Incomplete Information with Recursive Preferences

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Abstract

We consider the case of an agent with recursive preferences (Stochastic Differential Utility or SDU) who cannot observe the random drift of the stock price process. This partial information problem can be transformed into a problem with full information, in which the drift is replaced by its expected value conditional on the information given by the stock price history. We are then in a position to use recent results for maximizing SDUs under full information and their connection to Forward-Backward Stochastic Differential Equations. They enable us to study qualitative differences between the cases with full and partial information. We also analyze some special cases in which we obtain semi-explicit results and compute the SDUnumerically.

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1 Introduction

A generalization of a standard utility function, Stochastic Differential Utility (SDU), has been introduced by economists to model the preferences which are not time-additive. We adopt the continuous-time framework of Duffie and Epstein (1992). Recently, there has been a significant progress in characterizing optimal strategies for maximizing SDUs in Schroder and Skiadas (1999) and El Karoui, Peng and Quenez (1999) . It has been shown, under technical conditions, that the optimal wealth process, if exists, is a part of a solution to a Forward-Backward Stochastic Differential Equation (FBSDE), together with the corresponding SDU process and (backward) adjoint processes.

Also recently, methods have been developed to solve the problem of maximizing (standard) utility for an agent with partial information. Namely, it is assumed that the drift of the underlying stock price process is driven by a noise other than the Brownian motion driving the stock price equation, and that it is unobserved. The pioneering papers on this topic are Detemple (1986), Dothan and Feldman (1986) and Genotte (1986). More recently, papers by Lakner (1995, 1999), Karatzas and Zhao (1999), Zohar (1999), show how solutions can be obtained by using a combination of filtering and martingale/duality methods. The latter are the methods developed by Cox and Huang (1989), Karatzas and Shreve (1987) and Pliska (1986), to solve possibly non-Markovian utility maximization problems. They cannot be applied to our problem of maximizing SDUs under partial information. Instead, we combine the filtering approach with the above mentioned approach of Schroder and Skiadas (1999) and El Karoui, Peng and Quenez (1999) to characterize the optimal solution. It turns out, as in the case of standard utility, that the filtering step enables us to transform the problem into one with full information, by replacing the drift with its expected value conditional on the available information (historical stock prices), and replacing the original Brownian motion by the so-called *innovation process* familiar from the filtering theory. Then we can use the known results of maximizing SDUs under full information.

We study some special cases separately, in order to gain some insight on qualitative differences between the partial and the full information case. In particular, we see that for logarithmic preferences the so-called "certainty equivalence" principle does not in general hold for SDUs (even though it holds for the standard logarithmic utility). Namely, the optimal portfolio for the partial information case cannot, in general, be obtained by replacing the conditional expected value of the unknown drift in the expression for the optimal portfolio in the full information case. Moreover, for the logarithmic SDU, we find an explicit formula for the SDU in the full information case, and a semi-explicit formula for the SDU in the partial information case, in terms of an expected value of a functional of the estimated drift. Using Monte Carlo, we then numerically compute the SDU for the case of a normally distributed drift. Depending on the parameter α that represents the preference for an early or late resolution of uncertainty, we get different comparisons between the SDU under partial information and the (average) SDU under full information. Numerical results seem to suggest that, for the logarithmic parametrization considered, the agent always prefers more information (as it is the case for standard time additive preferences).

Section 2 describes the model; Section 3 obtains the characterization of the optimal solution via filtering and FBSDEs; Section 4 looks at some particular examples of SDUs, Section 5 presents some numerical solutions while Section 6 concludes the paper.

2 The model

2.1 Financial Markets

Let us define a complete probability space $(\Omega, \mathcal{G}, P)$, $\mathcal{G} = \{\mathcal{G}_t;\, t \in [0,T]\}$ with a fixed terminal time T, and on it an $n-$dimensional Brownian motion $W^{(1)}(\cdot) = \{W^{(1)}(t) = (W_1^{(1)}(t), \cdots, W_n^{(1)}(t));\, t \in [0,T]\}$.

In the market, there are n risky securities that satisfy

$$\begin{aligned} dS_i(t) &= S_i(t)\left[\mu_i^{(1)}(t)dt + \sum_{j=1}^{n} \sigma_{ij} dW_j^{(1)}(t)\right], \\ S_i(0) &> 0, \qquad i = 1, \cdots, n. \end{aligned}$$

In the above equation, the random vector of stock appreciation rates $\mu^{(1)}(\cdot) = \{\mu^{(1)}(t) = (\mu_1^{(1)}(t), \cdots, \mu_n^{(1)}(t));$ $t \in [0,T]\}$ is adapted to the filtration $\mathcal{G}$, and the stock volatility $\sigma = (\sigma_{ij})_{i,j=1,\cdots,n}$ is a nonsingular matrix of constants. Besides, in the market, there is a riskfree asset (a bank account) that pays a constant continuous interest rate r.

