_{JI}+\frac{1}{2}\frac{1}{(2\pi\alpha')^2}\,\mathrm{Tr}\,Y^{[m}_{IJ}Y^{n]}_{JK}Y^{[m}_{KL}Y^{n]}_{LI}\right), \tag{4.19}$$

after adding the S^1 block labels[2)]. The notation $A^{[m}B^{n]}$ is used as $A^{[m}B^{n]}\equiv A^mB^n-A^nB^m$.

(4.18) shows that there is a large amount of redundancy in the Y^m_{IJ}'s: all of the Y^m_{IJ} can be expressed in terms of $Y^m_{0J}\equiv Y^m_J$. Straightforwardly expanding (4.19) in terms of these Y^m_J's and recollecting terms yields

$$S=\int \mathrm{d}t\,\left(\mathrm{Tr}\,\dot{Y}^m_I\dot{Y}^m_{-I}-\frac{1}{(2\pi\alpha')^2}\,\mathrm{Tr}\,S^\alpha_IS^\alpha_{-I}-\frac{1}{2}\frac{1}{(2\pi\alpha')^2}\,\mathrm{Tr}\,T^{\alpha\beta}_IT^{\alpha\beta}_{-I}\right),^{3)} \tag{4.20}$$

with

$$S^\alpha_I=\sum_J Y^{[1}_JY^{\alpha]}_{I-J}-2\pi RIY^\alpha_I,$$

1) Periodicity, that is, both of the particles and the strings.

2) Actually, for proper normalization, (4.19) should be divided by the (infinite) number of copies that have been made of the space-time.

3) In the process, the infinite factor mentioned above surfaces clearly and is taken care of: the normalization of (4.20) is correct.

and

$$T_I^{\alpha\beta} = \sum_J Y_J^{[\alpha} Y_{I-J}^{\beta]}.$$

The action (4.20) should be compared to the action for $\mathrm{U}(N)$ gauge theory on a D-string, which reads

$$S = \int \mathrm{d}t \int \frac{\mathrm{d}x}{2\pi R'} \left(\mathrm{Tr}\dot{X}^\alpha \dot{X}^\alpha + \mathrm{Tr}\dot{A}^1 \dot{A}^1 - \mathrm{Tr}\left(\partial_1 X^\alpha - i\left[A^1, X^\alpha\right]\right)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}\mathrm{Tr}\left[X^\alpha, X^\beta\right]^2 \right). \tag{4.21}$$

It is easy to show that (4.20) and (4.21) are equivalent under the identifications $R = \frac{\alpha'}{R'}$ and

$$\sum_K \mathrm{e}^{\mathrm{i}Kx/R'} Y_K^1 = 2\pi\alpha' A^1,$$
$$\sum_K \mathrm{e}^{\mathrm{i}Kx/R'} Y_K^\alpha = X^\alpha.$$

We have thus shown by direct calculation that a system of D-particles on $\mathrm{S}^1 \times \mathbb{R}^8$ is equivalent to Yang-Mills theory on D-strings on the same manifold, but with inverse compactification radius. (Obviously, the non-compact Yang-Mills theory emerges when we take R to zero.) As a side-remark, note that the steps performed above may be repeated: the algebra becomes more intricate, but it is obvious that compactifying other directions will show a connection between D-particle dynamics and Yang-Mills theory on any D-k-brane, including the (space-filling!) 9-brane.

4.5 Geometric interpretation of the D-particle gauge theory

At first sight, one might think that the above calculation amounts to an independent proof of T-duality. This, however, is not the case. T-dualizing is actually the same as Fourier transformation: both relate a theory on a space with radius R to a theory on a dual space with radius $R' \sim R^{-1}$ and exchange momentum with winding number (or – after decompactification – with position). We shall make this connection explicit by showing that the position and winding number operator Y_J^m is the Fourier dual of the momentum operator $D_m(x)$ in the dual space:

$$Y_{IJ} = Y_{0,(J-I)} + J\delta_{IJ} \qquad \leftrightarrow \qquad D = A(x) + \partial. \tag{4.22}$$

The rest of this section will be spent on making this connection more precise, and showing how a subset of the gauge transformations in the Yang-Mills theory corresponds to changing the positions of the particles in the D-brane theory.

4.5.1 *Y* is the Fourier dual of *D*

With x^1 a compact direction with radius R', the Fourier transform of $A^m(x^1)$ is

$$\tilde{A}_K^m = \int_0^{2\pi R'} \frac{\mathrm{d}x^1}{2\pi R'}\, \mathrm{e}^{\mathrm{i}Kx/R'} A^m(x^1),$$

in which the dependence of A^{μ} on the time $t \equiv x^0$ has been suppressed.

The integer K has an obvious interpretation as a momentum. However, we may also identify $\tilde{A}^m_K$ with the Y^m_{0K}'s of the previous section. In this picture, K labels the copies of S^1. We will find that it is also connected to winding number.

To complete the correspondence sketched in (4.22), note that Fourier transform of Y^m_{IJ} is

$$\tilde{Y}^m(x,y) = \left[\tilde{Y}^m\left(\frac{x+y}{2}\right) + 2\pi R\delta^{m1}R'\mathrm{i}\partial_1\right]2\pi\delta\left(\frac{x-y}{2R'}\right).$$

Identifying

$$\int\frac{\mathrm{d}y}{2\pi R'}\,Y^m\left(\frac{x+y}{2}\right)2\pi\delta\left(\frac{x-y}{2R'}\right) = 2\tilde{Y}^m(x) \equiv 2\pi\alpha' A^m(x),$$

and noting that $2\pi RR' = 2\pi\alpha'$, we find that the Fourier transform of Y^m_{ij} can be identified with $2\pi\alpha'\mathrm{i}\left[\delta^{m1}\partial_1 - \mathrm{i}A^m(x^1)\right]$. Both terms in this expression depend on one spatial coordinate only, which signifies that it lives on a D-string: fields on a D-brane only depend on longitudinal coordinates.

With this identification, it is obvious that $\frac{1}{(2\pi\alpha')^2}\sum\mathrm{Tr}[Y,Y]^2$ equals $\int\mathrm{Tr}[D,D]^2$, that is, the last term in (4.14) is equivalent to the spatial part of $\mathrm{Tr}F^2$ in a Yang-Mills theory. So this is what T-duality is about: replacing a component of the covariant derivative by a coordinate transversal to the smaller brane. Clearly, it is possible to repeat this exercise and Fourier-transform any of the other eight directions, ultimately relating D-particle quantum mechanics to Yang-Mills theory in ten-dimensional space-time.

4.5.2 The interpretation of gauge transformations in the D-particle theory

To really understand the correspondence, it is useful to investigate what happens to gauge transformations in the Y picture. For simplicity, let us consider $\mathrm{SU}(2)$ gauge theory on the D-string. This theory contains the fields $A^3_m(x)$ and $A^{\pm}_m(x)$[1)].

A gauge transformation of the form $\Lambda = \mathrm{e}^{\mathrm{i}pxT_3}$ sends

$$\begin{aligned} A^3_1 &\to A^3_1 + p \\ A^3_\alpha &\to A^3_\alpha \\ A^{\pm}_m &\to \mathrm{e}^{\pm\mathrm{i}px}A^{\pm}_m, \end{aligned} \tag{4.23}$$

as can be seen from the commutation relations $[T_3, T_{\pm}] = \pm T_{\pm}$. (Note that $T_{\pm} = \frac{1}{2}[\sigma_1 \pm i\sigma_2]$.)

For the moment, we shall limit our attention to the case $pR' = k \in \mathbb{Z}$, which renders the A^3_m transformation equal to the identity, while making the $A^{\pm}_m$ transformations single-valued in space. Taking the Fourier transform, the above transformation is then found to correspond to

$$\begin{aligned} (Y_1)^3_0 &\to (Y_1)^3_0 + 2\pi\alpha' k/R' \\ (Y_m)^3_j &\to (Y_m)^3_j \qquad\qquad \text{for } j \neq 0 \text{ or } m \neq 1 \\ (Y_m)^{\pm}_j &\to (Y_m)^{\pm}_{j\pm k}. \end{aligned}$$

1) We keep ignoring the A_0 fields, since they can always be gauged away.

