Monte Carlo Computation of Optimal Portfolios in Complete Markets*

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Abstract

We introduce a method that relies exclusively on Monte Carlo simulation in order to compute numerically optimal portfolio values for utility maximization problems. Our method is quite general and only requires complete markets and knowledge of the dynamics of the security processes. It can be applied regardless of the number of factors and of whether the agent derives utility from intertemporal consumption, terminal wealth or both. We also perform some comparative statics analysis. Our comparative statics show that risk aversion has by far the greatest influence on the value of the optimal portfolio.

Key Words: Utility maximization, Monte Carlo methods.
JEL classification: C15, G11.

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1 Introduction

The derivation of the optimal portfolio of a rational investor is a central problem in asset pricing. Although the interest of closed form solutions that would allow to derive equilibrium implications is obvious, the increase in computational power along with the lack of closed form solutions for many interesting cases have triggered an interest in numerical methods as a possible answer to the problem. In this paper we suggest a method purely based on Monte Carlo simulation that allows to solve the problem in complete markets.

Merton (1971) introduced a methodology to attack the problem of a rational investor with time additive preferences that chooses how to allocate her wealth between consumption and the existing securities. In his setting, computation of the optimal consumption and investment strategies requires the solution of a PDE (partial differential equation). However, that PDE only has a closed form solution in a handful of cases. Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) introduced martingale methods to solve the problem of an utility optimizing investor. Martingale methods allow to consider more general settings than dynamic programming methods and in general allow to compute the optimal consumption policy of the investor. But optimal portfolios, in general, cannot be computed in closed form when martingale methods are used.

A large number of papers have recently undertaken the problem of computation of optimal portfolios. Campbell and Viceira (1999) and Barberis (1998) use numerical approximations to find optimal portfolios in a discrete time setting. In continuous time, Kim and Omberg (1996) solve the PDE in closed form for a specific parametrization of the model. Liu (1998) finds a closed form solution for a general class of parameterizations for an agent whose utility depends only on terminal wealth. Wachter (1999) solves also a specific case but one that allows for intertemporal consumption. Brennan, Schwartz and Lagnado (1997) and Xia (1999) solve numerically the PDE also for specific (but more general) parameterizations of the utility function. Finally, Detemple, Garcia and Rindisbacher (1999) compute the Malliavin derivatives of the processes and then use Monte Carlo simulation in order to retrieve the optimal portfolio. In this paper we introduce a pure Monte Carlo simulation approach, very easy to implement and that can be applied whenever two conditions are met (this two conditions are also required by the method introduced in Detemple, Garcia and Rindisbacher (1999)):

- Markets are complete, that is, the number of non-redundant stocks and the number of Brownian motion processes that explain the uncertainty of the economy are equal.
- We know the dynamics of all the processes involved (the *expanded opportunity set* is markovian).

The method we introduce here can be applied to any type of time additive utility function and any parametrization of the security processes, regardless of whether the agent derives utility from final wealth, intertemporal consumption or both, and regardless of the number of Brownian motion processes that explain the uncertainty of the economy. The advantage of Monte Carlo simulation is that it is very easy to implement and converges reasonably fast. Monte Carlo simulation has been increasingly popular for pricing derivatives since its introduction in finance by Boyle (1977). However, it had not been considered as a tool to solve optimal portfolios until the work of Detemple, Garcia and Rindisbacher (1999).

Our method uses the fact that the optimal portfolio of the investor is part of the instantaneous standard deviation of the optimal wealth process (in a contemporaneous paper, Lioui and Poncet (2000) use that fact in order to derive optimal portfolios in a stochastic interest rate setting). The latter can be computed by finding second moments (the expected value of the squared change in the wealth level), and, therefore, Monte Carlo methods can be applied. It should be remarked that our method is not necessarily the most efficient one, but the combination of its simplicity and flexibility should prove attractive as a general tool, and future research could potentially further improve its properties. Moreover, since this is a numerical method, it does not provide an explicit dependence of the optimal portfolio on the current state; instead, it computes numerically the today's value of the optimal portfolio from the today's values of the state variables. Hence, this computation would have to be repeated from one day to the next.

The structure of the paper is as follows. In Section 2 we describe the setting and give an intuition for the method. In Section 3 we apply the general idea to the computation of the optimal portfolio. In Section 4 we do some exercises and perform some comparative statics. In Section 5 we explain the extension of the method to the multifactor case. We close the paper with some conclusions.

2 General Method

2.1 Securities

Here we describe the financial assets the investor can choose among. In order to illustrate the method we will define some specific, although fairly general dynamics. As it will be clear later the method is not restricted to the set of prices defined here. In order to simplify the notation, we consider real prices, expressed in terms of the unique consumption good. There are two types of securities. First, there are n stocks whose price satisfies the following dynamics,

$$\frac{dS_t^i}{S_t^i} = \mu_t^i dt + (\sigma_t^i)^\top dW_t \tag{1}$$

where W is a vector of n independent standard Brownian motion processes. Realizations of these n Brownian motion processes define the path followed by the economy. Additionaly, μ^i and σ^i represent the drift and volatility of the stock process i and are possibly stochastic (we discuss their dynamics later). The second type of security is a risk-free security, called "bank account", whose price B evolves according to the following dynamics,

$$dB_t = B_t r_t dt \tag{2}$$

where r is possibly stochastic interest rate.