We shall denote by $\mathcal{F} = \{\mathcal{F}_t;\, 0 \le t \le T\}$ the $P-$augmentation of the filtration

$$\mathcal{F}_t^S \triangleq \sigma(S(s); \quad 0 \le s \le t),$$

generated by the price process $S(\cdot) = \{S(t) = (S_1(t), \cdots, S_n(t));\, t \in [0,T]\}$. In our model, we assume that investors do not observe $\mu^{(1)}(\cdot)$ or $W^{(1)}(\cdot)$ but observe the price process $S(\cdot)$. Therefore, the price filtration $\mathcal{F}$ represents the "observation filtration".

We introduce the risk premium process

$$\theta^{(1)}(t) = \sigma^{-1}\left(\mu^{(1)}(t) - r\mathbb{1}\right),$$

assumed to satisfy the integrability condition

$$\int_0^T |\, \theta^{(1)}(t)\, |^2\, dt < \infty, \quad upa.s.$$

and the following assumption.

Assumption 1. *The process*

$$Z_t^{(1)} = \exp\left(-\int_0^t \theta^{(1)\prime}(s) dW^{(1)}(s) - \frac{1}{2}\int_0^t |\, \theta^{(1)}(s)\, |^2\, ds\right)$$

is a $\mathcal{G}$-martingale.

Thus, under Assumption 1,we can define

$$\tilde{P}(A) \triangleq E\left(Z_T^{(1)} \cdot 1_A\right), \quad A \in \mathcal{G}_T, \tag{1}$$

a probability measure equivalent to P on $\mathcal{G}_T$ and, by Girsanov Theorem, the n-dimensional process $\tilde{W}(\cdot) = \{\tilde{W}(t) = (\tilde{W}_1(t), \cdots, \tilde{W}_n(t));\, t \in [0,T]\}$ defined by

$$\tilde{W}(t) \triangleq W^{(1)}(t) + \int_0^t \theta^{(1)}(s) ds,$$

is a $(\tilde{P}, \mathcal{G})$ Brownian motion.

2.2 Filtering

We shall make the following assumption on the risk premium process $\theta^{(1)}(\cdot)$.

Assumption 2. *For all* $t \in [0,T]$, $E(|\, \theta^{(1)}(t)\, |) < \infty$.

We shall denote by

$$\theta(t) \triangleq E\left[\theta^{(1)}(t) \mid \mathcal{F}_t\right], \quad 0 \le t \le T$$

a measurable version of the conditional expectation with respect to the price filtration of the risk premium process.

Lemma 1 *The canonical decomposition of* $W^{(1)}(\cdot)$ *in the price filtration* $\mathcal{F}$ *is given by*

$$W^{(1)}(t) = W(t) + \int_0^t \left(\theta(s) - \theta^{(1)}(s)\right) ds, \tag{2}$$

where the process $W(\cdot) = \{W(t) = (W_1(t), \cdots, W_n(t));\ t \in [0,T]\}$ *defined by* (2) *is a* $\mathcal{F}$*-Brownian motion, which is called the innovation process. Furthermore, the filtration* $\mathcal{F}$ *admits a predictable representation property with respect to the innovation Brownian motion* $W(\cdot)$.

Proof. The first assertion is standard and can be found in Lipster and Shiryayev (1977, p.258, Theorem 7.12). The second assertion can also be found in Lipster and Shiryayev (1977, p.258, Theorem 7.12) and a different proof can be found in Lazrak and Quenez (1999). ■

From the second part of this Lemma, note that we are only able to prove a predictable representation property with respect to the innovation $W(\cdot)$ and we *do not* assert that $\mathcal{F}$ coincides with the canonical filtration of the innovation $W(\cdot)$.

Now, denoting by

$$\mu_i(t) \triangleq E\left[\mu_i^{(1)}(t) \mid \mathcal{F}_t\right], \quad 0 \le t \le T, \quad i = 1, \cdots, n,$$

a measurable version of the conditional expectation with respect to the price filtration of the appreciation rate process, one can easily obtain the canonical decomposition of the price process under its natural filtration

$$dS_i(t) = S_i(t)\left[\mu_i(t)dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)\right], \quad i = 1, \cdots, n. \tag{3}$$

Notice that in the above price decomposition, the appreciation rate process $\mu(\cdot) = \{\mu(t) = (\mu_1(t), \cdots, \mu_n(t));$ $t \in [0,T]\}$ becomes observable and hence, we are in a standard complete market context.

Now, notice that from Lipster and Shiryayev (1977, p.298, Theorem 8.1) (see also Lakner (1998) and Lazrak and Quenez (1999)), that the $\mathcal{F}$-martingale

$$Z_t := E\left[Z_t^{(1)} \mid \mathcal{F}_t\right], \quad 0 \le t \le T,$$

is an indistinguishable version of the process

$$Z_t = \exp\left(-\int_0^t \theta'(s) dW(s) - \frac{1}{2}\int_0^t |\, \theta(s)\,|^2\, ds\right), \quad 0 \le t \le T. \tag{4}$$

