ON OPTIMAL TERMINAL WEALTH UNDER TRANSACTION COSTS *

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Abstract

In this note we show that the modern approach to the problem of maximizing expected utility from terminal wealth in financial markets, namely martingale/duality methodology, works also in the presence of proportional transaction costs. More precisely, we show that the optimal terminal wealth is given as the inverse of marginal utility evaluated at the random variable which is optimal for an appropriately defined dual problem. We thereby resolve a question left open by Cvitanić & Karatzas (1996).

1 Introduction

The modern approach to utility maximization problems in non-Markovian, incomplete financial markets, on a finite time-horizon $[0, T]$, consists of solving an appropriate dual problem over a set of "state-price densities" corresponding to "shadow markets" associated with the incompleteness of the original market. Given the optimal solution $\hat{Z}$ to the dual problem, it is usually possible to show that the optimal terminal wealth for the primal problem is represented as the inverse of marginal utility evaluated at $\hat{Z}$ (see the book by Karatzas &

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Shreve 1998, for example). In Cvitanić and Karatzas (1996) (henceforth [CK96]) a similar technique was employed to formulate a dual problem for maximizing expected utility from terminal wealth in continuous-time markets with proportional transaction costs. However, the analysis was not completed, since the authors left open the question of existence of the optimal solution $\hat{Z}$ to the dual problem. We show here that, if one enlarges somewhat the domain of the dual problem, then, indeed, an optimal $\hat{Z}$ exists, and the usual relationship between the primal and the dual optimal solution is valid. This is the main new ingredient in the paper, and the idea is taken from the recent very important contribution Kramkov and Schachermayer (1998) (henceforth [KS98]) to the theory of utility maximization in incomplete markets. This enlargement of the dual domain is needed to "close" the set of shadow state-price densities that are natural for this market. Once this is done, the existence in the dual problem is an immediate consequence of a theorem by Komlós (1967) (for recent generalizations of this theorem and further references see Delbaen & Schachermayer 1999; see also Schwartz 1986). However, due to that enlargement, the optimal solution $\hat{Z}$ to the dual problem is not necessarily a density, and one would think that its usual interpretation as a pricing kernel in a shadow market becomes questionable. Nevertheless, as already noted in [KS98] in the context of incomplete markets, the optimal dual variable still gives rise to consistent pricing, provided one uses the optimal primal variable (the optimal terminal wealth) as the numeraire. Besides being inspired by the approach of [KS98], our approach is also facilitated by the bipolar theorem of Brannath & Schachermayer (1998).

The literature on utility maximization under transaction costs usually considers the infinite time-horizon problem and the PDE approach. For a comprehensive treatment and references for such an approach see Shreve & Soner (1994). A recent work in this vein is Cuoco & Liu (1997). The duality approach to markets with transaction costs has also proved to be fruitful for the problem of pricing and super-replicating contingent claims; see the pioneering work of Jouini & Kallal (1995), as well as Kusuoka (1995), Cvitanić & Karatzas (1996), Ortu (1996), Cvitanić, Pham & Touzi (1998), Kabanov (1999).

2 The model and the main results

We briefly recall here the model and the utility maximization problem of [CK96]. As in that paper, for simplicity we restrict ourselves to the classical setting of a model with one stock driven by a Brownian motion; however, the theory may be developed in much greater generality of semimartingale models as in [KS98] and Kabanov (1999). We refer to [CK96] for some of the known results that we need, and we collect the proofs of our results in

Appendix. Our financial market consists of a *bank account* with price

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1; \tag{2.1}$$

and of one *stock*, with price-per-share

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)] \ , \quad S(0) = p \in (0, \infty), \tag{2.2}$$

for $t \in [0,T]$. Here $T > 0$ is the time-horizon, and $W(\cdot)$ a standard, one-dimensional Brownian motion which generates (the augmented) filtration $\mathbf{F} = \{\mathcal{F}(t)\}$. The coefficients of the model $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot) > 0$ – are assumed to be bounded and $\mathbf{F}-$progressively measurable processes; furthermore, $\sigma(\cdot)$ is also assumed to be bounded away from zero (uniformly in (t, ω)).