Since

$$T_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we find that $(Y_m)^3_J = y^m_{(1)J} - y^m_{(2)J}$, while $Y_\pm$ are equal to the φ_{12} and φ_{21} of §4.3 respectively.

We thus find that in the D-particle picture the zero modes of the position elements transform according to

$$\left[y^1_{(1)0} - y^1_{(2)0}\right] \to \left[y^1_{(1)0} - y^1_{(2)0}\right] + 2\pi k R,$$

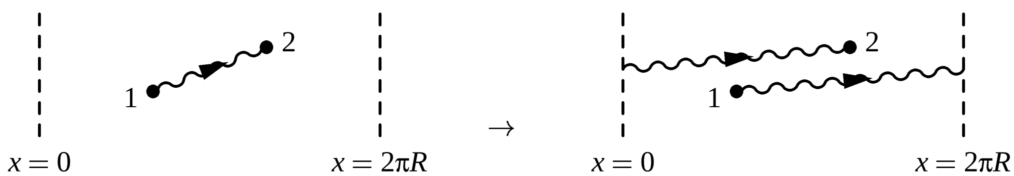
in other words: one of the D-particles is moved k times around the S^1. We would expect that such a move affects the strings stretching between these particles, and in fact it does, for

$$(\varphi^m_{12})_j \to (\varphi^m_{12})_{j+k} \tag{4.24}$$

and

$$(\varphi^m_{21})_j \to (\varphi^m_{21})_{j-k}, \tag{4.25}$$

that is, strings stretching between the two D-particles acquire extra winding when one of the particles is moved around the S^1. Graphically, this may be depicted as follows:



(The dotted lines symbolize periodic identification.)

Obviously, the gauge symmetry is spontaneously broken: the two configurations sketched above are clearly unequal, while the action (4.19) *is* invariant under the gauge transformation.

We have shown that the 'Abelian' gauge transformation $A^3 \to A^3 + k/R'$ has a clear geometric interpretation in the D-particle picture. It should be noted that the D-particle counterpart of a more general gauge transformation is less clear. For example, taking $\Lambda = e^{i\alpha T_\pm}$ mixes the A^3 and $A^\pm$ fields. On the D-particle side this corresponds to mixing the $y_{(i)}$'s with the φ_{ij}'s. It is difficult to visualize such mixing of D-particle positions and string excitations, and in fact, when we consider the diagonal and off-diagonal components of the matrices Y on equal footing as required by these gauge transformations, the expansion (4.17) breaks down, and we should be very careful in trying to describe the theory from a classical point of view.

For non-integer pR', the $(Y^\pm_m)_J$'s mix non-trivially. This is caused by the fact that moving a D-particle changes the mass ratio of strings stretching between that particle and its various neighbours[1)]. The expectation values of these strings are altered correspondingly.

Finally note that although we have only investigated the case of $SU(2)$, the considerations above are equally applicable to situations with more D-particles. In such cases there are more gauge transformations that act only on the diagonal components of A. All of these can be interpreted as moving some of the D-particles around. In all cases the strings stretching between these particles have their winding numbers changed as in (4.24) and (4.25).

[1)] In particular, moving particle no. 2 to the right increases the mass of the 1-2-string, and decreases the mass of the 2-1′-string.

4.6 N D-particles in superstring theory

The calculation of the previous sections were performed within the context of bosonic string theory, but – as shown by the example in appendix 3.B – it can also be regarded as the bosonic part of the calculation in superstring theory. We shall not perform the fermionic part of the calculation here, but simply state that supersymmetrization of (4.14) yields [29]

$$\Gamma = \int \frac{dx^0}{\sqrt{2\pi\alpha'}} \operatorname{Tr}\left(\frac{1}{g_c}\left\{(\dot{Y}^m)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}[Y^m,Y^n]^2\right\} + i\psi\dot{\psi} + \frac{1}{2\pi\alpha'}\psi\gamma_m[Y_m,\psi]\right), \tag{4.26}$$

in which $\psi = \psi^a T_a$ are 16-component spinors in the adjoint of SU(N), and γ_i are symmetric (16×16)-matrices acting on these spinors, satisfying

$$\{\gamma_i,\gamma_j\} = 2\delta_{ij}.$$

Explicit expressions for these matrices may be found in [7], appendix 5.B. Note that many authors prefer to work with 32-component spinors Ψ, related to our ψ's by

$$\Psi^a = \sqrt{g_c}\,\psi^a \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Also note that we use the conventions of §4.1 for covariant derivatives, in particular

$$D_\mu\psi = \partial_\mu\psi - i[A_\mu,\psi],$$

and that transposition of the left-hand side spinors is left implicit.

The above action has been written down explicitly in temporal gauge ($A^0 = 0$), but it is not difficult to obtain the complete expression from the Lagrangian for supersymmetric Yang-Mills theory:

$$\mathcal{L} = -\operatorname{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} - i\bar{\Psi}\Gamma^\mu D_\mu\Psi\right),$$

where $\bar{\Psi} = \Psi\Gamma^0$, and $\Gamma^0 = \mathbb{1}_{16}\otimes\sigma^2$ and $\Gamma^i = \gamma_i\otimes i\sigma^1$.

Gauss's law can then be found to read

$$G_a \equiv f_{abc}\left[\frac{1}{(2\pi\alpha')^2}Y_b^m Y_c^m - \frac{i}{2}\psi_b\psi_c\}\right] = 0,$$

with f_{abc} the structure constants of SU(N): $[T_a,T_b] = i f_{abc}T_c$. This is the equation of motion for A^0 in temporal gauge. Physical states – which have to be gauge invariant – must satisfy Gauss's law: they must be annihilated by the G_a's.

Upto gauge transformation, (4.26) is invariant under the supersymmetry:

$$\begin{aligned}\delta A^m &= -i\sqrt{g_c}\,\varepsilon\gamma^m\psi,\\ \delta\psi &= \sqrt{1/g_c}\left[\frac{1}{2\pi\alpha'}\dot{Y}^m\gamma_m - \frac{i}{2}\frac{1}{(2\pi\alpha')^2}[Y^m,Y^n]\gamma_{mn}\right]\varepsilon,\end{aligned}$$

with ε a constant 16-component spinor.

Using these formulas, it is not difficult to find the supersymmetric extension of the calculation presented in the previous sections.

CHAPTER 5

Scattering and bound states

Nobody would believe in the world
if they hadn't spent years getting used to it.

— J. Gaarder, 'The solitaire mystery'

In this chapter we shall investigate some aspects of the spectrum of the system presented in the previous chapter. We shall consider the dynamics both in the bosonic case and in the supersymmetric case, and find that these are very different: a system of bosonic D-particles does not allow for scattering states, since quantum mechanically two D-particles are held together by an attractive potential that rises linearly as the particles are moved away from one another. On the other hand, supersymmetric D-particles do not feel such an attractive force in the ground state, and may escape to infinity unhindered. This raises the question of the existence of bound states of such particles. The final section of this chapter describes a scattering experiment of supersymmetric D-particles, which suggests that there may be bound states of these particles after all.

5.1 Bosonic D-particles cannot escape

Let us return to the action for N bosonic D-particles as derived in §4.2:

$$S = \frac{1}{g_c} \int \frac{\mathrm{d}t}{\sqrt{2\pi\alpha'}} \, \mathrm{Tr} \left((\dot{Y}_m)^2 + \frac{1}{2} \frac{1}{(2\pi\alpha')^2} [Y_m, Y_n]^2 \right) . \tag{5.1}$$

As noted previously, the potential is zero not only in the origin $Y_m = 0$, but more generally when the Y_m's commute, eg when they are diagonal. Therefore, in the classical theories there are many paths along which the potential is zero, so there are many classical solutions in which particles approach one another from infinity and escape to infinity after interacting. Although it is always difficult to sketch high dimensional pictures, an attempt to depict the potential is made in figure 5.1.