Therefore, by Girsanov Theorem, the process $\tilde{W}(\cdot)$, identified via Lemma 2.1 by,

$$\tilde{W}(t) \equiv W^{(1)}(t) + \int_0^t \theta^{(1)}(s) ds = W(t) + \int_0^t \theta(s) ds, \tag{5}$$

is also a $(\tilde{P}, \mathcal{F})$ Brownian motion. Additionally, note that, under the assumption of constant interest rates, the price filtration $\mathcal{F}$ is generated by the $\tilde{P}-$Brownian motion $\tilde{W}(\cdot)$, since the price decomposition under $(\tilde{P}, \mathcal{F})$ is given by

$$dS_i(t) = S_i(t) \left[rdt + \sum_{j=1}^{n} \sigma_{ij} d\tilde{W}_j(t) \right], \quad i = 1, \cdots, n.$$

Therefore, the restriction of the measure $\tilde{P}$ on the probability space $(\Omega, \mathcal{F}_T)$ is the equivalent martingale of the market model (3). Finally, it will be convenient to introduce the *state price density process* given by

$$H_t \triangleq e^{-rt} Z_t.$$

2.3 Portfolio and consumption strategy

In the present paper we require the consumption and investment processes to be adapted to the price filtration $\mathcal{F}$.

The set $\mathcal{C}$ of consumption processes is formed by any $\mathcal{F}$-progressively measurable process of the form $c : \Omega \times [0, T] \longrightarrow \mathbb{R}$ satisfying

$$E\left(\int_0^T \mid c(t) \mid^2 dt \right) < \infty, up\ and c(t) > 0, \quad dP \otimes dtup\ a.s.$$

The quantity $c(t)$ represents the rate of consumption of a single numeraire good at time t.

A portfolio strategy is a $n-$dimensional, $\mathcal{F}$-progressively measurable process $\pi(\cdot) = (\pi_1(\cdot), \pi_2(\cdot), \cdots, \pi_n(\cdot)) : \Omega \times [0, T] \longrightarrow \mathbb{R}^n$ satisfying

$$\int_0^T \mid \pi(t) \mid^2 dt < \infty, \quad upa.s.$$

The quantity $\pi_i(t)$ represents the amount of money invested in the ith stock at time t.

Given an initial wealth $x > 0$, consumption plan c, and portfolio strategy π, the corresponding wealth $X^{x,c,\pi}$ is defined by the linear stochastic differential equation

$$dX^{x,c,\pi}(t) = \sum_{i=1}^{n} \pi_i(t) \left(\frac{dS_i(t)}{S_i(t)} - rdt \right) + rX^{x,c,\pi}(t)dt - c(t)dt, \quad t < T, \quad X^{x,c,\pi}(0) = x. \tag{6}$$

A portfolio-consumption strategy (c, π) is said to be feasible for a given level $x > 0$ of initial wealth if

$$X^{x,c,\pi}(t) \geq 0, \quad 0 \leq t \leq T, \quad dP \otimes dtup\ a.s. \tag{7}$$

We denote by $\mathcal{A}(x)$ the space of portfolio-consumption strategies (c, π) feasible for the initial wealth x. Additionally, we denote by $\mathcal{B}(x)$ the space of portfolio-consumption strategies (c, π) feasible for some initial wealth $y < x$, that is

$$\mathcal{B}(x) \triangleq 0 < y \leq x \cup \mathcal{A}(y). \tag{8}$$

Finally, we denote by $\mathcal{C}(x)$ the set of consumption processes,

$$\mathcal{C}(x) = \left\{ c \in \mathcal{C} : E\left(\int_0^T H_t c(t) dt \right) \leq x \right\}. \tag{9}$$

Lemma 2.1 and the decomposition (3) allow us to transfer the classical transformation of the trajectorial wealth constraint (7) into a static constraint of the form (9) that holds in a standard complete information framework (Karatzas et al. (1987)):

Lemma 2 *For any consumption process $c \in \mathcal{C}$ and for any $x \in (0, \infty)$, there exists a portfolio strategy π such that $(c, \pi) \in \mathcal{B}(x)$ if and only if $c \in \mathcal{C}(x)$.*

2.4 Preferences

Stochastic differential utility (Duffie and Epstein (1992)) is defined in terms of a continuously differentiable function $f : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$, called the *intertemporal aggregator.* The first argument of f is the consumption (c) and the second argument is the utility (y), and we write f_c and f_y as the partial derivative of f with respect to the consumption and the utility arguments. We suppose that f is increasing with respect to c.For each $c \in \mathcal{C}$, investor preferences are represented by a *utility process* $Y^c_{(\cdot)} = \{Y^c_t;\, t \in [0, T]\}$ that satisfies the recursion

$$Y^c_t = E\left[\int_t^T f(c_s, Y^c_s)ds \mid \mathcal{F}_t\right]. \tag{10}$$

Equivalently, the utility represented by (10) may be characterized by the Backward Stochastic Differential Equation (BSDE, see Pardoux and Peng (1992) and El Karoui, Peng and Quenez (1997))

$$-dY^c_t = f(c_t, Y^c_t)dt - Z^{c'}_t dW(t), \quad Y^c_T = 0, \tag{11}$$

where the *intensity process* $Z^c_{(\cdot)} = \{Z^c_t = (Z^c_{1t}, \cdots, Z^c_{nt});\, t \in [0, T]\}$ is $\mathcal{F}$-adapted and is a part of the solution of the BSDE (11). Note that the BSDE (11) is not standard because the solution is not required to be adapted to the canonical filtration of the innovation $W(\cdot)$ but is required to be only adapted to the *bigger* filtration $\mathcal{F}$. However, we know from Lazrak and Quenez (1999), that the El Karoui, Peng and Quenez (1997) existence and uniqueness proofs of standard BSDE do extend easily to BSDE of the type (11).