A *trading strategy* is a pair (L, M) of $\mathbf{F}-$adapted processes on $[0,T]$, with left-continuous, nondecreasing paths and $L(0) = M(0) = 0$; $L(t)$ (respectively, $M(t)$) represents the total amount of funds transferred from bank account to stock (respectively, from stock to bank account) by time t. We relax the requirement of left-continuity at the terminal time T, i.e., we allow one final trade at that time. Given *proportional transaction costs* $\lambda \geq 0$, $0 < \mu \leq 1$ for such transfers, and initial holding $x > 0$ in bank, the portfolio holdings $X(\cdot) = X^{x,L,M}(\cdot)$ in bank and $Y(\cdot) = Y^{y,L,M}(\cdot)$ in stock satisfy

$$X(t) = x - (1+\lambda)L(t) + (1-\mu)M(t) + \int_0^t X(u)r(u)du, \ 0 \leq t \leq T, \tag{2.3}$$

$$Y(t) = L(t) - M(t) + \int_0^t Y(u)[b(u)du + \sigma(u)dW(u)], \ 0 \leq t \leq T. \tag{2.4}$$

A *contingent claim* is a pair (C_0, C_1) of $\mathcal{F}(T)-$measurable random variables. We say that a trading strategy (L, M) *hedges the claim* (C_0, C_1) *starting with* x *as initial capital*, if $X(\cdot), Y(\cdot)$ of (2.3), (2.4) satisfy

$$X(T) + (1-\mu)Y(T) \geq C_0 + (1-\mu)C_1; \quad X(T) + (1+\lambda)Y(T) \geq C_0 + (1+\lambda)C_1. \tag{2.5}$$

For the portfolio optimization problem introduced below we will consider only the strategies that hedge $(0,0)$. We denote by $\mathcal{A}^+(x)$ the set of terminal holdings $(X(T), Y(T))$ corresponding to such *admissible* strategies. Denote by $\mathbf{L}^0_+$ the set of non-negative $\mathcal{F}(T)-$measurable random variables. For the associated dual problem we need to introduce the set

$$\mathcal{H} := \left\{ Z \in \mathbf{L}^0_+ \ / \ E\left[\frac{Z}{B(T)}(X(T) + f(Y(T))\right] \leq x, \ \forall \ (X(T), Y(T)) \in \mathcal{A}^+(x) \right\}. \tag{2.6}$$

Here, $f(y) = (1-\mu)y$ if $y \geq 0$, and $f(y) = (1+\lambda)y$ if $y < 0$. We assume that at time T we transfer all the funds to the bank account and note that the corresponding terminal holdings $X(T+)$ satisfy

$$X(T+) := X(T) + f(Y(T)) \geq 0 \ , \quad \forall \ (X(T), Y(T)) \in \mathcal{A}^+(x). \tag{2.7}$$

We consider the following "retirement" utility maximization problem:

$$V(x) := \sup_{\mathcal{A}^+(x)} EU(X(T+)). \tag{2.8}$$

Here, $U : (0,\infty) \to \mathbf{R}$ is a strictly concave, strictly increasing, continuously differentiable utility function which satisfies $U'(0+) = \infty$, $U'(\infty) = 0$ and

Assumption 2.1 The utility function $U(x)$ has asymptotic elasticity strictly less than 1, i.e.

$$AE(U) \triangleq \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1. \tag{2.9}$$

It is shown in [KS98] that this condition is basically necessary and sufficient to ensure nice properties of value function $V(x)$ and the existence of an optimal solution.

Let us denote by $I(\cdot) : (0,\infty) \to (0,\infty)$ the continuous, strictly decreasing inverse function of $U'(\cdot)$. We have

$$I(0+) = \infty\ , \quad I(\infty) = 0\ , \quad \tilde{U}'(z) = -I(z)\ , \quad 0 < z < \infty, \tag{2.10}$$

where $\tilde{U}$ is strictly convex, nonincreasing dual function of U,

$$\tilde{U}(z) := \max_{x>0}[U(x) - xz] = U(I(z)) - zI(z).$$

For a given $z > 0$, the *auxiliary dual problem* associated with (2.8) is given by

$$\tilde{V}(z) := \inf_{Z\in\mathcal{H}} E\tilde{U}(zZ/B(T)). \tag{2.11}$$

More precisely, similarly as in [CK96], for every $z > 0$, $Z \in \mathcal{H}$ and $(X(T), Y(T)) \in \mathcal{A}^+(x)$ we have

$$EU(X(T+)) \leq E[\tilde{U}(zZ/B(T)) + X(T+)Z/B(T)] \leq E\tilde{U}(zZ/B(T)) + zx. \tag{2.12}$$

