

10/3/2005

Let  $M$  be a closed 3-manifold, and let  $\mathcal{F}$  be a taut foliation of  $M$ . An *automorphism* of  $(M, \mathcal{F})$  is a homeomorphism  $\phi : M \rightarrow M$  taking  $\mathcal{F}$  to itself.

**Question 1.** *Classify automorphisms of tautly foliated 3-manifolds.*

If  $M$  is hyperbolic, then the mapping class group of  $M$  is finite. In fact, if  $M$  is atoroidal, the existence of  $\mathcal{F}$  implies that the mapping class group of  $M$  is finite. So after replacing  $\phi$  by a finite power if necessary, we may assume that  $\phi$  is isotopic to the identity.

Then  $\phi$  lifts to a homeomorphism

$$\tilde{\phi} : \tilde{M} \rightarrow \tilde{M}$$

which is a bounded distance from the identity map. That is, there is a constant  $C$  such that

$$d_{\tilde{M}}(p, \tilde{\phi}(p)) \leq C$$

for all  $p \in \tilde{M}$ .

Let  $\tilde{\mathcal{F}}$  denote the pullback of  $\mathcal{F}$  to  $\tilde{M}$ . Denote the leaf space of  $\tilde{\mathcal{F}}$  by  $L$ . Then  $L$  is a (typically non-Hausdorff) 1-manifold. Two leaves  $\lambda, \mu \in L$  are *comparable* if there is a transversal to  $\tilde{\mathcal{F}}$  from  $\lambda$  to  $\mu$ . Equivalently, if there is an embedded interval in  $L$  between  $\lambda$  and  $\mu$ . If the region of  $L$  between  $\lambda$  and  $\mu$  is homeomorphic to  $I$ , then we say that  $\lambda$  and  $\mu$  cobound a *product pocket*. If we fix a co-orientation on  $\mathcal{F}$ , it makes sense to write  $\lambda < \mu$  or  $\lambda > \mu$  whenever  $\lambda, \mu$  are comparable.

For each leaf  $\lambda$  of  $\tilde{\mathcal{F}}$ , our estimate implies that  $\tilde{\phi}(\lambda)$  and  $\lambda$  are a finite Hausdorff distance apart.

**Lemma 2.** *Let  $\mathcal{F}$  be taut, and let  $M$  be atoroidal. Suppose  $\lambda, \phi(\lambda)$  are a finite Hausdorff distance apart in  $\tilde{M}$ . Then  $\lambda$  and  $\phi(\lambda)$  are comparable as leaves of  $\tilde{M}$ , and they cobound a product pocket.*

*Proof.* For simplicity, we assume that  $L$  branches in both directions.

Since  $\lambda$  and  $\phi(\lambda)$  are a finite distance apart, we can take a separated net  $N \subset \lambda$  and join points  $p \in N$  to  $\phi(p) \in \phi(N)$  by paths  $\gamma_p$  where

$$\text{length}(\gamma_p) \leq C$$

for some uniform constant  $C$ . Since  $M$  is compact,  $\phi : \lambda \rightarrow \phi(\lambda)$  is a quasi-isometry. So we can take  $N$  to be the 0-skeleton of a triangulation  $\tau$  of  $\lambda$ , let  $\phi(\tau)$  be the image triangulation in  $\phi(\lambda)$ . If  $p, q \in N$  are joined by an edge  $e$  of  $\tau$ , then

$$\alpha_e := e \cup \gamma_p \cup \phi(e) \cup \gamma_q$$

is a loop of uniformly bounded length, so we can fill it by a disk of uniformly bounded diameter. Together with the 2-cells of  $\tau$  and  $\phi(\tau)$ , we get the 2-skeleton of a 2-complex. Filling this in with 3-cells of bounded diameter, we get a proper map

$$\Phi : \lambda \times I \rightarrow \tilde{M}$$

such that the length of  $\Phi(p \times I)$  is uniformly bounded, and such that

$$\Phi(\lambda \times 0) = \lambda, \quad \Phi(\lambda \times 1) = \phi(\lambda)$$

It follows by properness that there is a uniform constant  $K > 0$  such that if  $r \in \tilde{M}$  is in the region cobounded by  $\lambda$  and  $\phi(\lambda)$  then

$$d_{\tilde{M}}(r, \lambda) \leq K, \quad d_{\tilde{M}}(r, \phi(\lambda)) \leq K$$

Now, if  $\lambda$  and  $\phi(\lambda)$  do not cobound a product pocket, then without loss of generality there is some  $\nu_1 > \lambda$  which is contained between  $\phi(\lambda)$  and  $\lambda$ , for which  $\nu_1$  and  $\phi(\lambda)$  are incomparable. Notice that this implies that  $\nu'_1$  is contained between  $\phi(\lambda)$  and  $\lambda$  in  $\tilde{M}$  whenever  $\nu'_1 > \nu_1$ .

Since  $\mathcal{F}$  is taut and branches in both directions, there is some  $\nu'_1 > \nu_1$  for which there is  $\nu_2 < \nu'_1$  contained between  $\nu_1$  and  $\nu'_1$ , and incomparable with  $\nu'_1$ . Notice that  $\nu_2$  is contained between  $\lambda$  and  $\phi(\lambda)$ . Proceeding inductively, we construct a sequence  $\nu_i$  where  $\nu_i$  and  $\nu_{i+1}$  are incomparable, all contained between  $\lambda$  and  $\phi(\lambda)$ .

Since  $M$  is compact, there is a constant  $\epsilon > 0$  called a *separation constant* such that any two points which are distance  $\leq \epsilon$  apart are on comparable leaves. From the definition of  $\epsilon$ , it follows that  $\nu_i$  is distance at least  $i\epsilon$  from either  $\lambda$  or  $\phi(\lambda)$ . Choosing  $i$  sufficiently large, we get  $i\epsilon > K$ , contradicting our earlier estimate.

If  $L$  branches in at most one direction, then any two leaves which are a finite Hausdorff distance apart cobound a product pocket, for general reasons, and the lemma follows in this case too.  $\square$

It follows that the orbit of any leaf  $\lambda$  in  $L$  under  $\phi$  is contained in a totally ordered subset. More generally, if  $M$  is atoroidal, for any group  $G$  of automorphisms of  $M$ , there is a finite index subgroup  $G'$  whose image is trivial in  $\text{MCG}(M)$ . If  $G$  acts by automorphisms of  $(M, \mathcal{F})$ , then  $G$  acts on  $L$ , and the orbit of any point  $\lambda \in L$  under  $G'$  is a totally ordered subset of  $L$ .

This gives a homomorphism from  $G'$  to a direct product of copies of  $\text{Homeo}^+(\mathbb{R})$ , one for each product pocket of  $\mathcal{F}$ , and the kernel fixes  $\mathcal{F}$  leafwise. Of course, if  $\mathcal{F}$  has no product pockets (e.g. if  $\mathcal{F}$  is minimal and not  $\mathbb{R}$ -covered) then all of  $G'$  fixes  $\mathcal{F}$  leafwise.

**Corollary 3.** *If  $\mathcal{F}$  is taut and minimal and not  $\mathbb{R}$ -covered, and  $M$  is atoroidal, then any group  $G$  of automorphisms of  $(M, \mathcal{F})$  has a finite index subgroup which fixes  $\mathcal{F}$  leafwise.*