When the intertemporal aggregator satisfies for some $K > 0$, the uniform Lipschitz condition

$$\mid f(c, y) - f(c, y') \mid \leq K \mid y - y' \mid, \quad \forall c \in \mathbb{R}^+, \forall (y, y') \in \mathbb{R} \times \mathbb{R}, \tag{12}$$

Duffie and Epstein (1992) established the existence and the uniqueness of the recursive utility (10) (or equivalently (11)) for any consumption plan $c \in \mathcal{C}$.

In the second part of the paper we will consider a logarithmic intertemporal aggregator of the form

$$f(c, y) = (1 + \alpha y)\left[\log(c) - \log(1 + \alpha y)\right], \tag{13}$$

with parameter restrictions: $\alpha \geq 0$. Unfortunately, the logarithmic intertemporal aggregator (13) violates the Lipschitz condition (12) and therefore a specific treatment is needed. Schroder and Skiadas (1999) established the existence and uniqueness of the logarithmic SDU that solves (10) (or equivalently (11)) when the intertemporal aggregator has the form (13) under appropriate restrictions on the consumption process.

3 SDU Optimization under incomplete information: the general theory

In this section, we will suppose that the intertemporal aggregator satisfies the Lipschitz condition (12) and consider the optimal consumption and portfolio selection problem under incomplete information,

$$V(x) \triangleq \sup_{(c,\pi) \in \mathcal{A}(x)} Y^c_0. \tag{14}$$

Assumption 3: The value function V is finite for all $x > 0$.

Remark 1 *We know from Karatzas et al. that when the utility is time additive and of the power form,*

$$E\left(\int_0^T \frac{c_t^\alpha}{\alpha}dt\right),$$

for some $\alpha \in (0,1)$, *the value function of the problem* (14) *is finite and has the form* $V(x) = kx^\alpha$, *for some positive constant* $k > 0$. *In our case, if we impose a growth condition of the form*

$$| f(c,y) | \leq C(\frac{c^\alpha}{\alpha} + 1), \quad \forall c \in \mathbb{R}^+, \forall y \in \mathbb{R},$$

for some constants $C > 0$ *and* $\alpha \in (0,1)$, *the classical BSDE comparison Theorem gives*

$$Y_0^c \leq E\left(\int_0^T C(\frac{c_t^\alpha}{\alpha} + 1)dt\right),$$

and therefore,

$$V(x) \leq C(kx^\alpha + T),$$

and hence Assumption 3 *is satisfied.*

Proposition 1 *For all* $x > 0$, *we have*

$$V(x) = \sup_{c\in\mathcal{C}(x)} Y_0^c. \tag{15}$$

Proof. For all $x > 0$, it is clear from the wealth dynamic (6), that $\forall (c,\pi) \in \mathcal{A}(y)$, where $0 < y < x$, we have, $X^{y,c,\pi}(t) \leq X^{x,c,\pi}(t)$, $\quad dP \otimes dt$ a.s. It follows that $\mathcal{A}(y) \subset \mathcal{A}(x)$ and consequently $V(y) \leq V(x)$ and therefore,

$$V(x) = \sup_{0<y\leq x} V(y) = \sup_{(c,\pi)\in\mathcal{B}(x)} Y_0^c,$$

where the set $\mathcal{B}(x)$ is defined in (8). Finally, from Lemma 2.2, we are done.■

We introduce now the following assumption

Assumption 4: The functional $Y_0^\bullet : \mathcal{C} \longrightarrow \mathbb{R}$ where for each $c \in \mathcal{C}$, Y_0^c is the initial value of the BSDE (11) is concave.

Remark 2 *It is well known that if the intertemporal aggregator is concave with respect to* (c,y) *then, Assumption* 4 *is satisfied.*

Now the Lagrangian of (15) is given by $\mathcal{L}(\nu, c) = Y_0^c + \nu\left(x - E\left(\int_0^T H_t c(t)dt\right)\right)$, $\nu > 0$, $c \in \mathcal{C}$. Standard results of convex analysis (see Luenberger (1969)) yield

Proposition 2 *If the maximum is attained in* (15) *by* $c^* \in \mathcal{C}(x)$, *then there exists* $\nu > 0$ *such that*

$$\sup_{c\in\mathcal{C}} \mathcal{L}(\nu, c) = \mathcal{L}(\nu, c^*). \tag{16}$$

Conversely, let $\nu > 0$ *and* $c^* \in \mathcal{C}$ *be such that* (16) *is satisfied with* $E\left(\int_0^T H_t c^*(t)dt\right) = x$, *then* c^* *is optimal for Problem* (15).