Consequently, we have

$$V(x) \leq \inf_{z>0}[\tilde{V}(z) + zx] =: \inf_{z>0} \gamma(z). \tag{2.13}$$

Remark 2.1 Standard duality approach for incomplete markets (see Karatzas & Shreve 1998, for example) suggests that we should look for pairs $(\hat{z}, \hat{Z}) \in (0,\infty)\times\mathcal{H}$ and $(\hat{X}(T+), 0) \in \mathcal{A}^+(x)$ such the inequalities in (2.12) and (2.13) become equalities. The pair $(\hat{X}(T+), 0)$ is then optimal for (2.8). It is easily seen that this is the case (namely that those inequalities become equalities) if and only if

$$(\hat{X}(T+), 0) = I(\hat{z}\hat{Z}/B(T)), 0) \in \mathcal{A}^+(x), \quad E\left[\hat{Z}I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right] = x.$$

Proposition 2.1 *For every $z > 0$ there exists $\hat{Z}_z \in \mathcal{H}$ that attains the infimum in* (2.11).

Proposition 2.2 *For every $x \in (0, \infty)$ there exists $\hat{z} \in (0, \infty)$ that attains the infimum of $\gamma(z)$ in* (2.13).

Denote $\hat{Z} := \hat{Z}_{\hat{z}}$ the optimal solution to (2.11) with $z = \hat{z}$ denoting the optimal solution to $\inf_{z>0} \gamma(z)$ of (2.13). Our main result is the following

Theorem 2.1 *The pair $(\hat{C}_0, \hat{C}_1) := (I(\hat{z}\hat{Z}/B(T)), 0)$ belongs to the set $\mathcal{A}^+(x)$ of (nonnegative) terminal holdings that can be hedged starting with initial wealth $x > 0$ in the bank account. Furthermore,*

$$E\left[U\left(I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right)\right] = V(x) = \inf_{z>0}[\tilde{V}(z) + zx] = \tilde{V}(\hat{z}) + x\hat{z}.$$

In particular, the strategy that hedges $(\hat{C}_0, \hat{C}_1)$ is optimal for the utility maximization problem (2.8).

Remark 2.2 The set $\mathcal{H}$ is an extension of the set of state-price densities $Z_0(T)$ used in [CK96], Jouini & Kallal (1995), and associated with this market: a positive martingale $Z_0(\cdot)$, $Z_0(0) = 1$, gives rise to a pricing density if there exists another positive martingale $Z_1(\cdot)$ such that $(1 - \mu)S(t) \le Z_1(t)/Z_0(t) \le (1 + \lambda)S(t)$, $0 \le t \le T$. If this is the case, then $Z_0(T) \in \mathcal{H}$. In our case, $Z_0(T) \in \mathcal{H}$ may not be associated with a martingale starting at $Z_0(0) = 1$. Nevertheless, as observed in [KS98], it is possible to consistently price a contingent claim $(C, 0)$ maturing at time T by setting the price to be equal to $\hat{E}[C/(B(T)\hat{C}_0)]$, where $\hat{E}$ is expectation taken under the probability measure $\hat{P}$, defined by $(d\hat{P})/(dP) = (\hat{Z}\hat{C}_0)/(xB(T))$. This, indeed, is a probability measure, by Lemma 3.2 below. The interpretation here is that we use the product $B(T)\hat{C}_0$ as a numeraire for the claim C, rather than just $B(T)$.

Remark 2.3 Under Assumption 2.1, there exist $z_0 > 0$, $0 < \gamma, \mu < 1$ and $0 < c < \infty$ such that

$$zI(z) < \frac{\gamma}{1-\gamma}\tilde{U}(z) \qquad \text{and} \qquad \tilde{U}(\mu z) < c\tilde{U}(z), \qquad \forall\ 0 < z < z_0; \tag{2.14}$$

see [KS98] Lemma 6.3 and Corollary 6.1 for details.

