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1. MINIMAL SURFACES

Definition 1.1. Let S be a smoothly embedded oriented 2-dimensional surface in an oriented Riemannian 3-manifold M. Then S is *minimal* if it is a critical point for area with respect to all smooth variations of S, and S is *locally least area* if it is a local minimum for area.

Let $S = S(t)$ be a smooth variation of $S = S(0)$. For convenience, we normalize so that $S_t := \frac{\partial S}{\partial t}\Big|_{t=0} = \zeta\nu$ for some function ζ on S. Then one has the following formula for the first variation of area:

$$\frac{d}{dt}\Big|_{t=0}\text{area}(S(t)) = \int_S \zeta\mu \; d\text{area} \tag{1}$$

where μ is the mean curvature

$$\mu = \text{tr}(X \in TS \to \langle \nabla_X X, \nu\rangle) \tag{2}$$

and ν is the positive unit normal vector field to S.

It follows that S is minimal iff μ is identically zero. For such S, one has the following formula for the second variation of area:

$$\frac{d^2}{dt^2}\Big|_{t=0}\text{area}(S(t)) = \int_S |\nabla\zeta|^2 - (\text{Ric}(\nu) + |A|^2)\zeta^2 d\text{area} \tag{3}$$

where A is the second fundamental form

$$A(e_1, e_2) := \langle \nu, \nabla_{e_1} e_2\rangle \tag{4}$$

If K_S denotes the intrinsic sectional curvature of S, and K_M the sectional curvature of TS in M, then for a minimal surface,

$$K_S = K_M - \frac{1}{2}|A|^2 \tag{5}$$

For M a hyperbolic 3-manifold, Ric(ν) is identically equal to -2. It follows that for a locally least area minimal surface S in a hyperbolic 3-manifold, one has the following pointwise estimate

$$-1 \geq K_S \geq -2 \tag{6}$$

See e.g. [2] for more details.

2. HYPERBOLIC GEOMETRY

2.1. **Incompressible surfaces.** A closed orientable surface T in an orientable 3-manifold M is *incompressible* if the inclusion map $i : T \to M$ induces a monomorphism on fundamental groups:

$$i_* : \pi_1(T) \to \pi_1(M) \tag{7}$$

After replacing M with the cover corresponding to the image of $\pi_1(T)$, we can assume i_* is an isomorphism.

2.2. **Quasifuchsian surfaces.** Suppose that M is a complete hyperbolic 3-manifold. The hyperbolic structure determines a representation

$$\rho : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C}) \tag{8}$$

By composing with the isomorphism i_*, we get a representation, which by abuse of notation we also denote by ρ:

$$\rho : \pi_1(T) \to \mathrm{PSL}(2, \mathbb{C}) \tag{9}$$

Denote by Γ_T the image $\rho(\pi_1(T))$.

Given $p \in \mathbb{H}^3$, we can consider the orbit $\Gamma_T p \subset \mathbb{H}^3$. Topologically, $\mathbb{H}^3$ is an open ball, which can be compactified to a closed ball by adding the sphere at infinity S^2_∞. The *limit set of* T, denoted $\Lambda(T)$, is the intersection

$$\Lambda(T) = \overline{\Gamma_T p} \cap S^2_\infty \tag{10}$$

Definition 2.1. T is *quasifuchsian* if the limit set $\Lambda(T)$ is a quasicircle.

Let $\widetilde{C}(T)$ denote the convex hull (in $\mathbb{H}^3$) of $\Lambda(T)$. Then $\widetilde{C}(T)$ is invariant under Γ_T and covers a convex subset $C(T) \subset M$ which is homotopic to $i(T)$. Since $C(T)$ is convex, its boundary is a barrier surface, and we can find a locally least area surface S in M in the isotopy class of T.

Note that $C(T)$ is *convex*. It follows that Γ_T is an example of what is known as a *convex cocompact* Kleinian group.

2.3. **Poincaré series.** We let $\widetilde{S}$ denote a connected cover of S in the universal cover $\widetilde{M} = \mathbb{H}^3$. Since S is incompressible, $\widetilde{S}$ is topologically a plane. We assume without loss of generality that $0 \in S$.

We let $d(\cdot, \cdot)$ denote the distance function in $\mathbb{H}^3$, and $d_{\widetilde{S}}(\cdot, \cdot)$ the distance function of $\widetilde{S}$ induced by the Riemannian metric on S.

The *Poincaré series* $P(\Gamma_T, s)$ of the Kleinian group Γ_T is defined by the formula

$$P(\Gamma_T, s) = \sum_{\gamma \in \Gamma_T} e^{-sd(0,\gamma(0))} \tag{11}$$

The *critical exponent* c of $P(\Gamma_T, s)$ is the infimal value of $s \in \mathbb{R}$ for which $P(\Gamma_T, s)$ converges. Note that by an easy argument, $1 \leq c \leq 2$

One has the following theorem:

Theorem 2.2. *Suppose* Γ *be a convex cocompact Kleinian group. Let* c *be the critical exponent of* $P(\Gamma, s)$*. Then* c *is equal to the Hausdorff dimension* $\dim_{\mathcal{H}}$ *of the limit set* $\Lambda(\Gamma)$.

See e.g. [1] page 84 for a proof.