3.1 Necessary conditions and sufficient conditions for optimality

Let $c^* \in \mathcal{C}$ be an optimal consumption plan for (16) for a given $\nu > 0$ and, let Y^{c^*} be the associated utility. Now, for an arbitrary consumption plan $c \in \mathcal{C}$, let us consider for each $\alpha \in [0,1]$ the consumption plan $(c_\alpha(t) = c^*(t) + \alpha(c(t) - c^*(t));\quad 0 \leq t \leq T)$ and let Y^{c_α} the associated utility. From the optimality of c^*,we have

$$Y_0^{c_\alpha} - \nu E\left(\int_0^T H_t(c^*(t) + \alpha(c(t) - c^*(t)))dt\right) \leq Y_0^{c^*} - \nu E\left(\int_0^T H_t c^*(t)dt\right),$$

and dividing by α, and then letting α go to 0 we get,

$$\partial Y_0^{c^*} - \nu E\left(\int_0^T H_t(c(t) - c^*(t))dt\right) \leq 0, \tag{17}$$

where $\partial Y_0^{c^*}$ denotes the right derivative of $\alpha \to Y_0^{c_\alpha}$ at $\alpha = 0$.

Assumption 5: The function f_c is supposed to be bounded or to satisfy for all $(c, y) \in \mathbb{R}^+ \times \mathbb{R}$, $f_c(c, y) \leq K(c+ \mid y \mid)$ for some constant $K > 0$.

By classical results on differentiability of BSDE, $\partial Y_0^{c^*}$ exists and is given by

$$\partial Y_0^{c^*} = E\left(\int_0^T \exp(\int_0^t f_y(c^*(s), Y_s^{c^*})ds) f_c(c^*(t), Y_t^{c^*})(c(t) - c^*(t))dt\right) \tag{18}$$

under Assumption 5.

Exploiting the fact that (17) and (18) hold for an arbitrary consumption process $c \in \mathcal{C}$, Duffie and Skiadas (1994) or El Karoui, Peng and Quenez (1999) establish a necessary condition of optimality that holds under the Inada Assumption,

Assumption 6: For all $y \in \mathbb{R}$, $\lim_{c \downarrow 0} f_c(c, y) = \infty$.

Proposition 3 *Suppose that assumptions $A1 - A6$ are satisfied. Let $c^* \in \mathcal{C}$ be an optimal consumption plan for* (14). *Then, there exists $\nu > 0$ such that*

$$\exp(\int_0^t f_y(c^*(s), Y_s^{c^*})ds) f_c(c^*(t), Y_t^{c^*}) = \nu H_t, \qquad 0 \leq t \leq T, \qquad dP \otimes dtupa.s., \tag{19}$$

where Y^{c^*} is the utility associated to c^*.

Conversely, if a given consumption plan $c^* \in \mathcal{C}$ satisfies (19) for a given $\nu > 0$ and $E\left(\int_0^T H_t c^*(t)dt\right) = x$, then (18) gives

$$\partial Y_0^{c^*} = \nu(E\left(\int_0^T H_t c(t)dt\right) - x) \leq 0,$$

for each $c \in \mathcal{C}(x)$. Additionally, from the concavity of the functional $Y_0^\bullet$ we have for each $c \in \mathcal{C}(x)$,

$$Y_0^c \leq Y_0^{c^*} + \partial Y_0^{c^*} \leq Y_0^{c^*},$$

and therefore c^* is optimal for (14).

Proposition 4 *Suppose that assumptions $A1 - A6$ are satisfied. Let $c^* \in \mathcal{C}$ be a consumption plan satisfying $E\left(\int_0^T H_t c^*(t)dt\right) = x$ and suppose that there exists some $\nu > 0$ such that* (19) *is satisfied. Then, c^* is optimal for* (14).

3.2 A forward backward characterization

To summarize, we obtain necessary and sufficient conditions for optimality and these conditions may be expressed as a forward backward system under the following assumption (see Ma and Yong (1999) for the general existence theory).

Assumption 7: f is strictly increasing with respect to c and for all $y \in \mathbb{R}$, $\lim_{c\uparrow\infty} f_c(c,y) = 0$.

Under Assumption 7, we denote for each $y \in \mathbb{R}$, by $I(\cdot,y) : (0,\infty) \longrightarrow (0,\infty)$ the inverse of the function $f_c(\cdot,y)$ that is, $f_c(I(c,y),y) = c$, $\forall(c,y) \in (0,\infty) \times \mathbb{R}$. Thus, under Assumption 7, Equation (19) may be expressed as $c^*(t) = I(e^{A_t}, Y_t^{c^*})$, where $A_t = \log(\nu H_t) - \int_0^t f_y(c^*(s), Y_s^{c^*})ds$ and therefore we can characterize optimality by a forward backward system.

