Hedging with Monte Carlo Simulation

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Abstract

Monte Carlo simulation provides a simple procedure to price securities numerically; however, it does not immediately yield the weights necessary to construct a replicating portfolio. The procedure commonly used consists in approximating the derivatives by perturbing the price of the underlying. Here we suggest an alternative procedure that exploits the semimartingale representation of the dynamics of any redundant security and consists in retrieving the volatility term of that representation. Compared to the alternative procedure, the method we suggest has the advantage that the computational cost is independent of the number of dimensions (unlike the traditional procedure where the cost increases linearly with the number of dimensions).

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1 Introduction

Monte Carlo simulation is a convenient numerical method (it is easy to implement and converges reasonably fast) to price securities for which an analytical expression does not exist. In fact, it has been traditionally argued that it was suitable only for European-type securities, but recently there has been a renewed interest in adapting it to American style securities (see, for example, Ibáñez and Zapatero (1999) and Longstaff and Schwartz (1999)). For a general overview of pricing using Monte Carlo see Boyle, Broadie and Glasserman (1997).

One problem with Monte Carlo simulation is that (unlike binomial trees) it does not provide automatically the weights on the underlying security/securities required to form a replicating portfolio (a portfolio of the underlying security/securities and a risk-free asset). In the one-dimensional case (when the underlying is stochastic but all other parameters are constant or deterministic) this weight can also be interpreted as the sensitivity of the price of the derivative to the price of the underlying security ("delta"). The procedure used in practice consists in perturbing the price of the underlying security and computing the effect of this perturbation on the price of the derivative. However, as the number of dimensions of the problem increases (for example, if the interest rate is also stochastic) the computational cost of this procedure increases, since other sensitivities will have to be estimated (in the case of stochastic interest rate, the sensitivity of the derivative to the interest rate or "rho"). In general, the computational cost will grow linearly with the number of dimensions. In fact, for most interesting models the number of dimensions will be more than one.

In this paper we suggest an alternative procedure that relies on the semimartingale representation of the dynamics of the securities involved. We show that the procedure we suggest provides an accurate computation of the "delta" of the derivative, similar to the method mentioned in the previous paragraph in the one-dimensional case; however, the computational cost of the procedure we suggest in this paper is independent of the number of dimensions of the problem. Therefore, it seems to be superior to the method mentioned above when the problem has more than one dimension. Since the method we suggest consists in retrieving the volatility parameter of the semimartingale representation of the derivative we call this method RVM.

It should be mentioned that a recent paper Fournie et al. (1999) provides a very efficient method for computing hedging portfolios using Monte Carlo, generalizing a method suggested by Broadie and Glasserman (1996). However, the method is, in general, more complex since it involves finding Malliavin derivatives of the stochastic processes on which pricing depends (such as underlying security, interest rates, volatility, and so on). Moreover, it cannot be applied to American options, and some exotic options such as barrier options. Our method to compute hedging portfolios can be applied to any type of options, as long as we can simulate the option price.

The structure of this paper is as follows: in Section 2 we review the basics of the application of Monte Carlo simulation to pricing of derivatives; in Section 3 we study the problem of constructing a replicating portfolio as well as the solution commonly employed; in Section 4 we describe the procedure suggested in this paper; Section 5 is devoted to some examples; Section 6 extends the procedure to a multi-factor setting; we close the paper with some Conclusions.

2 Asset Pricing using Monte Carlo Simulation

Consider an economy where uncertainty is given by the realizations of a standard Brownian Motion Process whose value we represent by W. For simplicity reasons, at this stage we will assume that it is a one-dimensional Brownian Motion Process. We assume that markets are complete. More precisely, in our economy there is a bank account whose value B evolves according to the following dynamics,

$$dB_t = B_t r_t dt \tag{1}$$

where r is the interest rate process, possibly stochastic. There is also another security whose price S behaves according to the following stochastic differential equation (under risk-neutral probability measure),

$$dS_t = S_t[r_t dt + \sigma_t dW_t] \tag{2}$$

where σ represents volatility of the security. The parameters r and σ are possibly stochastic and themselves satisfy stochastic equations driven by Brownian Motion W; we do not require a Markovian setting in order to price options in this market.