Uncertainty in this economy is given by realizations of the n dimensional Brownian motion process. We assume that the number of stocks and the number of Brownian motion processes is the same. Besides, we assume that the matrix Σ formed by stacking the n σ^i vectors is non-singular at every point in time t: this is equivalent to assuming that our *markets are complete.* In order to simplify the notation, we will assume that $n = 1$ and therefore, $\Sigma = \sigma$. In the last section of the paper we consider the case of $n = 2$ (and $n > 2$ would be analogous).

We now consider the dynamics of the different parameters of the model. We first define the market price of risk θ as

$$\theta = \frac{\mu - r}{\sigma} \tag{3}$$

As we mentioned above, we consider the possibility of μ, r and σ stochastic. The only restriction that we impose is that all the parameters, plus those of any existing state variables depend on the n dimensional Brownian motion process that describes the uncertainty in this economy. In other words, *we only require the expanded opportunity set to be markovian.* In our examples we will restrict ourselves to the following dynamics. With respect to the interest rate, we assume

$$dr_t = \left(a_r + b_r (r_t)^{l_r} + c_r (\theta_t)^{p_r}\right) dt + \left(d_r + f_r (r_t)^{q_r} + g_r (\theta_t)^{v_r}\right) dW_t \tag{4}$$

where a_r, b_r, c_r, d_r, f_r, g_r, l_r, p_r, q_r and v_r are constant. With respect to θ, we assume that it satisfies

$$d\theta_t = \left(a_\theta + c_\theta (\theta_t)^{p_\theta}\right) dt + \left(d_\theta + g_\theta (\theta_t)^{v_\theta}\right) dW_t \tag{5}$$

where a_θ, c_θ, d_θ, g_θ, p_θ and v_θ are constant. A subset of the previous dynamics are the "affine" models studied in Duffie, Pan and Singleton (2000).

2.2 General Idea

We now explain the general idea of the method we will use to compute the optimal portfolio of the individual. In the previous economic setting, consider the expression

$$C_t = E\left[\int_t^T f(r_s, \theta_s, W_s) ds \;\middle|\; \mathcal{F}_t\right] \tag{6}$$

where the information up to moment t, represented by $\mathcal{F}_t$, is the path of the Brownian motion process up to t. It is well known that the expression on the right-hand side of equation (6) satisfies a stochastic differential equation of the type

$$dC_t = \alpha_t dt + \nu_t dW_t \tag{7}$$

where α and ν are again possibly stochastic and path-dependent (this is due to the martingale representation theorem; see Karatzas and Shreve 1992, for example). Although in general a closed form expression for ν does not exist, the computation of that parameter is the key in many problems in finance, like hedging of contingent claims, or (the problem we consider in this paper) the optimal portfolio of an utility maximizing investor. In this paper we suggest to use Monte Carlo simulation in order to compute the process ν (and, therefore, the optimal portfolio of the individual). The method we introduce here can be applied whenever Monte Carlo simulation is possible and two requirements are satisfied: *complete markets* and *markovian expanded opportunity set* (regardless of the number of parameters). Monte Carlo simulation has the advantage that it is very easy to implement and converges fairly quickly.

Monte Carlo simulation was introduced in finance for the pricing of derivatives by Boyle (1977). Boyle, Broadie and Glasserman (1997) offer a detailed survey of the application of Monte Carlo simulation to the pricing of derivatives. In fact, the use of numerical methods in finance has been restricted until very recently to the pricing and hedging of derivatives. Only recently numerical methods have started to be used as a way to solve the problem of finding the optimal portfolios. We mention in discrete time, Campbell and Viceira (1999) and Barberis (1998). In continuous time, Brennan, Schwartz and Lagnado (1997) and Xia (1999) use numerical methods to solve the PDE's that result from the dynamic approach. Detemple, Garcia and Rindisbacher (1999) use Monte Carlo simulation combined with the computation of the Malliavin derivatives. In this paper we introduce a method based exclusively on Monte Carlo simulation.

When Monte Carlo simulation is applied to financial problems, an expression of the type of (1) is discretized in the following way,

$$S_{t+\Delta t} - S_t = S_t \left(\mu_t \Delta t + \sigma_t z_t\right) \tag{8}$$

where z is a pseudo-random number drawn from a hypothetical normal distribution with zero mean and standard deviation $\sqrt{t}$ (for more on generating pseudo-random numbers see Press, Teukolsky, Vetterling and Flannery (1992)). The time horizon T is divided in n intervals of size Δt and by generating n values of z we will have a discretized version of a possible path of S.