Proposition 5 *Suppose that assumptions $A1 - A7$ are satisfied. Then every optimal plan c for Problem* (14) *takes the form*

$$c_t = I(e^{A_t}, Y_t), \quad dP \otimes dtupa.s., \tag{20}$$

where the pair $(A;(Y,Z))$ solves a Forward-backward system that has a forward component,

$$\begin{cases} dA_t = -(r + f_y(I(e^{A_t}, Y_t), Y_t) + \frac{1}{2}|\theta_t|^2)dt - \theta_t^{'} dW_t, \\ A_0 = \log(\nu), \end{cases} \tag{21}$$

and a backward component

$$-dY_t = f(I(e^{A_t}, Y_t), Y_t)dt - Z_t^{'} dW_t, \qquad Y_T = 0, \tag{22}$$

for some constant $\nu > 0$ which is fixed in a such way that $E\left[\int_0^T H_t I(e^{A_t}, Y_t)dt\right] = x$.

Conversely, if $\nu > 0$ and $(A;(Y,Z))$ solve the forward backward system (21) $-$ (22), *then the process $c_{(\cdot)} = \{c_t = I(e^{A_t}, Y_t); t \in [0,T]\}$ is optimal for Problem* (14).

4 The case of a random drift under logarithmic SDU.

In this section, the risk premium $\theta^{(1)}$ is modeled as in Karatzas and Zhao (1998) as a $\mathcal{G}_0-$measurable random variable: $\Omega \longrightarrow \mathbb{R}^n$. Note that since $W^{(1)}$ is a Brownian motion with respect to $\mathcal{G}$, it turns out that $\theta^{(1)}$ is independent from $W^{(1)}$, under P.

We also specify the intertemporal aggregator that takes the logarithmic form,

$$f(c,y) = (1+\alpha y)\left[\log(c) - \log(1+\alpha y)\right],$$

with parameter restrictions: $\alpha \geq 0$.

Under our environment, letting $k_t = e^{\alpha(T-t)}$, and $J_{(\cdot)}$ be the process defined by

$$\exp(J_t) = E\Big[\exp\left(-\int_t^T (1-k_s)\theta'(s)dW(s) - \tfrac{1}{2}\int_t^T (1-k_s)^2 \mid \theta(s) \mid^2 ds\right) * \\ \exp\left(\int_t^T (1-k_s)(\alpha - r - \tfrac{k_s}{2} \mid \theta(s) \mid^2)ds\right) \mid \mathcal{F}_t\Big],$$

we know form Schroder and Skiadas (1999) that the optimal consumption process is given by

$$c_t = \exp(J_t - k_t A_t),$$

where the solution A to the forward SDE (21) is given by

$$k_t A_t = k_0 \log(\lambda) + (1 - r/\alpha)(k_0 - k_t) - \int_0^t k_s \theta(s) dW(s) - \frac{1}{2}\int_0^t k_s \mid \theta(s) \mid^2 ds,$$

for $0 \leq t \leq T$.

In fact, a straightforward computation shows that

$$\exp(J_t) = L_t e^{(\alpha - r)(T-t)+(1-k_t)(1-r/\alpha)},$$

where

$$L_t = E\left[\exp\left(-\int_t^T (1-k_s)\theta'(s)dW(s) - \frac{1}{2}\int_t^T (1-k_s) \mid \theta(s) \mid^2 ds\right) \mid \mathcal{F}_t\right],$$

and, it is not very difficult to check that L is the solution of the following *linear* BSDE:

$$dL_t = [\frac{1}{2}L_t\theta^2(t)k_t(1-k_t) + (1-k_t)\theta(t)\xi(t)]dt + \xi(t)dW(t), \quad \mathrm{L_T} = 1. \tag{23}$$

Then, by Schroder and Skiadas (1999), we know also that the process of optimal utility is given by

$$1 + \alpha Y_t = \exp(J_t + (1-k_t)A_t) = L_t e^{(1-k_t)A_t} e^{(\alpha-r)(T-t)+(1-k_t)(1-r/\alpha)},$$

and hence, the value function $V(x) \equiv Y_0$ is given by

$$1 + \alpha V(x) = L_0 e^{(1-k_0)\log(\lambda)} e^{(\alpha-r)T+(1-k_0)(1-r/\alpha)},$$

and the Lagrange multiplier λ is easily computed from the budget constraint,

$$E\left[\int_0^T H_t L_t e^{-k_t A_t} e^{(\alpha-r)(T-t)+(1-k_t)(1-r/\alpha)} dt\right] = x.$$

Therefore, in order to calculate V, all we have to compute numerically is L_0, since both λ and J_0 can be determined once we know L_0. In fact, the following is easily deduced from above

$$1 + \alpha V(x) = \left(L_0 \exp((\alpha - r)(T + \frac{1-k_0}{\alpha}))\right)^{1/k_0} \left(\frac{1-k_0^{-1}}{x\alpha}\right)^{\frac{1}{k_0}-1}.$$

Now, for the numerical section (see below), it is more convenient to calculate L_t, taking conditional expectations under the $\tilde{P}$-measure. To that end, we know by Bayes rule that, for a given $\mathcal{G}_T-$measurable random variable X,

$$E[X|\mathcal{F}_t] = \frac{\tilde{E}[(Z_T^{(1)})^{-1}X|\mathcal{F}_t]}{\tilde{E}[(Z_T^{(1)})^{-1}|\mathcal{F}_t]}.$$