Harrison and Kreps (1979) introduced a classical result that implies that in this setting any new security will be redundant and can be priced according to non-arbitrage techniques. As a consequence of the non-existence of arbitrage possibilities, the price C of a European derivative security will be given by the following expression

$$C_t = E_t e^{-\int_t^T r_u du} C_T \tag{3}$$

where r is the stochastic interest rate of (1) whose dynamics are known, C_T is the final payoff of the security at time T (possibly state-dependent) and E_t represents expected value with respect to the "equivalent risk neutral probability" conditional on information available at t.

The continuous time version of Harrison and Kreps (1979) states that in our complete markets setting any security (whose price we represent by C) will follow a stochastic differential equation of the type (see Harrison and Pliska (1981)),

$$\frac{dC_t}{C_t} = r_t dt + \gamma_t dW_t \tag{4}$$

where r is the interest rate of equation (1) and γ is the volatility of the security.

When we know the dynamics of the different stochastic processes involved in the pricing formula (3) we can use Monte Carlo simulation in order to approximate the value C of the asset. In order to do that we discretize the continuous time processes involved in the following way: assume that the dynamics of a given stochastic process whose value we denote by V are given by

$$dV_t = \alpha_t dt + \nu_t dW_t \tag{5}$$

where α and ν are parameters known at t (they might themselves satisfy stochastic equations of the type expressed in (5)). We discretize equation (5) in the following way,

$$\Delta V_t = \alpha_t \Delta t + \nu_t z_t \tag{6}$$

where Δt represents the appropriate time period and z_t are independent normal random variables with zero mean and standard deviation $\sqrt{\Delta t}$. If we consider a sequence of time steps $\{t, t+\Delta t, ..., T\}$ by dividing the total time interval $T-t$ in n subperiods of length $\Delta t = \frac{T-t}{n}$ equation (6) becomes

$$\Delta V_t = V_{t+1} - V_t = \alpha_t \frac{T-t}{n} + \nu_t z_t \tag{7}$$

In order to implement the procedure, a large number of paths of the different stochastic processes involved are generated and then the expectation in (3) is computed by averaging over the different paths. There are different procedures to speed up the convergence to the true value of the security ("variance reduction" techniques). For a detailed analysis of the application of Monte Carlo simulation and the different "variance reduction" techniques see Boyle (1977) and Boyle, Broadie and Glasserman (1997).

It is well known that as the number of steps and the number of runs increases the result of the Monte Carlo simulation converges to the "true" value of (3). Also, it is important to allocate optimally the available computational power between the number of steps and the number of runs. Duffie and Glynn (1995) have an important result in that respect. In fact, the previous discretization is the easiest of all possible discretizations, but it is not the only possible discretization. The discretization given by (6) is know as the Euler (first order) discretization. More sophisticated (higher order) discretizations are reviewed in Kloeden and Platen (1992).

3 Construction of a Hedging Portfolio

As we tried to make clear in the previous section, Monte Carlo simulation provides a procedure to approximate numerically the price of a contingent claim. In order to use Monte Carlo simulation we need a non-arbitrage setting where the dynamics of the underlying processes are known. That allows us to represent the price of the contingent claim as the expectation of a functional of the Brownian Motion Process that is the source of uncertainty. By replicating a large number of paths of the functional of the Brownian Motion Process we can approximate the price of the contingent claim.

In practice, constructing a hedging portfolio is often as important as pricing the derivative; a hedging portfolio is a portfolio of securities (in general the underlying security and the risk-free asset) that behaves (in theory) exactly as the derivative. A long (short) position in the derivative will then be offset by a short (long) position in the hedging portfolio. The existence of a hedging portfolio is guaranteed by the assumption of complete markets (that allows us to price derivatives using Monte Carlo simulation) and, therefore, the derivative is a "redundant security." In order to construct the hedging portfolio we need to compute the weight of the portfolio that will have to be invested in the underlying security (the remainder will go to the risk-free asset).

More explicitly, consider the price of a contingent claim that we represent by C (for example, a call option). This contingent claim depends on the dynamics of an underlying security that satisfies equation (2). Suppose that we know that the contingent claim is a function of the underlying security, time and other constant parameters, that is,

$$C = C(t, S_t, \cdot) \tag{8}$$

From Ito's lemma, equations (2) and (8) imply that

$$dC = C_t dt + C_S S(rdt + \sigma dW) + \frac{1}{2} C_{SS} S^2 \sigma^2 dt \tag{9}$$

where we have changed the notation so that subscripts now represent partial derivatives (with respect to t and to S) and we drop the time indicators.

