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A *Kleinian group* is a discrete finitely generated subgroup

$$\Gamma < \mathrm{PSL}(2, \mathbb{C})$$

If G is an abstract finitely generated group, and $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$ is a discrete, faithful embedding, then $\Gamma := \rho(G)$ is Kleinian.

If G is not virtually abelian, The representation ρ can usually be recovered up to conjugacy from the character $\chi_\rho : G \to \mathbb{C}/\pm 1$ defined by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g))$$

For ease of notation, we abbreviate χ_ρ to χ in what follows.

The *trace field* K of $\rho(G)$ is the field generated over $\mathbb{Q}$ by $\chi(g)$. Since G is finitely generated, this field is finitely generated over $\mathbb{Q}$. Let $\sigma : K \to K^\sigma \subset \mathbb{C}$ be another algebraic embedding of K. Define $\chi^\sigma : G \to \mathbb{C}$ by

$$\chi^\sigma(g) = \sigma(\chi(g))$$

Then χ^σ is the character of some representation

$$\rho^\sigma : G \to \mathrm{PSL}(2, \mathbb{C})$$

which is well-defined up to conjugacy in $\mathrm{PSL}(2, \mathbb{C})$, and is said to be obtained from ρ by Galois conjugacy. We call the image Γ^σ a Galois conjugate of Γ.

In this note, we study the following question:

Question 1. *For which finitely generated subgroups $\Gamma < PSL(2, \mathbb{C})$ are all Galois conjugates discrete?*

The following two examples are more or less trivial:

Example 2. If the trace field K is equal to $\mathbb{Q}$, then every Galois conjugate of Γ is conjugate to Γ (in $\mathrm{PSL}(2, \mathbb{C})$).

Example 3. If the trace field K is quadratic imaginary, then the only nontrivial σ is complex conjugation, and therefore Γ is discrete if and only if Γ^σ is.

The next non-example is less trivial, but very well known:

Example 4. If Γ is a lattice, then by Mostow-Prasad rigidity, there are no other discrete embeddings of Γ in $\mathrm{PSL}(2, \mathbb{C})$. It follows that if K is not quadratic imaginary, some Galois conjugate is indiscrete.

The point of this note is to describe the following source of nontrivial examples of arbitrary degree:

Example 5. Let T be a once-punctured torus, and let $\rho : \pi_1(T) \to \mathrm{PSL}(2, \mathbb{C})$ be discrete, faithful, with parabolic holonomy around the puncture of T. Since T is noncompact, ρ is conjugate into $\mathrm{PSL}(2, K)$ where K is the trace field. Suppose K is a number field of degree $d > 2$ whose only non-real embeddings are K and $\overline{K}$ (i.e.

complex conjugation). Then there are $d-2$ real embeddings $\sigma_1, \ldots, \sigma_{d-2} : K \to \mathbb{R}$ and resulting representations

$$\rho^{\sigma_i} : \pi_1(T) \to \mathrm{PSL}(2, \mathbb{R})$$

Each such representation has vanishing second Stiefel-Whitney class since ρ does (for geometric reasons) and therefore lifts to $\mathrm{SL}(2, \mathbb{R})$ where the trace of the lift of the boundary torus is -2. It follows that the Euler class of the representations is ± 1 which is maximal, and therefore ρ^{σ_i} are discrete for all i.

Another way to see this is to use the trace relations. If A and B are a standard pair of generators for $\pi_1(T)$, and if we denote $\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB)$ by x, y, z respectively, then we have the basic (Markoff) trace relation:

$$x^2 + y^2 + z^2 = xyz$$

If ρ^{σ_i} is indiscrete, then there are $x, y, z \in K^{\sigma_i} \subset \mathbb{R}$ for which this relation is satisfied, and for which $-2 < x < 2$ after a suitable change of basis. In this case, we have an inequality

$$(y-z)^2 \leq y^2 - xyz + z^2 \leq (y+z)^2$$

with equality if and only if $x = y = z = 0$. But this implies that

$$y^2 - xyz + z^2 \geq 0$$

From the trace relation

$$x^2 = -y^2 + xyz - z^2 \leq 0$$

and therefore $x = y = z = 0$ and the image of ρ^{σ_i} is finite, contrary to hypothesis.