Consider the problem of estimating numerically the value C_t of equation (6): a large numbers of paths of W will be simulated and used in the dynamics of all the relevant processes in the form explained above; in order to compute the expected value of (6) the average of all of the paths will be taken. Here we suggest to use a similar technique in order to derive ν, the volatility term of the expression (7). From (6), the volatility ν is[1]

$$\begin{aligned}\nu_t &= \lim_{\Delta t \to 0} \left(E \left[\frac{(C_{t+\Delta t} - C_t - \alpha_t \Delta t)^2}{\Delta t} \mid \mathcal{F}_t \right] \right)^{1/2} \\ &= \lim_{\Delta t \to 0} \left(E \left[\frac{(C_{t+\Delta t} - C_t)^2}{\Delta t} \,\middle|\, \mathcal{F}_t \right] \right)^{1/2} \end{aligned} \tag{9}$$

We can ignore the effect of the drift α because it multiplies Δt, that converges to 0 and does it faster than in the denominator because the numerator is squared. Alternatively, we can compute ν as

$$\begin{aligned}\nu_t &= \lim_{\Delta t \to 0} E \left[\frac{(C_{t+\Delta t} - C_t)(W_{t+\Delta t} - W_t)}{\Delta t} \mid \mathcal{F}_t \right] \\ &= \lim_{\Delta t \to 0} E \left[\frac{(C_{t+\Delta t} - C_t)(Z_t)}{\Delta t} \mid \mathcal{F}_t \right] \end{aligned} \tag{10}$$

where $Z_t := W_{t+\Delta t} - W_t$ is a normally distributed random variable with mean zero and variance Δt. Informally, in (9) we compute the instantaneous standard deviation of the stochastic process while in (10) we compute the instantaneous covariance between C and the Brownian motion process W (which, clearly, is also ν). As we will see, however, the expression in (10) is more convenient when there is more than one Brownian motion process (the "multifactor" case that we will consider in the last section of the paper).

At moment t we know the value C_t, but in order to compute numerically the expression in (10) we need to generate a number of values of $C_{t+\Delta t}$. We do not know the dynamics of C (that is in fact the problem we are trying to solve). However, from (6) we know that $C_{t+\Delta t}$ is the expected value of some function of the parameters of the model (that depend on the path of the Brownian motion process) *whose dynamics we know.* This is the fact that we will exploit and will allow us to compute optimal portfolios. We explain the exact procedure in the next section.

3 Computation of the Optimal Portfolio

We consider the problem of a rational, utility maximizing, investor. The utility of this investor is the result of a bequest target, intertemporal consumption or both. There is a single consumption good that we will use as numeraire. Individuals receive an initial endowment in units of the consumption good that they can either consume or invest in the financial markets.

[1] This is a heuristic derivation. For a formal treatment of the "quadratic variation" see Karatzas and Shreve (1992)).

In order to simplify the presentation, we will focus on the power utility case, i.e., on the following two problems:

$$
\begin{aligned}
U(X_T) &= \max_{\pi} E\left[e^{-\delta(T-t)}\frac{X_T^{\gamma}}{\gamma}|\mathcal{F}_t\right] && (11)\\
U(c) &= \max_{(\pi,c)} E\left[\int_t^T e^{-\delta(s-t)}\frac{c_s^{\gamma}}{\gamma}ds|\mathcal{F}_t\right] && (12)
\end{aligned}
$$

where π represents the trading strategy (to be described below), c is the consumption rate, X is the wealth level of the investor, and δ is the subjective discount rate (that in order to simplify the notation we will assume constant, and in fact zero in the examples, but this is not necessary for our method). The initial, exogenous wealth level of the investor is X_0. For simplicity, we only consider CRRA (constant relative risk aversion) utilities, with γ as the parameter that characterizes the degree of risk aversion. However, the same method can be applied to other, time additive utility functions.

The investor can allocate her wealth either in consumption c or in any of the securities described above. We denote by π the amount of wealth invested in the stock. The wealth process X of the investor satisfies,

$$dX_t = (\pi_t\mu_t + (X_t - \pi_t)r_t - c_t)\,dt + \pi_t\sigma_t dW_t. \qquad (13)$$

The previous problem was first considered in continuous time by Merton (1971), using dynamic programming. Subsequently, Karatzas, Lehoczky and Shreve (1987) and Cox and Huang (1989) introduced martingale methods that allow to solve the problem using Lagrange multipliers (as a static problem). We briefly recall the approach of those two papers. Using the notation introduced in (3), we can write (13) as

$$
\begin{aligned}
dX_t &= (r_tX_t - c_t)\,dt + \pi_t\,(\sigma_t dW_t + \theta_t dt)\\
&= (r_tX_t - c_t)\,dt + \pi_t\sigma_t d\tilde{W}_t && (14)
\end{aligned}
$$

where $\tilde{W}$ is a Brownian motion process with respect to Q, the "equivalent risk-neutral probability." We define the process

$$\xi_t = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dW_s\right). \qquad (15)$$