We can replicate the dynamics of the contingent claim by investing the amount $(C - C_S S)$ in the risk-free asset, and investing $C_S S$ in the underlying security. C_S is the "delta" of the derivative (or the number of units of the underlying security that have to be bought in order to replicate the dynamics of the derivative). It is clear from the analysis in the previous section that Monte Carlo simulation does not provide the value of the "delta."

Two major approaches have been suggested in the literature in order to estimate the "delta" of the contingent claim (that is, C_S) when Monte Carlo simulation is used for pricing purposes. On one hand we have the "indirect resimulation" procedure that approximates the derivative using finite differences (FDM). This approach relies on the fact that

$$\frac{\partial C}{\partial S} \simeq \frac{\triangle C}{\triangle S} \tag{10}$$

In order to implement this procedure, first, two different starting points of the underlying security are assumed and then the price of the derivative is computed for each of the prices of the underlying security: the estimate of delta is the relation between the change in prices of the derivative and the different prices of the underlying. This method requires a large number of simulations and is, therefore, costly. (see Boyle, Broadie and Glasserman 1997 for more on this method).

Alternatively, direct methods use the same simulations generated to price the security. "Pathways methods" rely on the structure of the payoff, while "likelihood ratios" rely on the probability density of the underlying security (see Broadie and Glasserman (1996), Fu and Hu (1995), Fournie et al. (1999) for a detailed explanation of these procedures). Although the computational advantages of the direct methods are undeniable, their application is only possible for certain types of payoffs of the underlying security. In general, then, "indirect resimulation" procedures (as the FDM) has been the only general method available.

However, another problem of the FDM is that the computational cost will increase linearly with the number of dimensions of the derivative considered. To be more specific, consider the following setting, more complex (but also more general) than the setting presented above. Suppose that instead of a constant interest rate r as in equation (1) the interest rate satisfied a stochastic differential equation as,

$$dr_t = \alpha(r_t)dt + \kappa(r_t)dW_t \tag{11}$$

with α and κ some functions of r_t. In that case, equation (8) becomes

$$C = C(t, r_t, S_t, \cdot) \tag{12}$$

and Ito's lemma yields,

$$\begin{aligned} dC &= (C_t + C_S S r + C_r \alpha + \frac{1}{2} C_{SS} S^2 \sigma^2 + \frac{1}{2} C_{rr} \kappa^2 + C_{Sr} S \sigma \kappa) dt \\ &+ (C_S S \sigma + C_r \kappa) dW \end{aligned} \tag{13}$$

This is called a two-dimensional problem because both the dynamics of the underlying security and the dynamics of the interest rate affect the dynamics of the derivative security. In order to construct a portfolio that replicates the dynamics of the call option we would have to estimate both the derivative of the call with respect to the underlying security as well as the derivative with respect to the interest rate (or "rho"), represented by C_r in the previous equation. We could use the FDM explained above to estimate both derivatives; the procedure is the same but should be performed twice, therefore the computational cost of the estimate would be twice as high. In fact, three-dimensional (or more) settings are quite common.

However, it has to be pointed that we have assumed that there is only a single Brownian Motion Process explaining the dynamics of this model. It could be possible to have a more general setting where the dynamics of the different stochastic securities and finance variables (as the interest rate) were explained by more than one Brownian Motion Process. In fact, Monte Carlo simulation is just as easy to implement (only that two independent series of "pseudo-random" numbers have to be generated). The procedure to construct the hedging portfolio would be exactly the same described above: the computational cost of the FDM is independent of the number of Brownian Motion Processes, it only depends on the number of dimensions of the problem. Certainly, assuming more than one Brownian Motion Process may yield a more realistic setting, however it is not common in practice because in that case the hedging portfolio may require more securities: for example, with two Brownian Motion Processes, the hedging portfolio would typically have to include at least two risky securities (the underlying and a different security).

In this paper we suggest an alternative "indirect method" that allows to construct a replicating portfolio in a simple way, similar to the FDM, regardless of the setting. As we have shown, the computational cost of the FDM increases with the number of dimensions of the problem, but is independent of the number of Brownian Motion Processes. The computational cost of the method we suggest is independent of the number of dimensions of the problem, but increases linearly with the number of Brownian Motion Processes. However, as we have argued before, in practice it is common to assume several dimensions, but only as many Brownian Motions as the available risky securities. That makes the method we suggest more attractive for many interesting cases. We devote the next section to explain that method that we call the "retrieval of volatility method."