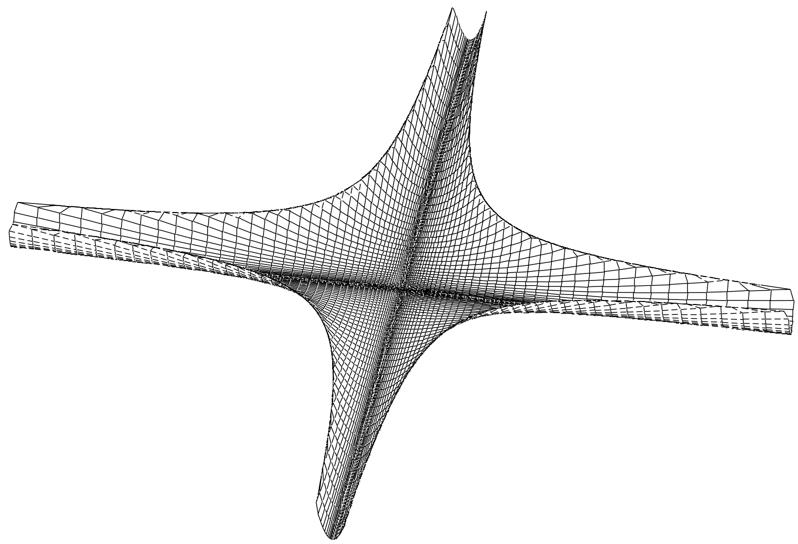


Figure 5.1: An impression of the potential term in (5.1): the plot shows $V = x^2y^2$. To make the picture more visually pleasing, the potential has been cut off at $V = 1$.

5.1.1 Analysis of the classical action

Even classically, the dynamics of (5.1) is highly non-trivial, and we shall give some indication of why this is so. For simplicity, consider the case $N = 2$, so Y can be written as

$$\vec{Y} = \begin{pmatrix} \vec{\xi} & \vec{\alpha} + i\vec{\beta} \\ \vec{\alpha} - i\vec{\beta} & -\vec{\xi} \end{pmatrix}.$$

Expressed in terms of $\vec{\xi}$, $\vec{\alpha}$ and $\vec{\beta}$, the action for this system is found to be

$$S = \frac{1}{g_c}\int \frac{dt}{\sqrt{2\pi\alpha'}} \left[2\left(\dot{\xi}^2 + \dot{\alpha}^2 + \dot{\beta}^2\right) - \frac{8}{(2\pi\alpha')^2}\left(\alpha_\perp^2 + \beta_\perp^2\right)\xi^2 - V^{(4)}(\alpha, \beta)\right]$$

with

$$V^{(4)} = \frac{2}{(2\pi\alpha')^2}\left[\Big((\vec{\alpha} + i\vec{\beta}) \cdot (\vec{\alpha} - i\vec{\beta})\Big)^2 - (\vec{\alpha} + i\vec{\beta})^2(\vec{\alpha} - i\vec{\beta})^2\right],$$

and $\vec{\alpha}_\perp$ is the part of $\vec{\alpha}$ perpendicular to $\vec{\xi}$, ie $\vec{\alpha}_\perp = \vec{\alpha} - (\hat{\vec{\xi}} \cdot \vec{\alpha})\,\hat{\vec{\xi}}$, and similarly for $\vec{\beta}_\perp$.

Upto order α^2 and β^2, the longitudinal components decouple, as is to be expected from the fact that these correspond to reparametrization of the strings stretched between the D-particles. We shall ignore them altogether. The remaining part of the action (upto order α^2 and β^2) can be written as

$$S = \int \frac{dt}{\sqrt{2\pi\alpha'}} \left[\frac{1}{2}g_c^{-1}\dot{y}^2 + \frac{1}{2}\dot{w}^2 - \frac{1}{2}\frac{1}{(2\pi\alpha')^2}y^2w^2\right], \tag{5.2}$$

where we have introduced $\vec{y} = 2\vec{\xi}$: the distance between the D-particles and $\vec{w} = 2g_c^{-1/2}(\vec{\alpha}_\perp, \vec{\beta}_\perp)$: a 'vector'[1)] containing the degrees of freedom of the string stretched between them.

There are obviously many solutions to the equations of motion derived from (5.2) with the particles escaping to infinity as time goes to infinity: for example $\vec{y}(t) = \vec{y}_0 + \vec{v}_0 t$ with $\vec{w}(t) = 0$ satisfies the equations of motion for any $\vec{y}_0$ and $\vec{v}_0$. However, there are far more solutions with $\vec{w}$ not always zero. If at $t = 0$ the particles are close together and $\vec{w}$ is small but non-zero, no matter how big their relative velocity, the particles will not be able to escape to infinity. An example of such a trajectory in (y_1, w_1)-space is shown in figure 5.2. For the present purpose

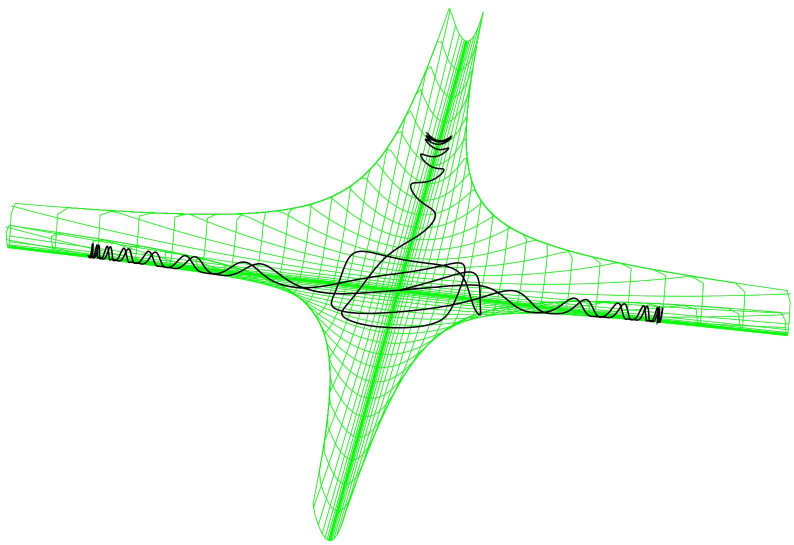


Figure 5.2: An example trajectory in the potential of figure 5.1, with initial values $x = 0$, $y = 0$, $\dot{x} = 1.1$, $\dot{y} = 0.7$.

though, the most important point is that there exist solutions with particles escaping to infinity. Before moving on to a quantum mechanical description, let us note that the above analysis straightforwardly generalizes to the $\mathrm{SU}(N)$ case: the matrices $\vec{Y}$ will contain more diagonal elements corresponding to relative positions and more off-diagonal elements corresponding to stretched strings, but the basic picture remains the same: there are solutions with particles coming in from, and escaping to, infinity. Again, most of these have the off-diagonal elements zero for all time, and constant 'velocities' for the diagonal elements.