On the other hand,

$$\hat{Z}_t := \tilde{E}[(Z_T^{(1)})^{-1}|\mathcal{F}_t] = \frac{1}{Z_t}. \tag{24}$$

Therefore, switching to expectations under $\tilde{P}$ in the expression for L_t, we get

$$L_t = \tilde{E}\left[\exp\left(\int_t^T k_s\theta(s)d\tilde{W}(s) - \frac{1}{2}\int_t^T k_s \mid \theta(s) \mid^2 ds\right) \mid \mathcal{F}_t\right]. \tag{25}$$

In the case of full information, i.e., in which we observe the random $\theta^{(1)}$, we find similarly as above

$$L_t = \tilde{E}\left[\exp\left(\int_t^T k_s\theta^{(1)}d\tilde{W}(s) - \frac{1}{2}\int_t^T k_s \mid (\theta^{(1)}) \mid^2 ds\right) \mid \mathcal{G}_t\right].$$

This can be explicitly calculated:

$$L_t = \exp(\frac{(\theta^{(1)})^2}{4\alpha}(k_t - 1)^2) \tag{26}$$

and therefore,

$$1 + \alpha V^f_{\theta^{(1)}}(x) = \left(\exp(\frac{\mid (\theta^{(1)}) \mid^2}{4\alpha}\frac{(k_0-1)^2}{k_0} + \frac{\alpha - r}{k_0}(T + \frac{1-k_0}{\alpha}))\right)\left(\frac{1-k_0^{-1}}{x\alpha}\right)^{\frac{1}{k_0}-1}, \tag{27}$$

where $V^f_{\theta^{(1)}}(x)$ is the value function corresponding to the case where $\theta^{(1)}$ is a Dirac measure.

Finally, form Schroder and Skiadas (1999), we have the expression of the optimal strategy under incomplete information.

Proposition 6 *The optimal portfolio strategy is given by*

$$\frac{\pi_t}{X^{x,c,\pi}(t)} = k_t\sigma^{-1}\theta(t) + \left(\sigma'\right)^{-1}\frac{\xi(t)}{L_t}, \tag{28}$$

where we recall that (L, ξ) *solves the linear BSDE* (23)

When the utility is additive ($\alpha = 0$), the BSDE (23) exhibits a deterministic degeneracy and therefore $\xi = 0$ and we obtain the usual formula of optimal portfolio for logarithmic utility.

When θ is deterministic , the BSDE (23) exhibits again a deterministic degeneracy and hence $\xi = 0$.

Remark 3 *Note that in general, the BSDE* (23) *has a stochastic dynamic and therefore, the certainty equivalent principle fails to hold for the logarithmic SDU (unlike for the logarithmic additively separable utility, see Karatzas and Zhao (*1998*)).*

5 Numerical example

In this section we consider the case where the risk premium $\theta^{(1)}$ is a random $\mathbb{R}$-valued variable having normal distribution with mean m and variance γ^2 and $\mathcal{G}_0-$measurable. From Lemma 2.2 of Karatzas and Zhao (1998), it turns out that $\theta^{(1)}$ has the same distribution under the probability P and $\tilde{P}$ and that $\theta^{(1)}$ is independent of the price filtration $\mathcal{F}_t$ under $\tilde{P}$. In order to calculate the optimal utility using Monte Carlo we need

Lemma 3

$$\theta(t) = E[\theta^{(1)}|\mathcal{F}_t] = \frac{\gamma^2\tilde{W}(t) + m}{\gamma^2 t + 1}.$$

Proof: By Bayes rule we have

$$E[\theta^{(1)}|\mathcal{F}_t] = \frac{\tilde{E}[(Z_t^{(1)})^{-1}\theta^{(1)}|\mathcal{F}_t]}{\tilde{E}[(Z_t^{(1)})^{-1}|\mathcal{F}_t]}.$$

Since $\theta^{(1)}$ is independent of the filtration $\mathcal{F}_t$ under $\tilde{P}$ and $\tilde{W}$ is adapted to it, the conditional expectations above can be found by integrating against the normal density of $\theta^{(1)}$. This is tedious but straightforward. For example, we have

$$\tilde{E}[(Z_t^{(1)})^{-1}|\mathcal{F}_t] = \frac{1}{\sqrt{1+\gamma^2 t}}\exp\{-\frac{m^2}{2\gamma^2} + \frac{(\gamma^2\tilde{W}(t) + m)^2}{2\gamma^2(1+t\gamma^2)}\}. \tag{29}$$

Similar calculation for the other conditional expectation gives the desired result.

We can now compute L_0 via Monte Carlo, using (25).