The present value of this process represents the continuous time Arrow-Debreu prices. The problem of the investor is equivalent to the maximization of the expression in equation (11) (with $t = 0$) subject to:

$$E^Q[e^{-\int_0^T r_s ds}X_T] = E[\xi_T e^{-\int_0^T r_s ds}X_T] = X_0 \qquad (16)$$

or, respectively, to the maximization of the expression in equation (12) subject to

$$E^Q\left[\int_0^T e^{-\int_0^s r_u du} c_s ds\right] = E\left[\int_0^T \xi_s e^{-\int_0^s r_u du} c_s ds\right] = X_0 \tag{17}$$

where ξ is given by equation (15). The dynamic problem has now become a static problem. Using standard optimization techniques we find that the respective optimal final wealth and optimal consumption strategies are given by,

$$X_T^* = \left(y e^{\int_0^T (\delta - r_s) ds} \xi_T\right)^{\frac{1}{\gamma-1}} \tag{18}$$

$$c_t^* = \left(y e^{\int_0^t (\delta - r_s) ds} \xi_t\right)^{\frac{1}{\gamma-1}} \tag{19}$$

where y is the Lagrange multiplier, the scaling constant that guarantees that the budget constraints (16) and (17) are satisfied. This number y is easily found using standard numerical techniques. We now know the optimal wealth and consumption levels, but not the optimal portfolio π^*, the main interest of this paper. We proceed to develop a method for computing π^*.

It is by now standard that in the complete markets specification that we consider here any given wealth process X can be expressed as an expectation of the type described in equation (6). More explicitly, the value of a wealth process at every point in time is the expected discounted value of future consumption and/or terminal wealth under the equivalent "risk-neutral" probability measure that depends on θ (this is in fact the result used to derive the budget constraints of (16) and (17)):

$$X_t = E^Q\left[e^{-\int_t^T r_s ds} X_T | \mathcal{F}_t\right] = \frac{1}{\xi_t} E\left[\xi_T e^{-\int_t^T r_s ds} X_T | \mathcal{F}_t\right] \ , \tag{20}$$

in the case of utility from final wealth, and

$$X_t = E^Q\left[\int_t^T e^{-\int_t^s r_u du} c_s ds | \mathcal{F}_t\right] = \frac{1}{\xi_t} E\left[\int_t^T e^{-\int_t^s r_u du} \xi_s c_s ds | \mathcal{F}_t\right] \ , \tag{21}$$

in the case of utility from intertemporal consumption, with ξ given by equation (15). But the right hand sides of (20) and (21) are expressions of the type of (6) and, therefore, satisfy

$$d\left(E^Q\left[\xi_T e^{-\int_t^T} X_T | \mathcal{F}_t\right]\right) = \alpha_t dt + \nu_t dW_t \tag{22}$$

for (20) and similarly for (21). But, comparing (13) and (22) we conclude that the corresponding portfolio is given by

$$\pi_t = (\sigma_t)^{-1} \nu_t \ . \tag{23}$$

Therefore, by retrieving numerically ν using the method explained in the previous section, we can derive the portfolio strategy.

More explicitly, from (10),

$$\begin{aligned}\nu_t &= \lim_{\Delta t \to 0} E\left[\frac{(X_{t+\Delta t} - X_t)(W_{t+\Delta t} - W_t)}{\Delta t} \mid \mathcal{F}_t\right] \\ &= \lim_{\Delta t \to 0} E\left[\frac{(X_{t+\Delta t} - X_t)(Z_t)}{\Delta t} \mid \mathcal{F}_t\right] \qquad (24)\end{aligned}$$

where $Z_t = W_{t+\Delta t} - W_t$. Therefore, the specific procedure to find the optimal portfolio π_t^*, is as follows. Denote, for $u > t$,

$$H_{t,u} := e^{-\int_t^u r_s ds} \xi_u / \xi_t. \qquad (25)$$

We have from (20)

$$\begin{aligned}X_{t+\Delta t}^* &= E[H_{t+\Delta t,T} X_T^* | \mathcal{F}_{t+\Delta t}] \\ &= E\left[H_{t+\Delta t,T}\left(y e^{\int_0^T (\delta - r_s) ds} \xi_T\right)^{\frac{1}{\gamma-1}} |\mathcal{F}_{t+\Delta t}\right] \qquad (26)\end{aligned}$$

for the the case of utility from final wealth, where we have used (18), and, from (21),

$$\begin{aligned}X_{t+\Delta t}^* &= E\left[\int_{t+\Delta t}^T H_{t+\Delta t,s} c_s^* ds |\mathcal{F}_{t+\Delta t}\right] \\ &= E\left[\int_{t+\Delta t}^T H_{t+\Delta t,s}\left(y e^{\int_0^s (\delta - r_u) du} \xi_s\right)^{\frac{1}{\gamma-1}} ds |\mathcal{F}_{t+\Delta t}\right] \qquad (27)\end{aligned}$$