5.1.2 Quantum mechanical analysis

Quantum mechanically the picture changes dramatically: although it remains true that there are axes in the Y^m_{ab} hyperplane where the potential is zero, this no longer means that particles can actually escape to infinity. This can be understood as follows: when the particles are not too

---

1) in the limited sense of an ordered set of numbers; not in the sense of Lorentz covariance.

near one another, the potential roughly takes the form of

$$V = \frac{1}{2}\sum |r|^2 \phi^2,$$

where $\vec{r}$ represents the spatial separation between two D-particles, and ϕ represents the transverse directions in the Y^m_{ab} hyperplane, ie the strings connecting the particles. The ϕ-dependent part of the Hamiltonian is then roughly

$$H_{(\phi)} = \frac{1}{2}\sum (p_\phi)^2 + \frac{1}{2}\sum |r|^2 \phi^2.$$

Treating $\vec{r}$ as a c-number for the moment[1)], we see that this Hamiltonian has the form of a set of harmonic oscillators, each with frequency $\omega = |\vec{r}|$. The ground state of such a system has energy $E = \mathcal{N} \times \frac{1}{2}\omega = \frac{1}{2}\mathcal{N}|r|$, with $\mathcal{N}$ the number of degrees of freedom in ϕ ($2D-2$ in the case of two particles.) The one loop effective potential for the diagonal degrees of freedom $\vec{r}$ thus is $V_{\text{eff}} = \frac{1}{2}\mathcal{N}|r|$. The conclusion must be that quantum corrections forbid the particles to be far away from one another: there will only be bound states.

5.2 Super D-branes *do* escape

In the supersymmetric case, the picture changes again: a fermionic term is added to the Hamiltonian which might cancel the bosonic term, so there may be solutions of the Schrödinger equation that describe scattering particles. On a very intuitive level, this conjecture is substantiated by the fact that the supersymmetrized version of the harmonic oscillator has zero ground state energy, so the argument which showed that there are no scattering states in the bosonic case, does not hold in the supersymmetric case.

We can be much more precise. De Wit etal [30] have shown that the spectrum of the Hamiltonian for (4.26) is continuous all the way down to $E = 0$. This proves that scattering is possible[2)], but it also raises a new problem: are there any bound states at all? In other words, does the Hamiltonian have a discrete spectrum with $E = 0$ eigenstates in addition to the continous spectrum? This question is not just an academic one: duality between type IIA string theory and 11D supergravity requires that there be particles in the IIA theory that correspond to the gravitons in the 11D theory. Since the duality maps momentum in the eleventh direction to charge on the IIA side, such particles should exist with any (integer[3)]) charge. The only possible candidate seems to be the bound state of a number of D-particles (since fundamental strings do not carry the required charge at all), so it is very important to prove that such a state can be stable.

Only recently, this proof has been found [31], but the details are rather technical.

In the following section we shall present a discussion about a special case of D-particle scattering as discussed in [32], and then comment briefly on the existence of bound states. D-particle scattering is also discussed in [33], in which Bachas investigates D-brane scattering at distances slightly larger than the ones we consider here, though still shorter than the string scale.

1) This is a reasonable thing to do, because for large r and small ϕ, the potential is flat in the r direction, and small quantum corrections in this direction would not change the picture drastically.

2) and, as Feynman said, when something is not actually forbidden in quantum mechanics, it will happen.

3) The duality is described by compactifying the 11th direction. In a compact direction, the momentum is quantized, and therefore so is the required charge in the IIA theory.

5.3 Super D-particle scattering: a semi-classical approximation

In this section we shall consider scattering of two D-particles off one another. We shall do so in a semi-classical approximation only, since a full quantum-mechanical description would be rather technical. Such a description may however be found in [32]. We shall find that the passage of two free D-particles at ultra-close distance is likely to create a number of open strings stretching between the particles. These will pull the particles back together after some time, at which point a different set of stretching strings may be produced. We shall find that it may take a relatively long time before a collision occurs that does not produce any strings, thus allowing the particles to escape. This resonant behaviour is evidence for the existence of bound states.

5.3.1 General relations

We begin with the action found in §4.6, which we repeat for convenience:

$$S = \int \frac{\mathrm{d}x^0}{\sqrt{2\pi\alpha'}}\, \mathrm{Tr}\left(\frac{1}{g_c}\left\{(\dot{Y}^m)^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}[Y^m, Y^n]^2\right\} + \mathrm{i}\psi\dot{\psi} + \psi\gamma^m[Y^m,\psi]\right). \tag{5.3}$$

It should be noted that this formula has been written down in temporal gauge, $A^0_a = 0$. This is a good starting point for quantization, but one should not forget that physical states must be gauge invariant. This constraint is implemented by demanding that they are annihilated by Gauss's law:

$$G_a = \frac{1}{g_c}\varepsilon_{abc}Y^m_b\dot{Y}^m_c - \frac{\mathrm{i}}{2}\varepsilon_{abc}\psi_b\psi_c.$$

($G_a = 0$ is the Euler-Lagrange equation for A^0_a.) Requiring that the G_a's annihilate physical state constrains the combinations of strings that can be formed between the particles, as we shall see below.

To avoid notational cluttering, we shall set $2\pi\alpha' = 1$ in the following. Noting that Y and ψ have dimensions length and (length)$^{1/2}$ respectively, while α' has dimension (length)^2, it is not difficult to re-insert the left-out factors. Furthermore, we shall in the following put space labels m and spinor labels α upstairs, while putting gauge group labels a downstairs. In this way, no confusion should arise when these labels are replaced by explicit numbers.

As the canonical momentum conjugates to Y and ψ are

$$P^m_a \equiv \frac{\delta\mathcal{L}}{\delta\dot{Y}^m_a} = \frac{1}{g_c}\dot{Y}^m_a$$

and

$$\Pi^\alpha_a \equiv \frac{\delta\mathcal{L}}{\delta\dot{\psi}^\alpha_a} = \mathrm{i}\psi^\alpha_a,$$

we find that the Hamiltonian corresponding to (5.3) is

$$H = \frac{g_c}{2}(P^m_a)^2 - \frac{1}{2}Y^m_a K^m_a + \frac{1}{4g_c}(\varepsilon_{abc}Y^m_b Y^n_c)^2, \tag{5.4}$$

where we have introduced

$$K^m_a = \mathrm{i}\varepsilon_{abc}\psi_b\gamma^m\psi_c.$$

(We used the fact that the structure constants for $\mathrm{SU}(2)$ are $f_{abc} = \varepsilon_{abc}$, the fully anti-symmetric tensor density with $\varepsilon_{123} = +1$.)

Noting that the canonical commutation relations read

$$[Y_a^m, P_b^n] = \mathrm{i}\delta^{mn}\delta_{ab} \qquad \text{and} \qquad \{\psi_a^\alpha, \psi_b^\beta\} = \delta^{\alpha\beta}\delta_{ab},$$

it is not difficult to show that

$$Q^\alpha = \sqrt{g_c}\,\gamma^m \psi_a^\alpha E_a^m - \frac{1}{2}\sqrt{1/g_c}\,\varepsilon_{abc}[\gamma^{mn}\psi_a]^\alpha A_b^m A_c^n$$

are the (sixteen) supersymmetry generators that yield H, although it should be noted that they do so only upto a term proportional to G, the Gauss's constraint. That is, $H = \frac{1}{2}\{Q^\alpha, Q^\alpha\}$ only for states that satisfy Gauss's law.

The system described by (5.4) is quite tractable outside the region where Y is small: see [29] for an in-depth discussion. However, since we are planning to do a scattering experiment, we cannot avoid the problematic region. Treating this region fully quantum mechanically is a difficult task, so we will use an approximation. Even this approximation is not quite trivial in ten dimensions, so we shall first consider a toy model: (5.4) restricted to 2+1 dimensions, the supersymmetric analogue of the system studied in §5.1.1.

5.3.2 Scattering in 1+2 dimensions

We shall investigate the scattering of two particles described by (5.4) limited to 2+1 dimensions in some detail. To begin with, let us note that in this restriction the Y_a^m's become two component vectors, $m = 1, 2$, while the ψ_a^α's become two component spinors, $\alpha = 1, 2$. The (2×2) Dirac matrices acting on these spinors may be taken to be $\gamma_1 \equiv \sigma^3$ and $\gamma_2 \equiv \sigma^1$. These obviously satisfy $\{\gamma^m, \gamma^n\} = 2\delta^{mn}$.