4 The Retrieval of Volatility Method

The "retrieval of volatility method" (from now RVM) relies on the fact that in order to apply Monte Carlo simulation we need a complete markets setting and that allows us to represent the dynamics of any security as a semimartingale. The key will be to use the martingale representation of the price of the option to estimate numerically the "diffusion parameter" of the semimartingale, which is the parameter we need to compute the hedge.

The advantage of the RVM is that it allows to construct the replicating portfolio with accuracy in a simple way regardless of the complexity of the model considered and at a computational cost that is independent of the number of dimensions of the model.

More explicitly, consider the call option price on the underlying security and its dynamics described by equation (9). The risk-neutral dynamics of the price of this security can be expressed as

$$d\bar{C}_t = \nu_t dW_t, \tag{14}$$

where

$$\bar{C}_t = e^{-\int_0^t} C_t$$

is the discounted value of the process C. Since we know the price of the underlying security S and its volatility σ, in order to construct a hedging portfolio with the risk-free asset and the underlying security, we would need to know the parameter ν

of equation (14). In fact, this is equivalent to knowing C_S from equation (9) or C_S and C_r from equation (13). The advantage of this method is that the computational cost is independent of the number of dimensions considered. For one dimension it is equivalent to the FDM, but, unlike with that method, the computational cost does not increase with the number of dimensions.

The RVM we suggest here aims at estimating the parameter ν instead of the sensitivity C_S. Observe that the parameter ν can be obtained from the quadratic variation of the option dynamics as given by equation (14). That is, we have

$$E_t[\bar{C}_{t+\Delta t} - \bar{C}_t]^2 = E_t \int_t^{t+\Delta t} \nu_u^2 du.$$

Therefore,

$$\nu_t = \left(\lim_{\Delta t \to 0} E_t \frac{(\Delta \bar{C}_t)^2}{\Delta t} \right)^{1/2} \tag{15}$$

where

$$\Delta \bar{C}_t = \bar{C}_{t+\Delta t} - \bar{C}_t.$$

Alternatively, we can get ν_t as an instanteneous covariance between the price process and the Brownian motion process. That is,

$$E_t \Delta \bar{C}_t \Delta W_t = E_t \int_t^{t+\Delta t} \nu_u du.$$

Therefore,

$$\nu_t = \lim_{\Delta t \to 0} E_t \frac{(\bar{C}_{t+\Delta t} - \bar{C}_t) z_t}{\Delta t} \tag{16}$$

where $z_t = \Delta W_t$ is a normal random variable with mean zero, variance Δt.

In order to estimate ν we proceed as follows: as a result of the Monte Carlo simulation we have computed the value of $\bar{C}$ at moment t. In order to do that we simulated a number K of paths of the Brownian Motion Process W. Each path was discretized into n steps of size $\Delta t = \frac{T-t}{n}$. For step k of path i we generated a "pseudo-random" number with distribution $\mathcal{N}(0, \Delta t)$, denoted z_k^i. We now take the step one "pseudo-normal " numbers, namely z_1^i. Using Monte Carlo simulation it is straightforward to compute the value of $\bar{C}$ at moment $t + \Delta t$ for an upgrade of the underlying financial parameters that uses z_1^i as the shock experienced at moment t. We denote that value by $\bar{C}_{t+\Delta t}(z_1^i)$. To be specific, assume for simplicity that we have a European security with payoff C_T at maturity. We compute $\bar{C}_{t+\Delta t}(z_1^i)$ by Monte Carlo computation of $E_{t+\Delta t}\bar{C}_T$ as follows: The initial condition of the process S is

$$S_{t+\Delta t}(z_1^i) = S_t r_t \Delta t + \sigma_t z_1^i.$$

We simulate M paths of S starting with $S_{t+\Delta t}(z_1^i)$. Similarly for other processes that have to be simulated. This gives us M values of the final payoff C_T, denoted by $C_T^j(z_1^i)$, $j = 1, \ldots, M$. Then the estimate for $\bar{C}_{t+\Delta t}(z_1^i)$ is

$$\frac{1}{M} \sum_{j=1}^{M} \bar{C}_T^j(z_1^i).$$

Therefore, we have a (approximate) value of C at moment $t + \Delta t$ conditional on a shock z_1^i happening at moment t, and we find it for all $i = 1, \ldots, K$. We now use (15) and compute the estimate

$$\hat{\nu}_t = \left(\frac{1}{K} \sum_{i=1}^{K} \frac{(\bar{C}_{t+\Delta t}(z_1^i) - \bar{C}_t)^2}{\Delta t} \right)^{1/2} . \tag{17}$$