in the case of utility from intertemporal consumption, where we have used (19). The right hand side of (26) and (27) will not have in general a closed form solution (exceptions are the logarithmic utility case $\gamma = 1$ and affine models considered in Duffie, Pan and Singleton (2000) and Liu (1998)). However, we can use Monte Carlo simulation to compute that right-hand side. At moment t we know initial conditions X_t, r_t, and θ_t. Next, we simulate a number K of paths of the Brownian Motion process W. Each path is discretized into n steps of size $\Delta t = \frac{T-t}{n}$. For step k of path i we generate a "pseudo-random" number with distribution $\mathcal{N}(0, \Delta t)$, denoted z_k^i. We now take the step one "pseudo-normal " numbers, denoted z_1^i. Using Monte Carlo simulation we can compute the value of $X_{t+\Delta t}^*$ at moment $t + \Delta t$ for an upgrade of the underlying financial parameters that uses z_1^i as the shock experienced at moment t. We denote that value by $X_{t+\Delta t}^*(z_1^i)$. To be specific, we compute $X_{t+\Delta t}^*(z_1^i)$ by Monte Carlo computation of $E[H_{t+\Delta t,T} X_T^* | \mathcal{F}_{t+\Delta t}]$ as follows: For the initial condition of the process ξ we can take $\xi_t = 1$, and we get the time $t + \Delta t$ value from

$$\xi_{t+\Delta t}(z_1^i) - \xi_t = -\xi_t \theta_t z_1^i.$$

We simulate M paths of ξ starting with $\xi_{t+\Delta t}(z_1^i)$. Similarly for other processes that have to be simulated. This gives us M values of the final wealth

$$X_T^* = \left(y e^{\int_0^T (\delta - r_s) ds} \xi_T\right)^{\frac{1}{\gamma-1}},$$

denoted by $X_T^{*,j}(z_1^i)$, $j = 1, \ldots, M$, and similarly, M values of $H_{t+\Delta t,T}$, denoted $H_{t+\Delta t,T}(z_1^i)$. Then the estimate for $X^*_{t+\Delta t}(z_1^i)$ is

$$\hat{X}^*_{t+\Delta t}(z_1^i) = \frac{1}{M}\sum_{j=1}^{M} H_{t+\Delta t,T}(z_1^i) X_T^{*,j}(z_1^i).$$

Therefore, we have a (approximate) value of X^* at moment $t + \Delta t$, conditional on a shock z_1^i happening at moment t, and we find it for all $i = 1, \ldots, K$. We now compute the estimate

$$\hat{\nu}_t = \left(\frac{1}{K}\sum_{i=1}^{K} \frac{(\hat{X}^*_{t+\Delta t}(z_1^i) - X_t^*) \cdot z_1^i}{\Delta t}\right) \quad . \tag{28}$$

In summary, the method consists of two steps:

1. We are at moment t and know the initial values X_t, r_t, θ_t, ξ_t. Then, for a given realization of the Brownian motion process z_1^i we upgrade the values of all the underlying stochastic variables, ξ, r, σ..., and compute the wealth value at moment $t+\Delta t$ contingent on that realization of the Brownian motion process at moment t, and we denote it by $X_{t+\Delta t}(z_1^i)$.

2. Then we compute the portfolio $\pi_t^* = {\sigma_t^{-}}^1\nu_t$ using the estimate in (28).

Clearly, this procedure is independent of the type of utility function, whether the investor derives utility only from terminal wealth, from intertemporal consumption or from both, and of the dynamics of the stochastic processes involved (as long as those dynamics are known).

Remark: The speed of computation can be significantly improved by noting the following: We have

$$X_{t+\Delta t} = E[H_{t+\Delta t,T} X_T \mid \mathcal{F}_{t+\Delta t}]$$

and by the law of iterated conditional expectations, (24) can be written as

$$\nu_t \;=\; \lim_{\Delta t \to 0} E\left[\frac{(H_{t+\Delta t,T} X_T - X_t)(Z_t)}{\Delta t} \mid \mathcal{F}_t\right] \quad . \tag{29}$$

Consequently, instead of computing two expected values, one conditional on information up to time t, and the other conditional on information up to time $t+\Delta t$, we only have to do the former. In other words, we can set $M = 1$. However, our numerical experiments show that it might be more efficient to compute both expected values by Monte Carlo. This is because $Var(E[X|Y]) \leq Var(X)$. The optimal ratio between the numbers K and M of paths simulated to compute the two expectations, seems to be an open problem. We set $K = 10,000$ and $M = 50$ in the numerical experiments reported below.

3.1 Error Bounds

We will dwell briefly on the precision of the approximation that results from the algorithm suggested in this paper. The algorithm consists of two steps and each of the two steps will induce an error. However, since the two steps are sequential (and not simultaneous) the error will be a sum (and not a product) of the errors of each of the steps. The first of the two steps, the estimate of $X_{t+\Delta t}$ as given by the representation in (26) is a plain vanilla Monte Carlo simulation procedure. When we use an Euler discretization scheme, it is well known that the size of the error is of the order $\frac{1}{\sqrt{K}} + \frac{1}{n}$ where K is the number of paths generated and n the number of discretization points (that is, $n = \frac{T}{\Delta t}$). Duffie and Glynn (1995) consider the problem of optimal allocation of computational time available and show that, under Euler discretization, it is optimal to take n proportional to $\sqrt{K}$.