The setup for the scattering experiment is as follows: we assume that for large negative time the particles are far away from each other, and are approaching each other with velocity v and impact parameter b. Explicitly, we prepare the system in such a way that

$$\begin{aligned} \langle Y_3^1 \rangle &= vt \\ \langle Y_3^2 \rangle &= b, \end{aligned} \tag{5.5}$$

where Y_3^m is identified with the spatial separation between the D-particles. According to [30] such states exist for general numbers of particles[1)].

We shall investigate the evolution of this state in a Born-Oppenheimer approximation, that is, we shall try to split the Hamiltonian in a term $H_{\mathrm{fast};(slow)}$ and a term H_{slow}. The first term – which describes the evolution of a set of fast modes – depends only parametrically on the slow modes. By this we mean that the slow modes may be taken as a classical background for the fast modes. The dynamics of the slow modes is ruled by the effective potential resulting from integrating the fast modes out. In the following, we shall expand the Hamiltonian in terms of the different bosonic and fermionic fields, before showing which of these are slow and which are fast.

1) We shall be more explicit when we have uncovered the Hamiltonian.

1. Analysis of the Hamiltonian

The bosonic part of (5.4) may be written explicitly as as

$$H_B = \frac{g_c}{2}\left[P_3^2 + (P_1^2 + P_2^2)\right] + \frac{1}{2g_c}\left[Y_3^2Y_1^2 - (Y_3\cdot Y_1)^2 + Y_3^2Y_2^2 - (Y_3\cdot Y_2)^2 + Y_1^2Y_2^2 - (Y_1\cdot Y_2)^2\right].$$

We shall ignore the terms that are quartic in Y_1 or Y_2, which we assume to be small compared to terms that contain only two factors of Y_1 or Y_2. Obviously, for small $|\vec{Y}_3|$ this assumption loses its validity, but – as we shall see in a moment – with the system prepared as in (5.5), the momentum terms completely dominate over the potential terms when $|\vec{Y}_3|$ is small.

The remaining potential terms can be rewritten as

$$V = \frac{1}{2}x^2w^2,$$

with $\vec{x} \equiv \vec{Y}_3$, and w a two component object defined by $w_{1,2} = g_c^{-1/2}\vec{Y}_{1,2}\cdot\hat{\imath}_\perp$, where $\hat{\imath}_\perp$ is a unit vector perpendicular to $\vec{x}$. (For definiteness, let's say that $\hat{\imath}_{||} = \hat{x}$ and $\hat{\imath}_\perp$ form a right-handed coordinate system.)

The full bosonic part of the Hamiltonian may now be written as

$$H_B = \frac{g_c}{2}p_x^2 + \frac{1}{2}p_w^2 + \frac{1}{2}p_u^2 + \frac{1}{2}x^2w^2,$$

where we have introduced the two component object u defined by $u_{1,2} = g_c^{-1/2}\vec{Y}_{1,2}\cdot\hat{\imath}_{||}$. At this point, we can see that the bosonic part of the ground state wave function is

$$\phi_t^B(\vec{x},\vec{w}) = \chi(x_1 - vt, x_2 - b)\phi_0(\vec{w};|\vec{x}|),$$

where $\chi(\vec{x})$ is any function that is smooth and non-zero in a small region around $\vec{x} = 0$ only, while ϕ_0 is the ground state function of a two dimensional harmonic oscillator,

$$\phi_0(\vec{w};|\vec{x}|) = |\vec{x}|^{1/4}\mathrm{e}^{-\frac{1}{2}|\vec{x}|(w_1^2+w_2^2)}.$$

When $|\vec{x}|$ is large and $|\vec{w}|$ and $|\vec{u}|$ are small, the fermionic term reduces to

$$H_F = -\frac{1}{2}|\vec{x}|K_3^{||},$$

with

$$K_3^{||} = i\varepsilon_{3ab}\psi_a\gamma^{||}\psi_b.$$

For fixed $\vec{x}$, $K_3^{||}$ depends on two out of the three fermions only. To make this clear, change conventions such that $\tilde{\gamma}_{||} = \sigma^3$ and $\tilde{\gamma}_\perp = \sigma^1$. We may then write

$$K_3^{||} \equiv \mathrm{i}\left[(\tilde{\psi}_1^1\tilde{\psi}_2^1 - \tilde{\psi}_1^2\tilde{\psi}_2^2) - (\tilde{\psi}_2^1\tilde{\psi}_1^1 - \tilde{\psi}_2^2\tilde{\psi}_1^2)\right].$$

Introducing fermion creation and annihilation operators

$$a_1 = \sqrt{1/2}\,(\tilde{\psi}_1^1 - \mathrm{i}\tilde{\psi}_2^1), \qquad a_2 = \sqrt{1/2}\,(\tilde{\psi}_1^2 + \mathrm{i}\tilde{\psi}_2^2),$$
$$a_1^\dagger = \sqrt{1/2}\,(\tilde{\psi}_1^1 + \mathrm{i}\tilde{\psi}_2^1), \qquad a_2^\dagger = \sqrt{1/2}\,(\tilde{\psi}_1^2 - \mathrm{i}\tilde{\psi}_2^2),$$

the fermionic term in the Hamiltonian is found to be

$$H_F = |\vec{x}| \left(\sum_{i=1,2} a_i^\dagger a_i - 1 \right) = |\vec{x}| \, (N_F - 1) \, .$$

The fermionic ground state wave function thus is the (four component) spinor ξ_t^F that is annihilated by both a_1 and a_2[1]. Summing up, the full (four component) wave function for the ground state at large $|\vec{x}|$ is $\psi_t(\vec{x},\vec{w}) = \psi_t^B(\vec{x},\vec{w})\xi_t^F$.

To see that this state actually has energy zero, note the bosonic ground state energy is $2 \times \frac{1}{2}\omega = |\vec{x}|$. This is exactly cancelled by the fermionic ground state energy, which is $-|\vec{x}|$. No potential term is associated with u, so we find that it is indeed possible for $|\vec{x}|$ to become large while keeping the energy zero.

2. Gauss's law

At this point it is appropriate to recall the Gauss constraint that we mentioned above: without it, there would be four possible fermion states: both oscillators could independently be in either of their two states. With it, only two of these four states survive: the fermionic part of G_3 reads

$$G_3^F = -\frac{\mathrm{i}}{2} \, [\psi_1^\alpha \psi_2^\alpha - \psi_2^\alpha \psi_1^\alpha] = -a_1^\dagger a_1 + a_2^\dagger a_2 .$$

Obviously, the bosonic and fermionic parts of the Gauss constraint do not need to annihilate physical states independently. However, if G^B annihilates a particular state, then so must G^F. In particular, if G^B is zero, then the fermions must either both be in their ground states, or both in their excited states. Below we shall find that this implies that only pairs of strings can be created between the scattering particles: states with an odd number of stretched strings do not obey Gauss's law[2].

3. The approach of the particles

Having investigated the Hamiltonian in some detail, we are now in a position to investigate the actual scattering. Starting with $\langle x_1 \rangle = vt$ and $\langle x_2 \rangle = b$ for large negative t, we see that a Born-Oppenheimer approximation can be used: w varies on a much shorter timescale than x does: $T_w \sim \omega^{-1} \sim 1/|\vec{x}|$, while $T_x \sim |\vec{x}|/|\dot{\vec{x}}| = |\vec{x}|/v$. Since there is no potential term associated with u, it decouples from the other operators, at least to first order, and may be ignored in the following analysis. Summing up, w and the fermions in $K^{\|}$ are fast modes, while x, u and the other fermions are slow. This justifies why we have quantized w and its fermionic superpartners, while considering x as a c-number and ignoring u altogether.

The energy of the excited states is proportional to $|\vec{x}|$. As the particles approach each other from infinity, w and the fermions must initially be in their ground states. The bosonic wave-function then takes the shape of an incoming Gaussian wave-packet (in (x,w)-space), which has a width of $\Delta w \equiv \sqrt{\langle w^2 \rangle} = |\vec{x}|^{-1/2}$ in each of the $\vec{w}$-directions. We shall assume that the

[1] Since a_1 and a_2 depend parametrically on x (through $\gamma^{\|}$), the ground state is not fully time independent, but it changes slowly.