Now, for each $\gamma \in \pi_1(T)$, we can take the geodesic representative g^S_γ in S, and g^M_γ in M, where the loops $g^\cdot_\gamma$ are based at the common base point 0 which is the image of $0 \in \widetilde{S}$ under the covering projection $\widetilde{S} \to S$. By the definition of a geodesic, we have the following trivial inequality:

$$\text{length}(g^S_\gamma) \geq \text{length}(g^M_\gamma) \tag{12}$$

We can rewrite $P(\Gamma_T, s)$ in terms of g^M_γ as

$$P(\Gamma_T, s) = \sum_{\gamma\in\pi_1(T)} e^{-s\,\text{length}(g^M_\gamma)} \geq \sum_{\gamma\in\pi_1(T)} e^{-s\,\text{length}(g^S_\gamma)} =: P^S(\Gamma_T, s) \tag{13}$$

and we observe that the critical exponent c^S for P^S satisfies

$$c^S \leq \dim_{\mathcal{H}}(\Lambda(T)) \tag{14}$$

2.4. **Gauss-Bonnet.** For $t \geq 0$, let $B^{\widetilde{S}}_t(0)$ denote the ball of radius t about 0 in $\widetilde{S}$. Using the coarea formula and Gauss-Bonnet, we derive the following:

$$\begin{aligned}
\text{area}(B^{\widetilde{S}}_t(0)) &= \int_0^t \text{length}(\partial B^{\widetilde{S}}_s(0))ds && (15)\\
&= \int_0^t\int_0^s \frac{d\text{length}(\partial B^{\widetilde{S}}_r(0))}{dr}dr\,ds && (16)\\
&= \int_0^t\int_0^s \Big(2\pi - \int_{B^{\widetilde{S}}_r(0)} K d\text{area}\Big)dr\,ds && (17)
\end{aligned}$$

We define the average curvature K_{av} by the formula

$$K_{\text{av}} = \frac{\int_S K d\text{area}}{\text{area}(S)} = \frac{2\pi\chi(S)}{\text{area}(S)} \tag{18}$$

Since S has pinched negative curvature, the geodesic flow is ergodic. It follows that for large r, the boundary of $B^{\widetilde{S}}_r$ projects under the covering projection $\pi : \widetilde{S} \to S$ to a curve which is equidistributed in the unit tangent bundle of S. So we can estimate

$$\left| \int_{B^{\widetilde{S}}_r(0)} K d\text{area} - K_{\text{av}}\text{area}(B^{\widetilde{S}}_r(0)) \right| \leq \text{const.} \tag{19}$$

and therefore

$$\text{area}(B^{\widetilde{S}}_t(0)) \sim e^{t\sqrt{-K_{\text{av}}}} \tag{20}$$

up to a bounded (multiplicative) constant.

On the other hand, defining

$$N(t) = \#\{\gamma \in \pi_1(T) \mid \text{length}(g^S_\gamma) \leq t\} \tag{21}$$

we have an estimate

$$|\,\text{area}(B^{\widetilde{S}(0)}_t) - \text{area}(S)N(t)\,| \leq \text{const.} \tag{22}$$

Putting these estimates together, we get the following:

$$\text{area}(S)N(t) \sim e^{t\sqrt{-K_{\text{av}}}} \tag{23}$$

It follows that

$$P^S(\Gamma_T, s) \sim \int_0^\infty \text{const.}\ e^{t\sqrt{-K_{\text{av}}}}e^{-st} \tag{24}$$

and therefore we estimate

$$K_{\mathrm{av}} \geq -(\dim_{\mathcal{H}}(\Lambda(T)))^2 \tag{25}$$

3. CURRENTS

To improve the estimate we obtained in the last section, we need to understand in more detail the relationship between length(g_γ^S) and length(g_γ^M) for typical $\gamma \in \pi_1(T)$.

For $p \in \widetilde{S}$, let U_p denote the unit tangent bundle in p. A geodesic ray in $\widetilde{S}$ emanating from p converges to a unique point in $\Lambda(S)$. This assignment defines a map $\exp$:

$$\exp : U_p \to \Lambda(S) \tag{26}$$

We let $PU_p = U_p / \pm 1$ denote the projectivization of U_p. A point $\theta \in PU_p$ defines a pair $\{\theta, -\theta\} \in U_p$ of antipodal points. The image under $\exp$ defines a pair of points in $\Lambda(T)$ which span a unique geodesic g_θ in $\mathbb{H}^3$.

Given an infinite geodesic $g \subset \mathbb{H}^3$ and a point $q \in \mathbb{H}^3$, we denote the distance from q to g by $d(q, g)$. Orthogonal projection from $\mathbb{H}^3$ to g is 1-lipschitz (i.e. length non-increasing) and has infinitesimal (operator) norm $\cosh^{-1}(d(q, g))$ on the tangent space Tq.

Definition 3.1. Given $p \in \widetilde{S}$, let $p' \in \mathbb{H}^3$ be the unique point which minimizes the function

$$h_p(q) = \frac{1}{\pi} \int_{PU_p} \cosh(d(q, g_\theta)) d\theta \tag{27}$$

Note that by strict convexity of distance to geodesics in hyperbolic space, and strict convexity of $t \to \cosh(t)$, the minimum is attained at a *unique* point p'. Note that $h_p(p') = 1$ iff S is totally geodesic.

From the definition of h_p and the ergodicity of the geodesic flow on S, we obtain the following improvement of the estimate in the last section:

Lemma 3.2. *Let S be a smooth surface in M in the isotopy class of T. Then*

$$K_{av} \geq \frac{-(dim_{\mathcal{H}}(\Lambda(T)))^2}{\inf_{p \in \widetilde{S}} h_p(p')} \tag{28}$$

Notice that for a minimal S, we have the estimate $-1 \geq K_{\mathrm{av}}$ from the other side.

REFERENCES

[1] P. Nicholls, *The ergodic theory of discrete groups*, London Mathematical Society Lecture Note Series, 143 *Cambridge University Press, Cambridge*, 1989

[2] L. Simon, *Survey lectures on minimal submanifolds, Seminar on minimal submanifolds*, 3–52, Ann. of Math. Stud., 103, *Princeton Univ. Press, Princeton, NJ*, 1983