In summary, the method consists of two steps:

1. First, for a given realization of the Brownian motion process z_1^i we upgrade the values of all the underlying stochastic variables, S, r, σ... and compute the discounted value of the option at moment $t + \Delta t$ contingent on that realization of the Brownian motion process at moment t, and we denote it by $\bar{C}_{t+\Delta t}(z_1^i)$. This step is similar to the first step of the FDM (for one dimension).

2. Then we compute the portfolio ν using equation (17). This is different (but similar in terms of the computational cost) to the second step of the FDM.

Of course, the difference in computational cost is clear for more than one dimension. For example, for the two-dimensional case represented by equation (13) in the FDM the corresponding two steps will have to be performed both in order to compute C_S and C_r while the RVM only requires it once. As we increase the number of dimensions, the computational cost of the FDM increases linearly while the computational cost of the RVM stays constant.

We also point out that the expression in (17) suggests that we can compute the alternative estimate

$$\hat{\nu}_t = \frac{1}{K} \sum_{i=1}^{K} \frac{(\bar{C}_{t+\Delta t}(z_1^i) - \bar{C}_t) z_1^i}{\Delta t} \tag{18}$$

Clearly, the cost of implementing (17) or (18) is the same; however, as we will see later, (18) will have to be used in the multi-factor case.

Remark: The speed of computation can be significantly improved for European options, namely those that pay an amount C_T at maturity T. Indeed, in that case

$$\bar{C}_{t+\Delta t} = E_{t+\Delta t} \bar{C}_T$$

and by the law of iterated conditional expectations, (16) can be written as

$$\nu_t = \lim_{\Delta t \to 0} E_t \frac{(\bar{C}_T - \bar{C}_t) z_t}{\Delta t} \tag{19}$$

Consequently, instead of computing two expected values, one conditional on information up to time t, and the other conditional on information up to time $t+\Delta t$, we only have to do the former. In other words, we can set $M = 1$. However, our numerical experiments show that it might be more efficient to compute both expected values by Monte Carlo. This is because $Var(E[X|Y]) \leq Var(X)$. It seems to be an open problem what the optimal ratio is, between the numbers K and M of paths simulated to compute the two expectations. We keep M quite low, between 10 and 50, in the numerical results reported below.

4.1 Error Bounds

We will dwell briefly on the precision of the approximation that results from the algorithm suggested in this paper. The algorithm consists of two steps and each of the two steps will induce an error.

The first source of error is a plain Monte Carlo simulation procedure. When we use an Euler discretization scheme, it is well known that the size of the error is of the order $\frac{1}{\sqrt{K}} + \frac{1}{n}$ where K is the number of paths generated and n the number of discretization points (that is, $n = \frac{T}{\Delta t}$). Duffie and Glynn (1995) consider the problem of optimal allocation of computational time available and show that, under Euler discretization, it is optimal to take n proportional to $\sqrt{K}$. We take $n = \sqrt{K}$ in our computations.

The second source of error is the limiting procedure of (15) or (16), when $\Delta t \to 0$. The size of this error is similar to that of using the perturbation method for computing "delta" of an option. In such a case (see Glynn (1989)) the size of the error is typically of magnitude $\frac{1}{K^{1/4}}$, when $n = \sqrt{K}$. This error is asymptotically larger than the one from the first source, and, therefore, the error that will determine the asymptotic precision of the results. The size of the error is, in any case, manageable. In the next section we discuss some numerical computations. All the exercises were performed on standard desktop PC's and computational times ranged (depending on T) between a few minutes for $T = .1$ to about an hour for $T = .5$, with a typical standard error of the order of .001. We used antithetic variables, but many other variance reduction techniques could be used to increase the speed/accuracy of computations.