3 Appendix: Proofs and Lemmas

Proof of Proposition 2.1: We first observe that $\mathcal{H}$ is convex, closed under a.s-convergence by Fatou's lemma, and bounded in $\mathbf{L}^1(P)$; the latter is seen by setting $(X(T), Y(T)) =$

$(xB(T), 0)$ in (2.6), implying $EZ \leq 1$ for $Z \in \mathcal{H}$. Fix $z > 0$ and let $\{Z_n\}$ be a minimizing sequence for (2.11). By Komlós theorem (see Schwartz 1986), there exists a subsequence Z'_k such that

$$\tilde{Z}_k := \frac{1}{k} \sum_{i=1}^{k} Z'_i \to \hat{Z}_z \in \mathcal{H}$$

as $k \to \infty$, almost surely. As in Lemma 3.4 of [KS98], Fatou's lemma is applicable here, so that $\liminf_{k\to\infty} E\tilde{U}(z\tilde{Z}_k) \geq E\tilde{U}(z\hat{Z}_z)$. In conjunction with convexity of $\tilde{U}$ this easily implies that $\hat{Z}_z$ is optimal for (2.11). □

For a given progressively measurable process $\theta(\cdot)$ introduce the local martingale

$$Z_\theta(t) := \exp\{\int_0^t \theta(s)dW(s) - \frac{1}{2}\int_0^t \theta^2(s)ds\}\ , \quad 0 \leq t \leq T. \tag{3.1}$$

Denote $Z_0 = Z_{\theta^*}(T)$, the risk-neutral density for the market without transaction costs, where $\theta^*(t) := (r(t) - b(t))/\sigma(t)$ is the risk-premium process. It is easy to check that $Z_0 \in \mathcal{H}$ (see [CK96]).

Lemma 3.1 *Value function $V(\cdot) : (0, \infty) \to \mathbf{R}$ is finite, decreasing and strictly convex.*

Proof: It is straightforward to check that $\tilde{V}(\cdot)$ is decreasing and strictly convex. Next, since $r(\cdot)$ is bounded, we have $k^{-1} \leq B(T) \leq k$ for some $k > 0$. In conjunction with Jensen's inequality, we obtain

$$E\tilde{U}(zZ/B(T)) \geq \tilde{U}(zkE[Z]) \geq \tilde{U}(zk), \tag{3.2}$$

hence $\tilde{V}(z) \geq \tilde{U}(zk) > -\infty$. On the other hand, Assumption 2.1 ensures the existence of $0 < \alpha < 1$, $z_1 > 0$ such that

$$\tilde{U}(\mu z_1) < \mu^{\frac{\alpha}{\alpha-1}}\tilde{U}(z_1) \qquad \text{for all} \quad 0 < \mu < 1;$$

see [KS98] Lemma 6.3 for the proof. We get, since $Z_0 \in \mathcal{H}$,

$$\begin{aligned} \tilde{V}(z) &\leq E\tilde{U}(zZ_0/B(T)) = E\tilde{U}(zZ_0/B(T))\mathbf{1}_{\{zZ_0/B(T)>z_1\}} + E\tilde{U}(zZ_0/B(T))\mathbf{1}_{\{zZ_0/B(T)\leq z_1\}} \\ &\leq |\tilde{U}(z_1)| + (z/z_1)^{\frac{\alpha}{\alpha-1}}|\tilde{U}(z_1)| \cdot E\left[(Z_0/B(T))^{\frac{\alpha}{\alpha-1}}\right] < \infty. \end{aligned}$$

Proof of Proposition 2.2: We have $\tilde{V}(0+) = \tilde{U}(0+)$, so $\lim_{z\downarrow 0}\gamma(z) = \tilde{U}(0+) = U(\infty)$. Therefore, if $U(\infty) = \infty$, the infimum $\gamma(z)$ on $[0, \infty)$ cannot be attained at $z = 0$. Suppose now that $U(\infty) < \infty$ and that the infimum is attained at $\hat{z} = 0$, namely that $\inf_{z>0}\gamma(z) = \tilde{U}(0+)$. Then we have

$$x \geq \frac{\tilde{U}(0+) - \tilde{U}(zH)}{z} \geq E[HI(zH)]$$

for all $H \in \mathcal{H}$ and $z > 0$. Leting $z \to 0$ we get $x \geq \infty$, a contradiction. Therefore, either the infimum of $\gamma(z)$ is attained at a (unique) number $\hat{z} = \hat{z}_x \in (0, \infty)$ or it is attained at $\hat{z} = \infty$. If the latter is the case, then there exists a sequence $z_n \to \infty$ such that for z_n large enough and a fixed $z < z_n$, we have (by (3.2))

$$x \leq \frac{\tilde{V}(z) - \tilde{V}(z_n)}{z_n - z} \leq \frac{\tilde{V}(z) - \tilde{U}(z_n k)}{z_n - z}.$$