The second of the two steps is the error in the limiting procedure of (24). The size of this problem is similar to that of hedging a position in options and, in such a case (see Glynn (1989)) the size of the error is of magnitude $\frac{1}{K^{1/4}}$ (when $n = \sqrt{K}$). This error is larger than the one derived of the first step and, therefore, the error that will determine the precision of the results. The size of the error is, in any case, manageable. In the next section we analyze some results for different values of the parameters of the model. All the exercises were performed on standard desktop PC's. We used C as the programming language. The computational times ranged from around five minutes ($T = 1 year$), half an hour minutes ($T = 5$), to one hour ($T = 10$), with $K = 10,000$, $M = 50$. The standard deviation of the error for these parameters is around 0.002. The method of Detemple, Garcia and Rindisbacher (1999) is likely to be faster in general, but it involves stronger conditions on the model and the computation of Malliavin derivatives of the underlying processes.

4 Analysis of Results

The basic model we study is a simplification of the general model presented in section 2. With respect to the interest rate, we consider the Cox, Ingersoll and Ross (1985) dynamics, that is,

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r\sqrt{r_t}dW_t \quad . \tag{30}$$

For the equity premium θ we first consider a simple mean-reverting process with constant volatity,

$$d\theta_t = \kappa_\theta(\bar{\theta} - \theta_t)dt + \sigma_\theta dW_t \tag{31}$$

This is in fact the model considered in Detemple, Garcia and Risdinbacher (1999). They calibrate this model to a given data set and find the following values for the parameters,

$$\bar{r} = .06, \quad \sigma_r = .0364, \quad \kappa_r = .0824 \quad \kappa_\theta = .6950, \quad \bar{\theta} = .0871,$$

$$\sigma_\theta = .21, \quad \sigma(t) \equiv .2, \quad r(0) = .06, \quad \theta(0) = .1.$$

With respect to the utility function, they only consider the case of utility from terminal wealth. We will consider both cases as expressed in equations (11) and (12). In the case of utility from terminal wealth, with time horizon $T = 1$, risk-aversion characterized by $\gamma = -1$ and initial wealth $X_0 = 1$ (so that the portfolio can be interpreted as the proportion of current wealth invested in the risky stock), which is the case considered in Detemple, Garcia and Risdinbacher (1999), we obtain an optimal investment in the risky security of $\pi^* = 0.252$. In other words, about 25% of the wealth is to be kept in stock.

In Table 1 we consider the sensitivity of the portfolio both to the changes in risk-aversion and horizon. Risk-aversion is measured by the parameter γ: the lower γ (which has to be smaller than 1 to guarantee the concavity of the utility function) the more risk averse the individual. The case $\gamma = 0$ corresponds to an agent with logarithmic utility. In that case the problem has closed-form solution and the optimal portfolio is $\frac{\theta}{\sigma}$. We consider both the case in which the individual only draws utility from terminal wealth (TW) and when the individual draws utility from intertemporal consumption (IC). As expected, when the horizon increases and/or the agent becomes less risk averse, the investor allocates a larger proportion of wealth to the risky asset. The investor allocates a smaller proportion of wealth to the risky security when she derives utility from intertemporal consumption.

In Tables 2 and 3 we consider changes in κ_r, the speed of mean-reversion, σ_r, the constant component of the volatility of the interest rate and σ_θ, the constant component of the volatility of the equity premium. An increase in the mean-reversion parameter of the interest rate slightly increases holdings of the risky security for a short horizon ($T = 1$) and decreases holdings when the investor faces a long horizon ($T = 5, 10$). The reason is that a higher mean-reversion coefficient makes the interest rate more volatile in the short term, but more stable in the long term and hedging takes that into consideration. Short term higher volatility results because shocks that push the interest rate away from $\bar{r}$ will trigger a stronger opposite "reaction"; however, over time the interest rate will fluctuate less around $\bar{r}$ and the long term hedging needs are lower. When volatility of the interest rate increases the agent invests more in the risky security since it becomes relatively more attractive. When volatility of the risk premium process increases, the agent invests less in the risky security (since it becomes less attractive).

Finally, we consider the following alternative dynamics of the interest rate and the equity premium process,

$$dr_t = \kappa_r(\bar{r} - r_t)dt - \sigma_r(r_t)^{v_r} dW_t \tag{32}$$

$$d\theta_t = \kappa_\theta(\bar{\theta} - \theta_t)dt + \sigma_\theta(\theta)^{v_\theta} dW_t \tag{33}$$

For $v_r \neq 0.5$ and/or $v_\theta \neq 0$ the problem does not belong to the class of affine models anymore. In Tables 4 and 5 we study the effect of changes in these parameters.

When v_r changes the volatility of the interest rate process is not proportional to the interest rate. When v_r increases, optimal investment in the risky security decreases. When v_θ increases, however, the proportion of wealth invested in the risky security increases. In fact, the investment allocation seems to be quite sensitive to changes in this parameter.