[2] In the 2+1 dimensional case, either no, or two strings can be created; in the ten dimensional case which we shall consider below, upto eight pairs may be produced.

Born-Oppenheimer approximation remains valid right upto the point where T_x and T_w become equal. This happens when $|\vec{x}| = \sqrt{v}$, that is, when $x_1 = \sqrt{v-b^2}$. In accordance with [32], the area $|\vec{x}| \leq \sqrt{v} \equiv R$ shall be called the *stadium*.

4. The passage through the stadium
On entering the stadium, the width of the wave-packet is $\Delta w = v^{-1/4}$, so the potential is roughly

$$V \simeq \frac{1}{2} x^2 (\Delta w)^2 = \sqrt{v}.$$

We shall assume that v is large enough[1)] to allow us to neglect V against the kinetic energy $T \simeq \frac{1}{2g_c} v^2$. The wave-packet will then propagate freely as long as $|\vec{x}| < R$, which will remain the case for a timespan $\Delta t = \frac{2\sqrt{v-b^2}}{v}$. During that time it will spread in the $\vec{w}$-directions by diffusion.

Expanding the bosonic wave-function at the time when the stadium is entered,

$$u_0(w) \sim \mathrm{e}^{-\frac{1}{2}|\vec{x}|w^2}$$

in terms of the eigenfunctions $\psi_k = \mathrm{e}^{\mathrm{i}kw}$ of the free Hamiltonian $H_0 = -\frac{1}{2}\nabla_w^2$, we find that after Δt, the w-dependent part of the wave-function has evolved into

$$u_1(w) \sim \int \mathrm{d}k \int \mathrm{d}w'\ \mathrm{e}^{-\mathrm{i}kw'} u_0(w') \mathrm{e}^{-\frac{1}{2}k^2 \Delta t} \mathrm{e}^{\mathrm{i}kw} \sim \mathrm{e}^{-\frac{1}{2}\frac{1}{\Delta t + 1/|\vec{x}|} w^2},$$

which has a width

$$\Delta w_1 = \sqrt{\langle w^2 \rangle_1} = \sqrt{\Delta t + 1/|\vec{x}|} \leq \sqrt{3}\Delta w_0,$$

since $\Delta t \leq 2\sqrt{1/v}$. This means that the assumption that the potential can be neglected does not cause internal inconstency. (If ignoring the potential would have caused the wave-function to spread significantly, ignoring V would be proven inconsistent, since for large Δw the potential eventually becomes larger than the kinetic energy.)

Since the bosonic part of the Hamiltonian is symmetric under reflection of x_1, the ground state for w will be the same on both sides of the stadium, so the outgoing bosonic wave-function will almost completely map down to the new ground state. For the fermionic part, things are very different: sending $x_1 \to -x_1$ mixes the fermionic ground state with the fermionic excited states. During the time spent in the stadium, the fermionic part of the wave function cannot change significantly, since the distance between the energy levels sets a timescale $T_{\text{fermion}} = \Delta E^{-1} = \sqrt{1/v}$, which is larger than the time spent in the stadium, when b is not too small. When the particles leave the stadium, the wave-function will be projected out into the eigenstates of H_{fast} again. We shall compute the expectation value for the fermionic oscillation number after leaving the stadium in terms of v, b. This expectation value is given by

$$\left\langle N_F^{(\text{after})} \right\rangle \approx \left\langle GS_{(\text{before})} \middle| N_F^{(\text{after})} \middle| GS_{(\text{before})} \right\rangle, \tag{5.6}$$

since upon leaving the stadium, the fermion will still be mostly in its original state, which is the ground state for $\vec{x} = (-x_0, b)$.

[1)]Compared to $(2g_c)^{2/3}$, so for weak coupling this is not a strong constraint.

Perhaps the easiest way to compute (5.6) is the following: just before entering the stadium, the fermionic Hamiltonian is given by

$$H^{(\text{before})}_{F,\text{fast}} = -\frac{1}{2}|\vec{x}|K^{\|}_3, \tag{5.7}$$

while immediately afterwards it is given by

$$H^{(\text{after})}_{F,\text{fast}} = -\frac{1}{2}\vec{x'}\cdot\vec{K}_3. \tag{5.8}$$

In order to compute (5.6) we must express this in terms of the original fermionic creation and annihilation operators. The first step is to note that $\vec{x'}$ is $\vec{x}$ rotated over an angle $\phi = \pi - \operatorname{atan}\left(\frac{b}{\sqrt{v}}\right)$, (see figure 5.3), and in particular that $|\vec{x}'| = |\vec{x}|$. We may thus write

$$\begin{aligned} H^{(\text{after})}_{F,\text{fast}} &= -\frac{1}{2}|\vec{x}|\varepsilon_{3bc}\psi_b\left[\cos\phi\gamma^{\|} + \sin\phi\gamma^{\perp}\right]\psi_c \\ &= |\vec{x}|\cos\phi\left[\sum_i a^\dagger_i a_i - 1\right] - |\vec{x}|\sin\phi\left[a_1a_2 + a^\dagger_2 a^\dagger_1\right]. \end{aligned}$$

Since $a_i\left|GS_{(\text{before})}\right\rangle = 0$ and $\left\langle GS_{(\text{before})}\right| a^\dagger_i = 0$, we find that

$$\left\langle N^{(\text{after})}_F\right\rangle = -\cos\phi + 1 \approx 2 - 2\frac{b^2}{v}.$$

Upon leaving the stadium, we should reformulate Gauss's law in terms of the fermionic operators $\tilde{a}_i$ and $\tilde{a}^\dagger_i$ which connect the new ground state and excited states, and which can be used to write $H^{(\text{after})}_{F,\text{fast}}$ as $|\vec{x}|\left(\sum_i \tilde{a}^\dagger_i\tilde{a}_i - 1\right)$. This rewriting does not change the functional form of G^F_3, and – noting that the bosons are still in their ground state which is annihilated by G^B_3 – we find that the outgoing state either should have both of the fermions in the ground states, or both in the excited states.

As b is made smaller, N_F increases, and for hard scattering[1)] we find that the outgoing state is almost entirely projected onto the new excited state. In plain English: as two free D-particles fly past one another at very short distance, two strings will nearly always be produced between them. These strings will be in one of their fermionic ground states.

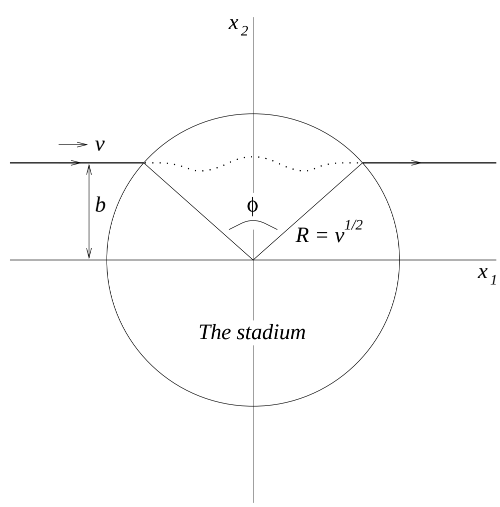
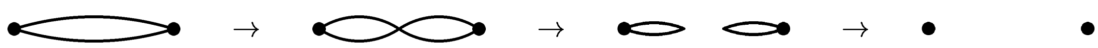


Figure 5.3: The passage through the stadium. The wavy line signifies the fact that a classical description of the motion is inappropriate inside the stadium.

[1)] It is doubtful, however, whether this regime may be successfully probed, since for small $b/v^{1/2}$ the characteristic time for the fermion evolution is not long compared to the time spent in the stadium.