Remark 4 *If we want to avoid computing the stochastic integral in L_t numerically, we can use the following identity: Denote*

$$f_s := \frac{k_s}{\gamma^2 s + 1}, \quad Y(s) := \frac{\gamma^2}{2}\tilde{W}^2(s) + m\tilde{W}(s).$$

Then

$$\int_t^T k_s \theta(s) d\tilde{W}(s) = f_T Y(T) - f_t Y(t) - \int_t^T \left(Y(s)(-\alpha f_s - \frac{\gamma^2 f_s}{\gamma^2 s + 1}) + \frac{1}{2}\gamma^2 f_s \right) ds.$$

Remark 5 *Note also that similar computation to that of Zohar (1999), allows us to obtain*

$$L_0 = \left(\frac{u_T}{u_0}\right)^{1/2} e^{\frac{m^2}{2\gamma^2}(\frac{1}{\gamma^2}\frac{u_0'}{u_0} - f_T)} e^{-\frac{1}{2}\int_0^T \left(\gamma^2 f_t + \frac{m^2}{\gamma^2}(\alpha f_s + \frac{\gamma^2 f_s}{\gamma^2 s+1})\right)},$$

where u denotes any positive solution on $[0,T]$ of the ODE

$$\frac{d^2 u_t}{dt^2} + \alpha\gamma^2 f_t u_t = 0, \tag{30}$$

that satisfies the boundary condition

$$\left(\frac{du_t/dt}{u_t}\right)_{t=T} = \gamma^2 f_T.$$

Note also that, up to now, we have not been able to find an explicit solution to the ODE (30).

In the case of full information, i.e., observing $\theta^{(1)}$, we can compute the optimal utility $V^f_{\theta^{(1)}}(x)$ directly using 26 and, after averaging we get,

$$1 + \alpha E(V^f_{\theta^{(1)}}(x)) = \frac{1}{\sqrt{1 - 2\gamma^2 \frac{(k_0-1)^2}{4\alpha k_0}}} \exp\left(\frac{m^2}{\frac{4\alpha k_0}{(k_0-1)^2} - 2\gamma^2} + \frac{\alpha - r}{k_0}(T + \frac{1-k_0}{\alpha}) \right) \left(\frac{1 - k_0^{-1}}{x\alpha} \right)^{\frac{1}{k_0} - 1},$$

under the condition that

$$\frac{(k_0 - 1)^2}{4\alpha k_0} < \frac{1}{2\gamma^2}.$$

Let us first consider the special, additive utility case $\alpha = 0$, i.e.,

$$U(x) = E\left[\int_0^T \log(c_t^*) dt\right],$$

where c^* is the optimal consumption process. In the case of full information it is easily seen that we then have

$$U^f_{\theta^{(1)}}(x) = T\log(x/T) + (\frac{(\theta^{(1)})^2}{2} + r)T^2/2,$$

and after averaging, we obtain

$$EU^f_{\theta^{(1)}}(x) = T\log(x/T) + (\frac{m^2 + \gamma^2}{2} + r)T^2/2.$$

Finally, we can also get the value function $U(x)$ in case of incomplete information, using the known form for the optimal consumption process:

$$\begin{aligned} U(x) &= T\log(x/T) + (\frac{m^2+\gamma^2}{2} + r)T^2/2 - \frac{1}{2}\int_0^T dt \int_0^t ds \frac{\gamma^2}{1+\gamma^2 s} \\ &= T\log(x/T) + (\frac{m^2+\gamma^2}{2} + r)T^2/2 - \frac{1}{2\gamma^2}\left[1 + (1+\gamma^2 T)(\log(1+\gamma^2 T) - 1)\right]. \end{aligned}$$

Note that, when the utility is additive ($\alpha = 0$), we have the following inequality:

$$U(x) \leq EU^f_{\theta^{(1)}}(x).$$

In other words, the utility under incomplete information is in this case smaller than the average utility under full information. That is, an investor that has incomplete information about $\theta^{(1)}$ would prefer more information, if given the choice. This investor compares her current utility (with incomplete information) with the utility she would achieve if she knew the true value of the parameter. Since she does not know that true value, she has to average (according to her current information) over all possible values of the parameter.

For $\alpha > 0$, we do numerical comparisons. We use Monte Carlo simulation to approximate the values of the utility in the case of complete and incomplete information. We generate draws of $\tilde{W}$ and average in order to compute the expression in (25). We fix the following parameter values: $m = 0.3$, $\gamma = 0.1$, $r = 0.04$, $T = 1$ and $x = 1$. Then we change the value of α. We compare the different levels of utility with incomplete information ($V(x)$) and expected utility with perfect information ($EV^f_{\theta^{(1)}}(x)$). The results are in the following table:

Utility levels		
α	$V(x)$	$EV^f_{\theta^{(1)}}(x)$
0.1	0.0425	0.441
2	0.116	0.119

It seems from our numerical results that the investor always prefers more information.

6 Conclusion

In this paper we show how the problem of optimizing Stochastic Differential Utility when the drift of the underlying asset prices is not observed can be transformed into an analogous problem with full information. In the special case of the logarithmic SDU we compare numerically the cases of partial and full information. In future research we plan to address the following questions:

- Can the comparison between the two cases be done analytically, rather than numerically?

- Can we get general existence results in the case of an unbounded unobserved drift process, for example a linear Gaussian process as in Lakner (1999) or Zohar (1999)?

- Can the computation of the optimal portfolio strategy be done in a numerically efficient fashion?

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