Overall, however, it seems that, for time additive CRRA preferences (which are the standard in the literature) portfolio allocation does not change very much, unless a very high degree of risk aversion is considered.

5 Multiple Factors

For ease of notation, in the previous section we have considered the case of a single Brownian motion process. However, as we stated before, the method that we suggest in this paper can be applied regardless of the number of Brownian motion processes that explain the dynamics of the model, as long as we stay in a complete markets setting, that is, the number of stocks is equal to the number of Brownian motion processes and the variance covariance matrix of all the stocks is non-singular.

Suppose, for example that we have two stocks and two independent standard Brownian motion processes W^1 and W^2. The price of each of the stock processes $S^i, i = 1, 2$ satisfies

$$\frac{dS_t^i}{S_t^i} = \mu_t^i + \sigma_t^{i1} dW_t^1 + \sigma_t^{i2} dW^2, i = 1, 2 \tag{34}$$

and we assume that the matrix

$$\Sigma = \begin{pmatrix} \sigma_t^{11} & \sigma_t^{12} \\ \sigma_t^{21} & \sigma_t^{22} \end{pmatrix} \tag{35}$$

is non-singular for all t.

In this setting, the agent can invest in both securities and, as a result, his/her wealth dynamics will be of the type

$$dX_t = \alpha_t dt + \nu_t^1 dW_t^1 + \nu_t^2 dW_t^2 \quad . \tag{36}$$

Of course the optimal portfolio of the individual will be now two-dimensional and, as in (23) equal to

$$\pi^* = (\Sigma)^{-1} \nu \tag{37}$$

where $\nu^\top = (\nu^1, \nu^2)$.

The implementation of Monte Carlo simulation is analogous to the one-dimensional case, but now we will have to generate two simultaneous series of random numbers z^1 and z^2. The discrete version of (34) is

$$S_{t+\Delta t}^i - S_t^i = S_t^i \left(\mu_t^i \Delta t + \sigma_t^{i1} z_t^1 + + \sigma_t^{i2} z_t^2 \right), i = 1, 2 \quad . \tag{38}$$

Analogously, we can replicate the paths of the stochastic processes involved, r, θ and ξ. In order to retrieve ν^1 and ν^2 we use, (this is similar to (24)),

$$\nu_t^i = \lim_{\Delta t \to 0} E\left[\frac{(X_{t+\Delta t} - X_t)(Z_t^i)}{\Delta t} \mid \mathcal{F}_t\right], i = 1, 2, \tag{39}$$

where $Z_t^i := W_{t+\Delta t}^i - W_t^i$. In order to implement this expression we use a two-step procedure as the one described in the one-dimensional case, but with the following modification: we start at t where we know X_t, r_t and θ_t. We generate realizations of each of the Brownian motion processes that we call $z_1^{1,i}$ and $z_1^{2,i}$, $i = 1, \ldots, N$, and compute $r_{t+\Delta t}(z_1^{1,i}, z_1^{2,i})$, $\theta_{t+\Delta t}(z_1^{1,i}, z_1^{2,i})$ and $\xi_{t+\Delta t}(z_1^{1,i}, z_1^{2,i})$. Then we compute $X_{t+\Delta t}(z_1^{1,i}, z_1^{2,i})$ by replicating the paths of all the stochastic processes involved. We then approximate each ν^j (whose estimate we denote by $\hat{\nu}^j$) by computing

$$\hat{\nu}_t^j = \frac{1}{N}\sum_{i=1}^{N} \frac{\left(\hat{X}_{t+\Delta t}^*(z_1^{1,i}, z_1^{2,i}) - X_t^*\right) z_1^{j,i}}{\Delta t}, \; j = 1, 2 \quad . \tag{40}$$

Obviously, the approach would be analogous if we had more than two factors.

In table 6 we report an example of an economy with two stocks and two independent Brownian motion processes. In other words (we change the notation so that now the subindex represents the stock, $i = 1, 2$).

$$\begin{aligned} \frac{dS_1(t)}{S_1(t)} &= \mu_1(t)dt + \sigma_1(t)dW_1(t), S_1(0) = 1; \\ \frac{dS_2(t)}{S_2(t)} &= \mu_2(t)dt + \sigma_2(t)dW_2(t), S_2(0) = 1. \end{aligned} \tag{41}$$

We assume that $\theta_1(t) = (\mu_1(t) - r(t))/\sigma_1(t)$ is given as in the first one-factor example, with $\sigma_1(0) = .2$ while $\theta_2(t)$ is constant. We take $\sigma_2(0) = .1$ The rest of the parameters are as in the previous section. We find the optimal portfolio $(\hat{\pi}_1(t), \hat{\pi}_2(t))$ for different degrees of risk aversion and different values of θ_2. The results are not surprising: increases in θ_2 result in larger investments in the second stock (investments in the first stock are almost unchanged).

