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Let M be a closed 3-manifold, and let \mathcal{F} be a taut foliation of M . An *automorphism* of (M, \mathcal{F}) is a homeomorphism $\phi : M \rightarrow M$ taking \mathcal{F} to itself.

Question 1. *Classify automorphisms of tautly foliated 3-manifolds.*

If M is hyperbolic, then the mapping class group of M is finite. In fact, if M is atoroidal, the existence of \mathcal{F} implies that the mapping class group of M is finite. So after replacing ϕ by a finite power if necessary, we may assume that ϕ is isotopic to the identity.

Then ϕ lifts to a homeomorphism

$$\tilde{\phi} : \tilde{M} \rightarrow \tilde{M}$$

which is a bounded distance from the identity map. That is, there is a constant C such that

$$d_{\tilde{M}}(p, \tilde{\phi}(p)) \leq C$$

for all $p \in \tilde{M}$.

Let $\tilde{\mathcal{F}}$ denote the pullback of \mathcal{F} to \tilde{M} . Denote the leaf space of $\tilde{\mathcal{F}}$ by L . Then L is a (typically non-Hausdorff) 1-manifold. Two leaves $\lambda, \mu \in L$ are *comparable* if there is a transversal to $\tilde{\mathcal{F}}$ from λ to μ . Equivalently, if there is an embedded interval in L between λ and μ . If the region of L between λ and μ is homeomorphic to I , then we say that λ and μ cobound a *product pocket*. If we fix a co-orientation on \mathcal{F} , it makes sense to write $\lambda < \mu$ or $\lambda > \mu$ whenever λ, μ are comparable.

For each leaf λ of $\tilde{\mathcal{F}}$, our estimate implies that $\tilde{\phi}(\lambda)$ and λ are a finite Hausdorff distance apart.

Lemma 2. *Let \mathcal{F} be taut, and let M be atoroidal. Suppose $\lambda, \phi(\lambda)$ are a finite Hausdorff distance apart in \tilde{M} . Then λ and $\phi(\lambda)$ are comparable as leaves of \tilde{M} , and they cobound a product pocket.*

Proof. For simplicity, we assume that L branches in both directions.

Since λ and $\phi(\lambda)$ are a finite distance apart, we can take a separated net $N \subset \lambda$ and join points $p \in N$ to $\phi(p) \in \phi(N)$ by paths γ_p where

$$\text{length}(\gamma_p) \leq C$$

for some uniform constant C . Since M is compact, $\phi : \lambda \rightarrow \phi(\lambda)$ is a quasi-isometry. So we can take N to be the 0-skeleton of a triangulation τ of λ , let $\phi(\tau)$ be the image triangulation in $\phi(\lambda)$. If $p, q \in N$ are joined by an edge e of τ , then

$$\alpha_e := e \cup \gamma_p \cup \phi(e) \cup \gamma_q$$

is a loop of uniformly bounded length, so we can fill it by a disk of uniformly bounded diameter. Together with the 2-cells of τ and $\phi(\tau)$, we get the 2-skeleton of a 2-complex. Filling this in with 3-cells of bounded diameter, we get a proper map

$$\Phi : \lambda \times I \rightarrow \tilde{M}$$

such that the length of $\Phi(p \times I)$ is uniformly bounded, and such that

$$\Phi(\lambda \times 0) = \lambda, \quad \Phi(\lambda \times 1) = \phi(\lambda)$$

It follows by properness that there is a uniform constant $K > 0$ such that if $r \in \tilde{M}$ is in the region cobounded by λ and $\phi(\lambda)$ then

$$d_{\tilde{M}}(r, \lambda) \leq K, \quad d_{\tilde{M}}(r, \phi(\lambda)) \leq K$$

Now, if λ and $\phi(\lambda)$ do not cobound a product pocket, then without loss of generality there is some $\nu_1 > \lambda$ which is contained between $\phi(\lambda)$ and λ , for which ν_1 and $\phi(\lambda)$ are incomparable. Notice that this implies that ν'_1 is contained between $\phi(\lambda)$ and λ in \tilde{M} whenever $\nu'_1 > \nu_1$.

Since \mathcal{F} is taut and branches in both directions, there is some $\nu'_1 > \nu_1$ for which there is $\nu_2 < \nu'_1$ contained between ν_1 and ν'_1 , and incomparable with ν'_1 . Notice that ν_2 is contained between λ and $\phi(\lambda)$. Proceeding inductively, we construct a sequence ν_i where ν_i and ν_{i+1} are incomparable, all contained between λ and $\phi(\lambda)$.

Since M is compact, there is a constant $\epsilon > 0$ called a *separation constant* such that any two points which are distance $\leq \epsilon$ apart are on comparable leaves. From the definition of ϵ , it follows that ν_i is distance at least $i\epsilon$ from either λ or $\phi(\lambda)$. Choosing i sufficiently large, we get $i\epsilon > K$, contradicting our earlier estimate.

If L branches in at most one direction, then any two leaves which are a finite Hausdorff distance apart cobound a product pocket, for general reasons, and the lemma follows in this case too. \square

It follows that the orbit of any leaf λ in L under ϕ is contained in a totally ordered subset. More generally, if M is atoroidal, for any group G of automorphisms of M , there is a finite index subgroup G' whose image is trivial in $\text{MCG}(M)$. If G acts by automorphisms of (M, \mathcal{F}) , then G acts on L , and the orbit of any point $\lambda \in L$ under G' is a totally ordered subset of L .

This gives a homomorphism from G' to a direct product of copies of $\text{Homeo}^+(\mathbb{R})$, one for each product pocket of \mathcal{F} , and the kernel fixes \mathcal{F} leafwise. Of course, if \mathcal{F} has no product pockets (e.g. if \mathcal{F} is minimal and not \mathbb{R} -covered) then all of G' fixes \mathcal{F} leafwise.

Corollary 3. *If \mathcal{F} is taut and minimal and not \mathbb{R} -covered, and M is atoroidal, then any group G of automorphisms of (M, \mathcal{F}) has a finite index subgroup which fixes \mathcal{F} leafwise.*