Math151c, Spring 2008

1. Suppose G is a Lie group and H is a closed Lie subgroup. Show that the left coset space $H\backslash G$ is a smooth manifold of dimension $\dim(G) - \dim(H)$, and that the projection $G \to H\backslash G$ is a principal H bundle, where H acts on the total space G on the left.

2. Let H be the subgroup of $\mathrm{GL}(n+k, \mathbb{R})$ consisting of matrices whose lower $n \times k$ block is empty; i.e. consisting of matrices of the form

$$\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$$

where $A \in \mathrm{GL}(n, \mathbb{R})$, $B \in \mathrm{GL}(k, \mathbb{R})$, and $*$ is arbitrary. Show that $\mathrm{GL}(n+k, \mathbb{R})$ is a principal H bundle over the Grassmann manifold $G_n(\mathbb{R}^{n+k})$. Let E denote the total space of this principal bundle. Let $\rho : H \to \mathrm{GL}(n, \mathbb{R})$ be the surjective homomorphism obtained by letting H act on the first n co-ordinates of $\mathbb{R}^{n+k}$, and let $E \times_H \mathbb{R}^n$ be the total space of the $\mathbb{R}^n$ bundle over $G_n(\mathbb{R}^{n+k})$ obtained from this action. Show that $E \times_H \mathbb{R}^n$ is isomorphic to the total space of the tautological bundle γ^n over $G_n(\mathbb{R}^{n+k})$.

3. Show that the space of *oriented* 2-planes in $\mathbb{R}^4$ is diffeomorphic to $S^2 \times S^2$ and therefore deduce that $G_2(\mathbb{R}^4)$ is double-covered by $S^2 \times S^2$. What is the diffeomorphism type of $G_2(\mathbb{R}^4)$?