5 Examples

In order to exemplify the basic procedure described in the previous section we consider

- (i) a European call option in the geometric Brownian Motion model

- (ii) a European call option in a model with stochastic volatility and stochastic interest rate

- (iii) a Barrier call option in the geometric Brownian motion model

As we discussed in Section 4, the RVM will be preferable to the FDM in cases of more than one dimension. The purpose of the exercise is to illustrate in detail the RVM. Computational cost and accuracy are independent of the number of dimensions.

Table 1 gives the results for a call option in the Black-Scholes model, with three different strike prices K and three different times to expiry T. Table 2 gives results in a model in which the interest rate and the volatility are stochastic and given by

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r\sqrt{r_t}dW_t \tag{20}$$

$$d\sigma_t = \kappa_\sigma(\bar{\sigma} - \sigma_t)dt + \sigma_\sigma dW_t \tag{21}$$

for some specific values of the parameters. Table 3 presents computations for an up-and-out barrier call option in the Black-Scholes model.

In case the explicit formula for the (discounted) payoff is known, in terms of a function of the terminal value of the Brownian motion W_T, such as the first, Black-Scholes example we are considering, we would not have in fact to discretize the horizon in order to implement Monte Carlo simulation (but also, Monte Carlo simulation would not be necessary, since there is a closed form expression for the option). In general, however, discretization will be necessary.

The exercise we perform is the following: we split the time horizon in n periods. We generate K paths and we average over them. We use antithetic variables in order to reduce the variance (see Boyle (1977)). We proceed as follows, taking advantage of (19): For the $i-$th path, we simulate n independent normal variables with mean zero and variance Δt, denoting the first of them z_i^1. Using these, we obtain the values of $\bar{C}_T$. We repeat this K times, obtaining corresponding values $\bar{C}_T^i$, z^i, $i = 1, \ldots, K$. We then find the volatility of the price proces via

$$\nu_t = \lim_{\Delta t \to 0} \frac{1}{K} \sum_{i=1}^{K} \frac{(\bar{C}_T^i - \bar{C}_t^i)z_t^i}{\Delta t} \tag{22}$$

Actually, as mentioned above, we get somewhat better results if we use (18), with $\bar{C}_{t+\Delta t}(z_1^i)$ computed via Monte Carlo using M paths, where M is less than a hundred.

6 Multi-factor Models

In the previous sections we have focused on cases where there is a single Brownian motion process. The number of Brownian motion processes that govern the dynamics of the economic setting is chosen independently of the number of dimensions. In fact, in order to hedge we will need as many securities (besides the cash account) as the number of Brownian motion processes modeled. In this Section we describe the implementation of the RVM in a setting with more than one Brownian motion process.

Suppose, for example, that we have an option on two underlying securities that follow stochastic differential equations of the type of (2), as follows,

$$\begin{aligned}\frac{dS_t^1}{S_t^1} &= rdt + \sigma^{11}dW_t^1 + \sigma^{12}dW_t^2 \\ \frac{dS_t^2}{S_t^2} &= rdt + \sigma^{21}dW_t^1 + \sigma^{22}dW_t^2\end{aligned} \tag{23}$$

where W^1 and W^2 are two independent standard Brownian motion processes (or factors) and we assume that σ^{11}, σ^{12}, σ^{21} and σ^{22} are constant. In fact they could be also stochastic and depend on W^1 and W^2, but the notation becomes more involved.

Now consider the option whose payoff depends on both securities. In a complete markets setting (that would require a risk-free asset as the one described by equation (1)) we could price the derivative using a non-arbitrage argument. In fact, the Monte Carlo simulation approach explained in Section 2 could also be used to price the derivatives in this model. Furthermore, we know that similarly to the one-dimensional case, the discounted option price will follow a stochastic differential equation of the type,

$$d\bar{C}_t = \nu_t^1 dW_t^1 + \nu_t^2 dW_t^2 \tag{24}$$

with the volatility $\nu = (\nu^1, \nu^2)$ stochastic in general. In theory, we could construct a replicating portfolio using the underlying securities of equation (23) and the risk-free asset However, we would have to compute first the parameters ν^1 and ν^2. The RVM provides us with a possible approach to do that. In the spirit of the method explained in Section 3, we would compute the estimate of the derivative, $\hat{C}$, using Monte Carlo simulation. Obviously, in this case we would need to generate two matrices of $K \times n$ (runs $\times$ steps) "pseudo-random" numbers, each of them corresponding to each of the Brownian Motion Processes W^1 and W^2. We would take the first two shocks of each of the matrices of the "pseudo-random numbers", say $z_1^{i,1}$ and $z_1^{i,2}$. We then would compute the values of S^1 and S^2 using the previous shocks, according to our discretization,