Letting $z_n \to \infty$ we get $x \leq 0$ by de l'Hospital's Rule, a contradiction. □

Lemma 3.2

$$\tilde{V}'(\hat{z}) = -E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = -x.$$

Proof: Let $h(z) := E[\tilde{U}(z\hat{Z}/B(T))]$. Then $h(\cdot)$ is convex, $h(\cdot) \geq \tilde{V}(\cdot)$ and $h(\hat{z}) = \tilde{V}(\hat{z})$. These three facts easily imply $\Delta^- h(\hat{z}) \leq \Delta^- \tilde{V}(\hat{z}) \leq \Delta^+ \tilde{V}(\hat{z}) \leq \Delta^+ h(\hat{z})$, where $\Delta^\pm$ denotes the left and the right derivatives. Because of this, it is sufficient to prove the lemma with $\tilde{V}$ replaced by h. It is easy to show, by monotone convergence theorem, that

$$\Delta^+ h(\hat{z}) \leq -E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right]. \tag{3.3}$$

On the other hand,

$$\Delta^- h(\hat{z}) \geq \limsup_{\varepsilon \to 0+} E\left[-\frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right)\right]$$

We claim that

$$\frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) = \frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) \mathbf{1}_{\{\hat{z}\frac{\hat{Z}}{B(T)} \geq z_0\}} + \frac{\hat{Z}}{B(T)} I\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right) \mathbf{1}_{\{\hat{z}\frac{\hat{Z}}{B(T)} < z_0\}}$$

is uniformly integrable when ϵ is small enough, where z_0 is the number from (2.14). Indeed, the first term is dominated by $\frac{\hat{Z}}{B(T)} I\left(\frac{\hat{z}-\varepsilon}{\hat{z}} z_0\right)$, which is uniformly integrable when ε is sufficiently small since $E[\frac{\hat{Z}}{B(T)}] \leq k \cdot E\hat{Z} \leq k$. It follows from (2.14) that the second term is dominated by

$$\frac{1}{\hat{z} - \varepsilon} \frac{\gamma}{1 - \gamma} \tilde{U}\left((\hat{z} - \varepsilon)\frac{\hat{Z}}{B(T)}\right),$$

which is in turn dominated by

$$\frac{1}{\hat{z} - \varepsilon} \frac{\gamma c}{1 - \gamma} \tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)$$

when ε is small. The uniform integrability follows from $E\left|\tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right| < \infty$.

Therefore, we can use the Mean Convergence Criterion to get the inequality

$$\Delta^- h(\hat{z}) \geq -E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right].$$

Together with (3.3) we establish $h'(\hat{z}) = -E[\frac{\hat{Z}}{B(T)} I(\hat{z}\frac{\hat{Z}}{B(T)})] = -x$. The latter equality follows from the fact that $\hat{z}$ attains $\inf_{z>0}[\tilde{V}(z) + xz]$. □

Lemma 3.3 *We have*

$$\sup_{Z\in\mathcal{H}} E\left[\frac{Z}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = E\left[\frac{\hat{Z}}{B(T)} I\left(\hat{z}\frac{\hat{Z}}{B(T)}\right)\right] = x.$$

Proof: For a given $Z \in \mathcal{H}$, $\varepsilon \in (0,1)$, let $Z_\varepsilon := (1-\varepsilon)\hat{Z} + \varepsilon Z \in \mathcal{H}$. By optimality of $\hat{Z}$ we get

$$0 \geq \frac{1}{\varepsilon} E\left[\tilde{U}\left(\frac{\hat{z}\hat{Z}}{B(T)}\right) - \tilde{U}\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right] \geq -\frac{1}{\varepsilon} E\left[\frac{\hat{z}(\hat{Z}-Z_\varepsilon)}{B(T)} I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right] = E\left[\frac{\hat{z}(Z-\hat{Z})}{B(T)} I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right]. \tag{3.4}$$

