8/20/2006

1. REMARK

The content of this note is an observation made in 2003 while I was working on [1]. However, I was encouraged to write it up after some useful conversations with Dusa McDuff at Oberwölfach in July 2006. Also see the reference [2].

2. THE MONOTONE TANGLE COMPLEX

Definition 2.1. An $(n+1)$-*tangle* is a configuration of $(n+1)$ ordered smooth properly embedded rays $\sigma_0, \dots, \sigma_n$ in $\mathbb{R}^2$ in general position. Two $(n+1)$-tangles $(\sigma_0, \dots, \sigma_n)$ and $(\sigma_0', \dots, \sigma_n')$ are *equivalent* if there is an orientation-preserving diffeomorphism ϕ from $\mathbb{R}^2$ to itself for which $\phi(\sigma_i) = \sigma_i'$ for all $0 \le i \le n$.

Example 2.2. There is only one 0-tangle up to equivalence.

Let $\sigma \subset \mathbb{R}^2$ be a properly embedded ray. A proper sequence of points

$$p_0, p_1, \dots \subset \sigma$$

is *monotone* if each p_i separates p_j from infinity in σ whenever $i > j$.

Definition 2.3. An $(n+1)$-tangle $(\sigma_0, \dots, \sigma_n)$ is *monotone* if for any distinct i, j the intersection $\sigma_i \cap \sigma_j$ can be ordered in a sequence which is monotone in both σ_i and σ_j.

Definition 2.4. The *tangle complex* $\mathcal{T}$ is the complex whose n-simplices are $n+1$-tangles up to equivalence, and whose boundary operator is given by

$$\partial(\sigma_0, \dots, \sigma_n) = \sum (-1)^i (\sigma_0, \dots, \hat{\sigma}_i, \dots, \sigma_n)$$

We identify several important subcomplexes:

- The *monotone tangle complex* $\mathcal{MT}$ is the subcomplex of $\mathcal{T}$ consisting of monotone tangles.
- The *finite tangle complex* $\mathcal{FT}$ is the subcomplex of $\mathcal{MT}$ consisting of monotone tangles whose rays intersect pairwise in finitely many points
- The *embedded tangle complex* $\mathcal{ZT}$ is the subcomplex of $\mathcal{FT}$ consisting of monotone tangles whose rays are disjoint

Lemma 2.5. *The complex $\mathcal{ZT}$ has $n!$ simplices in dimension n, and is homotopic to $\mathbb{CP}^\infty$.*

Proof. There is an obvious bijection between the n-simplices in $\mathcal{ZT}$ and the set of cyclic orderings of $(n+1)$-tuples of points. □

Lemma 2.6. *There is a retraction from $\mathcal{FT}$ to $\mathcal{ZT}$ which is a homotopy equivalence.*

Proof. The retraction maps each tangle to its germ at infinity. This retraction can be obtained as the composition of a sequence of elementary collapses which, at the level of tangles, removes the initial segment of each ray up to their first point of intersection. □

Theorem 2.7. *The complex $\mathcal{MT}$ is simply-connected.*

Proof. Since $\mathcal{MT}$ has only one 0-simplex, $\pi_1(\mathcal{MT})$ is generated by the set of 1-simplices, which are in bijective correspondence with equivalence classes of monotone 2-tangles.

A monotone 2-tangle (σ_0, σ_1) is determined up to equivalence by a sequence of $+$'s and $-$'s, corresponding to the sign of the intersections $\sigma_0 \cap \sigma_1$ ordered monotonely with respect to either ray.

Claim 1: Any sequence can be factorized as a product of two terms, one of which has only positive intersections, and the other of which has only negative intersections.

We use the following convention to depict tangles: think of $\mathbb{R}^2 \backslash 0$ as being obtained from a tubular neighborhood $\mathbb{R}^+ \times I$ of the positive x-axis by gluing the top and the bottom sides. For any tangle $T = (\sigma_0, \ldots, \sigma_n)$ we choose co-ordinates so that σ_0 is the positive x-axis. With this convention, Figure 1 shows how to perform the factorization promised by Claim 1 for a typical 2-tangle:

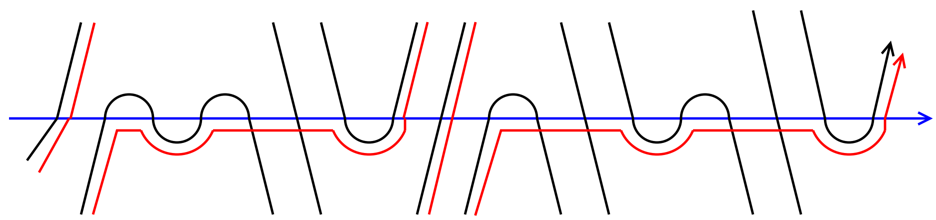

FIGURE 1. (blue, black, red) is a 2-simplex in $\mathcal{MT}$ whose boundary expresses a typical 2-tangle (blue, black) as a product of (blue, red) which has only positive intersections, and (red, black) which has only negative intersections.

Claim 2: The 1-simplex corresponding to the infinite sequence of $+$'s represents Id in $\pi_1(\mathcal{MT})$.

We denote the infinite sequence of $+$'s by $\dot{+}$, and similarly for $-$. Let $(\sigma_0, \sigma_1, \sigma_2)$ be a 3-tangle where r_0 is the positive x-axis, σ_1 is an infinite positive spiral, and σ_2 is another infinite spiral which spirals twice as fast as σ_1. Then

$$(\sigma_0, \sigma_1) = (\sigma_0, \sigma_2) = (\sigma_1, \sigma_2) = \dot{+}$$

and the claim follows.

Together with Lemma 2.6 and Lemma 2.5 this shows that $\mathcal{MT}$ is simply-connected. □

Problem 2.8. *Calculate the homotopy type of $\mathcal{MT}$. Is it homotopic to $\mathbb{CP}^\infty$?*

REFERENCES

[1] D. Calegari, *Circular groups, planar groups and the Euler class*, Geom. Topol. Monog.
[2] D. McDuff, *On tangle complexes and volume-preserving diffeomorphisms of open 3-manifolds*, Proc. London Math. Soc. (3) **43** (1983), no. 2, 321–333