$$\begin{aligned}S_{t+\Delta t}^1(z_1^{i,1}, z_1^{i,2}) &= S_t^1 + S_t^1[r\Delta t + \sigma^{11}z_1^{i,1} + \sigma^{12}z_1^{i,2}] \\ S_{t+\Delta t}^2(z_1^{i,1}, z_1^{i,2}) &= S_t^2 + S_t^2[r\Delta t + \sigma^{21}z_1^{i,1} + \sigma^{22}z_1^{i,2}]\end{aligned} \tag{25}$$

We would then compute the estimate of the discounted value of the derivative at $t + \Delta t$ based on the new values of the underlying securities for the given shocks, denoted $\bar{C}_{t+\Delta t}(z_1^1, z_1^2)$. We would repeat the procedure for $i = 1, \ldots, K$ to obtain K estimates $\bar{C}_{t+\Delta t}(z_1^{i,1}, z_1^{i,2}), i = 1, 2, ..., K$. Now we apply the equivalent of expression (18) for the case of two Brownian motion processes and estimate ν^1 as

$$\hat{\nu}_t^1 = \frac{1}{K}\sum_{i=1}^{K} \frac{(\bar{C}_{t+\Delta t}(z_i^{i,1}, z_i^{i,2}) - \bar{C}_t)z_1^{i,1}}{\Delta t} \tag{26}$$

and similarly for ν^2. It is clear that the incremental computational cost in the multifactor case comes from the need to generate two series of independent "pseudo-random" numbers to price the derivative: the computational cost of the hedging portfolio is similar for one or two factors. In particular, as before, the computational cost becomes much lower in the case of European-type payoffs.

7 Conclusions

In this paper we suggest a procedure that allows us to compute the replicating portfolio of a security priced using Monte Carlo simulation. Computation of the weights necessary to construct a replicating portfolio (the "delta" of a security in the one-dimensional case) is very often as important as the appropriate valuation of the security. However, Monte Carlo simulation does not immediately provide the hedging components. The method used in practice consists in perturbing the price of the underlying and approximating the derivative ("delta") by the change in price of the option by the perturbation in value of the security. The result is an approximation to the true sensitivity; the number of estimates to be computed using this procedure increases with the number of dimensions of the model. Here we suggest an alternative procedure by computing the quadratic variation parameter of the semimartingale representation of the derivative (we call it RVM or "retrieval of volatility method"). We show that it yields a very general and simple way to compute the weights of a replicating portfolio with the advantage over the classical method that the computational cost does not increase with the number of dimensions. The method applies (with no additional computational cost) to cases where there are several Brownian motion processes. It is also applicable to American and Exotic options.

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Table 1: Call option, Black-Scholes model

$r = 0.06; \sigma = 0.1; S_0 = 100$

	$T = .1$		$T = .3$		$T = .5$	
K	Δ	P	Δ	P	Δ	P
90	.997	10.539	.988	11.630	.972	12.736
100	.586	1.577	.636	3.170	.677	4.514
110	.002	.002	.083	.205	.187	.704

Table 2: Call option, Stochastic Parameters

$S_0 = 100; \kappa_\sigma = .695; \sigma_\sigma = .21; \bar{\sigma} = .1; \sigma_0 = .1;$
$\kappa_r = .0824; \sigma_r = .0364; \bar{r} = .06; r_0 = .06$

	$T = .1$		$T = .3$		$T = .5$	
K	Δ	P	Δ	P	Δ	P
90	.992	10.539	.975	11.820	.963	12.650
100	.705	1.578	.802	3.150	.825	4.540
110	.074	.082	.327	.880	.482	1.909

Table 3: Up-and-out call, B.S. model

$r = 0.06; \sigma = 0.1; S_0 = 100; L = 110$

	$T = .1$		$T = .3$		$T = .5$	
K	Δ	P	Δ	P	Δ	P
90	1.01	10.53	.75	11.11	.31	10.36
100	.58	1.58	.50	2.86	.27	3.05
110	.002	.002	.029	.09	.021	.14