5. After the collision

As the particles move apart again, the string stretched between them exerts a centripedal force on them, due to the effective potential $V = 2|x|$[1]. Therefore, the string must either decay quickly, or the particles will come together once more. The following gives some evidence indicating that the string does not decay quickly: consider a situation with two D-particles with two strings stretching between them. These strings may recombine as sketched in the sequence below, and then disappear altogether.



The probability of the strings meeting one another at some point along their length is proportional to that length. Furthermore, the total probability of such an event occurring is proportional to the available time. Finally, the recombination event can also be viewed as a closed string splitting, which assigns a factor g_c^2 to the probability (a factor g_c to the amplitude). All in all, the probability that the strings annihilate before the D-branes come together again is

$$P \sim g_c^2 \times (\text{typical length}) \times (\text{time between collisions}).$$

Very roughly, the 'typical length' is equal to the maximum separation, r_{max}, while the time is given by $\tau \sim r_{\text{max}}/v$. Conservation of energy yields $r_{\text{max}} \simeq \frac{1}{4g_c} v^2$, so $\tau \sim \frac{1}{4g_c} v$ and $P \sim v^3$. This indicates that for small velocities the strings will most probably survive until the branes collide again. Obviously, it is a very rough estimate, but actually computing the pre-factor is highly non-trivial.

For small g_c the maximum separation can be quite large, but eventually the particles are pulled together again. When they pass each other for the second time, the fermionic states may be exchanged once more[2], and the particles will be free to escape, almost in their original direction. The angular deviation is given by $\theta = f(g_c)\sqrt{b/v^3}$. It is non-trivial to determine the function $f(g_c)$ analytically, but for $g_c \sim 10^{-3}$, the following approximately holds:

$$\theta = 5.3\,(g_c)^{0.87} \times \sqrt{b/v^3}.$$

This completes our analysis of the toy model.

5.3.3 Scattering in ten dimensions

In the full ten dimensional theory, there are far more transversal directions: whereas in the simple case we had only one, now we have eight. Also, the number of fermions is increased from 3×2 to 3×16. However, this does not change the qualitative discussion of the previous section: there still are slow and fast components of the bosonic and fermionic oscillators when the particles are wide apart, and although the stadium changes from a disk to a 9-ball, the time

[1] The bosons and the fermions each contribute a term $\omega = |x|$.

[2] the amplitude for the fermions remaining in their excited state is much smaller, but non-zero. If they remain in that state, they will perform another swing.

spent inside it (as a function of the velocity and impact parameter) does not change. On the other hand, the most likely value for b changes from $\frac{1}{2}\sqrt{v}$ to $\left(\frac{1}{2}\right)^{1/8}\sqrt{v}$. Hard scattering (small b) becomes less likely. Should such an event occur, the force between the particles will be stronger, since there are 16 fermionic string states that may be excited instead of only two. The time between collisions will decrease accordingly. One important new aspect of the system, is that upon re-colliding, there are far more possibilities: in the low-dimensional case, the fermions could either return to the ground-state, allowing the particles to escape, or they could stay in the excited state, causing the particles to perform another swing. In the ten dimensional case, there are many possible combinations of fermion excitations, and it is much more likely that some new set of strings is stretched between the particles as they come out of the stadium for the second time. The total average number of collisions is thus increased tremendously, and it may be a long time before the particles escape to infinity again. When they finally do escape, their angular deviation may be large.

We may venture to estimate the average time the D-particles spend in each other's vicinity, but one should keep in mind that by now we have made so many approximations, that the result is a very rough estimate at best. Not letting this bother us, we may investigate the numbers: any state that is likely to be produced in the primary collision, also has a large likelyhood to be annihilated in the second collision. Therefore, the chance that the particles separate almost immediately is fairly large. However, if the system survives the first few swings, the estimated lifetime increases considerably: there are 16 fermions which are always excited in pairs, so there are $2^{16}/2 = 32768$ states. Only one of these allows the particles to escape, so the expectation value for the number of collisions is roughly 32768. The lifetime of the resonance could therefore be estimated at $\tau \sim 32768 \times g_c^{-1} v$, if we were to ignore the possibility of string evaporation. However, even for fairly small v evaporation is an important factor: with evaporation taken into account, τ becomes $\tau \sim 32768 \times g_c^{-1} v \times \frac{1}{1+(32v)^2}$. This peaks at $v \sim 1/40$ with $\tau \sim 500 g_c^{-1}$, which is still much longer than the inverse mass of the two particles together.

5.3.4 The existence of bound states

The scattering experiment described above does not of course prove that there are bound states. However, it does presents some evidence. When two particles in a scattering experiment can remain close to one another for a long time, the usual interpretation is that the particles temporarily combine and form a bound state. The qualitative treatment of D-particle scattering as presented above, certainly suggests that such particles can, with proper initial conditions, remain in each other's vicinity for a very long time. During this time they would perform thousands of swings as described in §5.3.2, thus gaining a considerable angular deviation. This is a strong hint at the presence of slowly decaying bound states. Whether these states are the marginally stable bound states required by the duality cannot be determined at this level of approximation.

CONCLUSIONS AND OUTLOOK

What is past
is prologue.

— *A. Pais*

We have found that particles exist in string theory, and that they are governed by the action of a free point particle with mass $g_c^{-1}\alpha'^{-1/2}$, that is,

$$S_{\mathrm{D-particle}} = -\frac{1}{g_c\sqrt{\alpha'}}\int \mathrm{d}t\,\sqrt{1-v^2}.$$

A system of N of these D-particles can – in low-energy approximation – be described by the dimensional reduction to $D = 0+1$ of ten dimensional supersymmetric Yang-Mills theory:

$$S_{N\ \mathrm{D-particles}} = \int dt\,\frac{1}{g_c}\,\mathrm{Tr}\left[\dot{Y}_m^2 + \frac{1}{2}\frac{1}{(2\pi\alpha')^2}[Y_m,Y_n]^2\right] + \mathrm{Tr}\left[\mathrm{i}\psi\dot{\psi} + \frac{1}{2\pi\alpha'}\psi\gamma_m[Y_m,\psi]\right].$$

This action describes the dynamics of N D-particles that interact through open strings that may be stretched between them: the diagonal components of the Y_m correspond to the positions of the particles, while the off-diagonal components represent the presence of stretched strings.

We have considered a scattering experiment of two D-particles as described by the above action, and found that supersymmetry is absolutely essential: without it, the possibility of stretched strings means that the one loop effective action for the particle coordinates contains a potential term that grows linearly with the distance between the particles. Bosonic D-particles are therefore confined, and scattering is impossible. Supersymmetry saves the day by cancelling the problematic effective potential: supersymmetric D-particles can escape to infinity without requiring any energy. Our analysis of the scattering showed that the particles may form resonances with a lifetime long compared to their inverse mass: evidence for the existence of bound states of N D-particles.

Outlook: M-theory from D-particles

If marginally bound states of N D-particles actually do exist (which a recent proof [31] has shown to be true), they have a mass of $m_{(N)} = \frac{N}{g_c\sqrt{\alpha'}}$. The mass shell relation for such a state reads

$$p_0^2 = m_{(N)}^2 + p_m^2. \tag{5.9}$$

As we saw in chapter two for strings, one may try to understand a mass term with m of the form $m = \frac{n}{R}$ as the remnant of a momentum term in a compactified direction. In the present context, we may view $g_c\sqrt{\alpha'} \equiv R_{11}$ as the radius of a compactified eleventh direction. From the eleven dimensional point of view, (5.9) is interpreted as

$$p_0^2 = p_{11}^2 + p_m^2 : \tag{5.10}$$

the mass shell relation for a massless particle in eleven dimensions.

This shows that a bound state of N D-particles in ten dimensions can also be viewed as a massless particle in a theory on an eleven dimensional space with compact eleventh dimension. This particle has momentum in the eleventh direction given by

$$p_{11} = \frac{N}{R_{11}}.$$

(For finite R_{11}, the momentum p_{11} is quantized in units of $1/R_{11}$, as is the mass of the bound state of D-particles. Upon decompactification, ie sending R_{11} to infinity[1)], the momentum can take on continuous values, but to obtain finite momentum, N must of course grow linearly with R_{11}.)