However, it follows that, as in the proof of Lemma 3.2,

$$\left(\frac{Z-\hat{Z}}{B(T)} I\left(\frac{\hat{z}Z_\varepsilon}{B(T)}\right)\right)^- \leq \frac{\hat{Z}}{B(T)} I\left(\frac{\hat{z}(1-\varepsilon)\hat{Z}}{B(T)}\right)$$

is uniformly integrable. We can now use Fatou's lemma in (3.4), to get

$$E\left[\frac{Z-\hat{Z}}{B(T)} I\left(\frac{\hat{z}\hat{Z}}{B(T)}\right)\right] \leq 0,$$

which completes the proof. □

Proof of Theorem 2.1: For fixed $x > 0$ define

$$\mathcal{C} := \{\xi \in \mathbf{L}^0_+ \ / \ xB(T)\xi \leq X(T) + f(Y(T)), \text{ for some } (X(T), Y(T)) \in \mathcal{A}^+(x)\}.$$

Denote by

$$\mathcal{C}^0 := \{Z \in \mathbf{L}^0_+ \ / \ E[Z\xi] \leq 1, \ \forall\, \xi \in \mathcal{C}\}$$

the polar of set $\mathcal{C}$. It is clear then that $\mathcal{H} = \mathcal{C}^0$. We also want to show $\mathcal{C} = \mathcal{H}^0 = \mathcal{C}^{00}$. By the bipolar theorem of Brannath & Schachermayer (1998), it is sufficient to show that $\mathcal{C}$ is convex, solid and closed under a.s.-convergence (a subset $\mathcal{C}$ of $\mathbf{L}^0_+$ is solid if $f \in \mathcal{C}$ and $0 \leq g \leq f$ imply $g \in \mathcal{C}$). It is obvious that $\mathcal{C}$ is convex and solid. On the other hand, from [CK96] (Theorem 4.1, in particular), we know that $\xi \in \mathcal{C}$ *if and only if*

$$E^*[(\xi B(T))^2] < \infty \quad \text{and} \quad \sup_{Z\in\mathcal{H}} E[Z\xi] \leq 1,$$

where E^* is expectation under the unique risk-neutral measure P^* corresponding to the market with no transaction costs ($\mu = \lambda = 0$). This implies (by Fatou's lemma) that $\mathcal{C}$ is closed under a.s-convergence, because the set $\{\xi B(T)\}_{\xi \in \mathcal{C}}$ is bounded in $\mathbf{L}^2(P^*)$. Indeed, the latter follows from [CK96] (as remarked in Appendix B of that paper, this can be shown by setting $U_n = V_n = 0$ in the arguments of its Appendix A; see (A.8)-(A.11) on p156). We conclude that $\mathcal{C} = \mathcal{H}^0$. Now, Lemma 3.3 implies $\frac{I(\hat{z}\hat{Z}/B(T))}{xB(T)} \in \mathcal{H}^0 = \mathcal{C}$, hence $(I(\hat{z}\hat{Z}/B(T)), 0) \in \mathcal{A}^+(x)$. This, in conjunction with Lemma 3.3 and Remark 2.1, implies the remaining statements of the theorem.

4 Conclusion

We have shown that the terminal wealth optimal for the utility maximization problem in a continuous-time market model with proportional transaction costs can be represented as the inverse of the marginal utility evaluated at a random variable optimal for a corresponding dual problem. We achieve this by appropriately defing the space of dual variables, and by using Komlós theorem on almost sure convergent subsequences. We leave open a difficult problem of finding an algorithm for actually constructing the trading strategy corresponding to the optimal terminal wealth. The problem of maximizing utility $E\int_0^T U(c(t))dt$ from consumption can be dealt with in a similar fashion, by considering the consumption process $c(\cdot)$ and corresponding dual processes as random variables on the probability space $(\Omega \times [0,T], \mathcal{F}(T) \times \mathcal{B}([0,T]), P \otimes \frac{1}{T} l)$, where l is the Lebesgue measure on $([0,T], \mathcal{B}([0,T]))$. We plan to consider the challenging problem of studying equilibrium in a market with transaction costs in future research.

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