6 Conclusions

In this paper we introduce a method that relies exclusively on Monte Carlo simulation in order to compute optimal portfolios. Our method is quite general and only requires complete markets and knowledge of the dynamics of the security processes. It can be applied regardless of the number of factors and of whether the agent derives utility from intertemporal consumption, terminal wealth or both. The implementation is very easy and allows us to perform some comparative statics. The method relies on the fact that the optimal portfolio is part of the instanteneous standard deviation of the wealth process and such standard deviation can be directly estimated. In fact, computing the instanteneous standard deviation through Monte Carlo simulation has other applications in finance, like the computation of the optimal hedge of an option, or optimal portfolio policies for an individual with a random income and random time of death (see Cvitanić, Goukasian and Zapatero (1999a,b)). To illustrate the method we perform some comparative statics analysis for the portfolio of an agent with CRRA. Our comparative statics show that risk aversion has by far the greatest influence on the value of the hedging portfolio.

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Table 1: Effect of Horizon and Risk Aversion

$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06; \theta_0 = 0.1; \sigma_0 = 0.2$
$\kappa_r = 0.0824; \kappa_\theta = 0.6950; \sigma_r = 0.0364; \sigma_\theta = 0.21$

	$T = 1$		$T = 5$		$T = 10$	
γ	IC	TW	IC	TW	IC	TW
0.5	1.042	1.095	1.221	1.236	1.272	1.279
0	0.5	0.5	0.5	0.5	0.5	0.5
-1	0.244	0.252	0.270	0.295	0.297	0.328
-2	0.174	0.175	0.210	0.230	0.239	0.270
-5	0.104	0.110	0.135	0.170	0.164	0.190
-10	0.056	0.059	0.125	0.139	0.143	0.167

Table 2: Effect of Parameter Dynamics for $\gamma = -1$

$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06;$
$\theta_0 = 0.1; \sigma_0 = 0.2; \kappa_\theta = 0.6950$

			$T = 1$		$T = 5$		$T = 10$	
κ_r	σ_r	σ_θ	IC	TW	IC	TW	IC	TW
0.0824	0.0364	0.21	0.244	0.252	0.270	0.295	0.297	0.328
0.12	0.0364	0.21	0.250	0.252	0.269	0.285	0.287	0.306
0.0824	0.05	0.21	0.257	0.257	0.282	0.311	0.313	0.337
0.0824	0.0364	0.3	0.234	0.243	0.262	0.283	0.280	0.298

Table 3: Effect of Parameter Dynamics for $\gamma = -2$								
$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06;$ $\theta_0 = 0.1; \sigma_0 = 0.2; \kappa_\theta = 0.6950$								
			$T = 1$		$T = 5$		$T = 10$	
κ_r	σ_r	σ_θ	IC	TW	IC	TW	IC	TW
0.0824	0.0364	0.21	0.174	0.175	0.210	0.232	0.239	0.270
0.12	0.0364	0.21	0.175	0.181	0.194	0.205	0.234	0.248
0.0824	0.05	0.21	0.177	0.183	0.213	0.241	0.259	0.290
0.0824	0.0364	0.3	0.166	0.170	0.192	0.207	0.219	0.242

Table 4: Non-affine models with $\gamma = -1$							
$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06; \theta_0 = 0.1; \sigma_0 = 0.2$ $\kappa_r = 0.0824; \kappa_\theta = 0.6950; \sigma_r = 0.0364; \sigma_\theta = 0.21$							
		$T = 1$		$T = 5$		$T = 10$	
v_r	v_θ	IC	TW	IC	TW	IC	TW
0.5	0	0.244	0.252	0.270	0.295	0.297	0.328
0.5	0.5	0.246	0.260	0.286	0.308	0.322	0.346
0.75	0	0.245	0.246	0.248	0.258	0.264	0.280

Table 5: Non-affine models with $\gamma = -2$					
$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06$ $\theta_0 = 0.1; \sigma_0 = 0.2; \kappa_r = 0.0824;$ $\kappa_\theta = 0.6950; \sigma_r = 0.0364; \sigma_\theta = 0.21$					
		$T = 1$		$T = 5$	
v_r	v_θ	IC	TW	IC	TW
0.5	0	0.174	0.175	0.210	0.230
0.5	0.5	0.184	0.189	0.221	0.244
0.75	0	0.168	0.170	0.171	0.190

Table 6: Two-stock portfolio				
$\bar{r} = 0.06; \bar{\theta} = 0.0871; r_0 = 0.06$ $\theta_1(0) = 0.1; \kappa_r = 0.0824;$ $\kappa_\theta = 0.6950; \sigma_r = 0.0364; \sigma_\theta = 0.21$ $\sigma_1(0) = 0.2; \sigma_2(0) = 0.1$				
	$\gamma = -1$		$\gamma = -2$	
θ_2	π_1	π_2	π_1	π_2
0	0.252	0	0.175	0
0.03	0.253	0.156	0.176	0.090
0.06	0.254	0.295	0.176	0.197
0.09	0.256	0.451	0.177	0.298