If we consider the fact that the ground state multiplet of the D-particle must represent the supersymmetry algebra of type IIA string theory, which is ten dimensional $\mathcal{N} = 2$ supersymmetry, we are led to a most beautiful connection: it is well-known that 10D $\mathcal{N} = 2$ supersymmetry can be viewed as the dimensional reduction of 11D $\mathcal{N} = 1$ supersymmetry, and that there is only one eleven dimensional supersymmetric point particle theory, which is eleven dimensional supergravity. Therefore, the marginally bound states of D-particles in ten dimensions must correspond to the supergravity multiplet in eleven dimensions.

Thus, the existence of bound states of D-particles, and the fact that D-particle scattering and graviton scattering yield the same amplitudes, are key ingredients for a successful attempt at formulating M-theory. Over the last year, a large number of articles have appeared on this subject. We mention [12] and [34], but there are many more, and the subject is developing rapidly.

[1)]which corresponds to taking the strong coupling limit of the ten dimensional theory.

APPENDIX A

CONVENTIONS AND NOTATION

Although most of the notational conventions are introduced in the main text, it seemed appropriate to include the following as a quick reference.

A.1 Units and metric

Right from the beginning we have set

$$\hbar = c = 1.$$

We have opted not to fix the string length scale. Many authors set $\alpha' = \frac{1}{2}$, but some set $\alpha' = 1$ or $\alpha' = 1/2\pi$. To avoid such confusion, α' is explicitly included in all formulas in this thesis, except in chapter 5, where $2\pi\alpha' \equiv 1$ to avoid excessive notational cluttering. The only other concession we have made, is that we have used the string length scale (l_s) and string tension (T), where this made equations more readable. Therefore we note that these are related by

$$T = \frac{1}{2\pi\alpha'} = \frac{1}{\pi l_s^2}.$$

The major part of the text uses Minkowski metric on space-time:

$$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1),$$

and on the world-sheet:

$$\eta_{\alpha\beta} = \mathrm{diag}(-1, +1).$$

In chapter 3 we change to Euclidean time to ease calculations. This is done by replacing τ with $\bar{\tau} = i\tau$.

A.2 Indices

The text uses indices to label many different things: world-sheet and space-time coordinates, spinor components, gauge group generators and spaces orthogonal and longitudinal to D-branes. In general we have used the following conventions:

- Space-time labels are μ, ν, etc.
- World-sheet labels are α, β, etc.

- Spinor components are labelled A, B, etc.
- Gauge group labels are a, b, etc.
- m, n, etc are used to enumerate spatial coordinates.
- i, j, etc enumerate particles and strings.
- In chapter 4 copies of S^1 are indexed by I, J, etc.

No difference is implied between upstairs or downstairs placement of indices, except for space-time and world-sheet labels, which are raised and lowered by $\eta_{\mu\nu}$ and $\eta_{\alpha\beta}$ and their inverses.

Sommation over these indices is always implied unless the text specifically states otherwise.

A.3 Other conventions

Although it may be obvious, note that time runs from left to right in all Feynman diagrams, and that i and i are very different: the former is $\sqrt{-1}$, while the latter is an integer dummy. Similarly, $\mathrm{e} = \exp(1)$, while e is an einbein.

BIBLIOGRAPHY

O brave new world,
That has such physics in 't!

— after Shakespeare, 'The Tempest'

[1] A. Pais, *Inward bound – of matter and forces in the physical world*, Oxford University Press, 1986.

[2] P. Ramond, *Dual theory for free fermions*, Phys.Rev. **D3** (1971) 2415.

[3] A. Neveu and J. H. Schwarz, *Factorizable dual model of pions*, Nucl.Phys. **B31** (1971) 86.

[4] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nucl.Phys. **B70** (1974) 39.

[5] F. Gliozzi, J. Scherk, and D. Olive, *Supersymmetry, supergravity theories and the dual spinor model*, Nucl.Phys. **B122** (1977) 253.

[6] P. W. Higgs, *Spontaneous symmetry breaking without massless bosons*, Phys.Rev. **145** (1966) 1156, (or one of many textbooks on quantum field theory).

[7] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring theory*, Cambridge University Press, Cambridge, 1987.

[8] J. Polchinski, *String duality – a colloquium*, hep-th/9607050.

[9] J. Dai, R. G. Leigh, and J. Polchinski, *New connections between string theories*, Mod.Phys.Lett. **A4** (1989) 2073.

[10] R. G. Leigh, *Dirac-Born-Infeld action from Dirichlet* σ*-model*, Mod.Phys.Lett. **A4** (1989) 2767.

[11] J. Polchinski, *Dirichlet-branes and Ramond-Ramond charges*, hep-th/9510017.

[12] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, *M theory as a matrix model: a conjecture*, hep-th/9610043.

[13] E. S. Fradkin and A. A. Tseytlin, *Fields as excitations of quantized coordinates*, Pis'ma Zh. Eksp. Teor. Fiz. **41** (1985) 169, *Translated in JETP Lett., vol 41, no 4, p 206*.

[14] E. S. Fradkin and A. A. Tseytlin, *Effective field theory from quantized strings*, Phys.Lett. **B158** (1985) 316.

[15] E. S. Fradkin and A. A. Tseytlin, *Effective action approach to superstring theory*, Phys.Lett. **160B** (1985) 69.

[16] E. S. Fradkin and A. A. Tseytlin, *Non-linear electrodynamics from quantized strings*, Phys.Lett. **163B** (1985) 123.

[17] M. Kaku, *Quantum field theory – a modern introduction*, Oxford University Press, 1993.

[18] D. Bailin and A. Love, *Supersymmetric gauge field theory and string theory*, Institute of Physics Publishing, Bristol, 1994.

[19] J. Polchinski, *TASI lectures on D-branes*, hep-th/9611050.

[20] D. Bailin and A. Love, *Introduction to gauge field theory*, Adam Hilger, Bristol, 1986.

[21] M. Born and L. Infeld, *Foundations of the new field theory*, Proc.R.Soc. **144** (1934) 425.

[22] G. H. Hardy, *Divergent series*, Oxford University press, 1949, chapter XIII.

[23] W. Nahm, *Functional integrals for the partition functions of dual strings*, Nucl.Phys. **B124** (1977) 121.

[24] O. D. Andreev and A. A. Tseytlin, *Partition-function representation for the open superstring effective action: cancellation of Möbius infinities and derivative corrections to Born-Infeld lagrangian*, Nucl.Phys. **B311** (1988) 205.

[25] C. Itzykson and J.-B. Zuber, *Quantum field theory*, McGraw-Hill, New York, 1980.

[26] A. A. Tseytlin, *On non-abelian generalisation of Born-Infeld action in string theory*, hep-th/9701125.

[27] E. Witten, *Bound states of strings and p-branes*, hep-th/9510135.

[28] W. Taylor IV, *D-brane field theory on compact spaces*, hep-th/9611042.

[29] U. H. Danielsson, G. Ferretti, and B. Sundborg, *D-particle dynamics and bound states*, hep-th/9603081.

[30] B. D. Wit, M. Lüscher, and H. Nicolai, *The supermembrane is unstable*, Nucl.Phys. **B320** (1989) 135.

[31] S. Sethi and M. Stern, *D-brane bound states redux*, hep-th/9705046.

[32] M. R. Douglas, D. Kabat, P. Pouliot, and S. H. Shenker, *D-branes and short distances in string theory*, hep-th/9608024.

[33] C. Bachas, *D-brane dynamics*, hep-th/9511043.

[34] R. H. Dijkgraaf, E. P. Verlinde, and H. L. Verlinde, *Matrix string theory*, hep-th/9703030.