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In [3] and [4], Chung and Yau introduce the following property for a generating set of a finitely generated group:

Definition. Let G be a group, and S a generating set, closed under inverses. We say G *satisfies Ric* ≥ 0 *with respect to* S, or just say G satisfies Ric ≥ 0 if S is understood, if for each $s \in S$ we have $sS = Ss$ as a set.

They show in [3] that for Cayley graphs of finite groups, with respect to such a generating set, one can derive a lower bound for the (nontrivial) Neumann eigenvalues of the Laplace operator. A similar inequality holds for the eigenvalues of the Laplacian on a compact Riemannian manifold with non–negative Ricci curvature; this motivates their terminology. Yau [5] raised the question of which groups G satisfy Ric ≥ 0 with respect to some finite generating set.

Example. If G is abelian, then G satisfies Ric ≥ 0 with respect to any generating set.

Example. If G is the symmetric group, and S is the set of all transpositions, then G satisfies Ric ≥ 0 with respect to S. To see this, observe that $(yz) \circ (xy) = (zx) \circ (yz)$ for any x, y, z. Note that this makes sense even when G is an infinite symmetric group.

Example. If

$$G = \langle x, y, z | [x, z] = [y, z] = z^2 = 1, [x, y] = z \rangle$$

then G satisfies Ric ≥ 0 with respect to the symmetric generating set

$$S = \{x, y, xz, yz, x^{-1}, y^{-1}, x^{-1}z, y^{-1}z\}$$

We show in the following two theorems that a finitely generated group G satisfies Ric ≥ 0 with respect to some finite generating set iff it is virtually abelian.

Theorem. *Let G be a group which satisfies Ric ≥ 0 with respect to S, a symmetric generating set with $2k$ elements. Then G contains a central abelian subgroup H of index at most $2^k k!$*

Proof. Let

$$S = \{e_1, e_1^{-1}, e_2, e_2^{-1}, \ldots, e_k, e_k^{-1}\}$$

Let F denote the subgroup of permutations of S subject to the constraint that if $s \in F$, then $s(e_i) = e_j^{\pm 1}$ iff $s(e_i^{-1}) = e_j^{\mp 1}$. Then the conjugation action of G on the generating set S defines a homomorphism

$$\phi : G \to F$$

This is true because in any group, $xy = yz$ iff $x^{-1}y = yz^{-1}$. Let H be the kernel $\ker(\phi)$. Then H commutes with $\langle S \rangle = G$, so H is a central abelian subgroup.

It suffices to estimate the size of the image $\phi(G)$. Let $\sim$ be the equivalence relation on S defined by $e_i \sim e_i^{-1}$. Then F descends to an action on $S/\sim$, so there is a surjective homomorphism $F \to \mathsf{S}_k$ whose kernel has order 2^k. It follows that the order of F is $2^k k!$ □

Theorem. *Let G be a finitely generated group with a central abelian subgroup H of finite index k. Then G satisfies Ric ≥ 0 with respect to a finite symmetric generating set with at most $kh_k 2\text{rank}(H)$ elements, where h_k denotes the cardinality of the k–torsion subgroup of H, and rank(H) denotes the minimal size of a generating set for H.*

Proof. Let $E = \{e_1, \dots, e_k\}$ be a set of coset representatives of H in G, where the e_i are chosen so that each e_i^{-1} is some e_j. Then for any j there is a permutation s_j such that $e_i e_j H = e_j e_{s_j(i)} H$. It follows that there are $h_{ij} \in H$ such that

$$e_i e_j = e_j e_{s_j(i)} h_{ij}$$

On the other hand, since k is finite, there is a n_j such that $e_j^{n_j} \in H$. It follows that $h_{ij}^{n_j} = 1$. Note that each n_j divides k, so that each h_{ij} is contained in the k–torsion subgroup H_k of H.

Let $H_0 < H$ be the *finite* subgroup generated by the h_{ij} for all i, j. Then the set of products EH_0 is finite, and by construction it is invariant under conjugation by any element of G. The union of EH_0 with a symmetric generating set for H is the desired subset S. □

Question. *Find sharp estimates on the index of H in terms of the size of the generating set S and vice versa.*

In view of the results in [3], [4] one would like to find sharp bounds on the size of the generating set for a given virtually abelian group, in order to get good estimates on spectral invariants of the group.

Compare this with the classical Cheeger–Gromoll splitting theorem [1],[2]:

Theorem (Cheeger–Gromoll)**.** *Let (M, g) be a compact connected Riemannian manifold with non–negative Ricci curvature. Then $\pi_1(M)$ is virtually abelian.*

References

[1] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of non–negative Ricci curvature*, J. Diff. Geom. **6** (1971) 116–128

[2] J. Cheeger and D. Gromoll, *On the structure of complete manifolds of non–negative curvature*, Ann. Math. **96** (1972) 413–443

[3] F. Chung and S.–T. Yau, *A Harnack inequality for homogeneous graphs and subgraphs*, Comm. Anal. Geom. **2** (1994) no. 4, 627–640

[4] F. Chung and S.–T. Yau, *Logarithmic Harnack inequalities*, Math. Res. Lett. **3** (1996), no. 6, 793–812

[5] S.–T. Yau, *private communication*