# Dynamics of Contract Design with Screening *

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**Abstract**

We analyze a novel principal-agent problem of moral hazard and adverse selection in continuous time. The constant private shock revealed at time zero when the agent selects the contract has a long-term impact on the optimal contract. The latter is based not only on the continuation value of the agent who truthfully reports, but also contingent upon the continuation value of the agent who misreports, called temptation value. The good agent is retired when the temptation value of the bad agent becomes large, because then it is expensive to motivate the good agent. The bad agent is retired when the temptation value of the good agent becomes small, because then the future payment does not provide sufficient incentives. We also compare the efficiency of the shutdown contract and the screening contract and find that the screening contract can bring more profit to the principal only when the agent's reservation utility is sufficiently small.

*Key words*: Adverse selection, constant private shock, principal-agent model, continuous-time, continuation value, temptation value, dynamic moral hazard.

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# 1 Introduction

We investigate a novel problem of optimal dynamic contracting under moral hazard and adverse selection, in which the agent's preference is subject to a constant private shock. Adverse selection is important in practice, as agents (managers) may vary in terms of productivity and preferences, privately known only to themselves. Similarly, investors may lack information on the quality and the future profitability of a project of an entrepreneur who seeks financing.

Despite the important role of adverse selection in managerial compensation and contract design, dynamic contracting literature on this topic is very scarce.[1] To the best of our knowledge, there are no discrete-time models with infinite periods investigating constant private shock and dynamic moral hazard. Sung (2005) was the first to investigate continuous-time contracting with both adverse selection and moral hazard with constant private type, in a model in which a risk-neutral principal hires agents with CARA preferences, and all the decisions are taken at time zero; then, the optimal contract is linear, and effort is constant. Continuous-time papers dealing with pure adverse selection problem subject to repeated persistent shock include Zhang's (2009) Markov chain model in which the agent's utility is affected by a persistent random shock observed only by the agent, and Williams (2009) who considers a persistent private random shock with a continuum of states. Fong (2009) considers a dynamic mixed model with instantaneous payment in analyzing the dynamic environment of health care provision; however, different from our model, that paper imposes the assumption that the good agent has no conflict of interest with the principal. In Sannikov (2007a) a principal employs an agent to manage a project whose drift and outcome are observed only by the agent; the paper assumes that the agent consumes only at a finite horizon and uses a non-standard methodology.[2] Different from those papers, our main methodological contribution is that we extend the continuation value based approach to

[1]Dynamic models without adverse selection include the seminal paper by Holmstrom and Milgrom (1987), the first to explore continuous-time moral hazard models. Their work was generalized and extended by many authors, including Schättler and Sung (1993, 1997), Sung (1995, 1997). Cvitanić, Wan and Zhang (2008) generalized Holmstrom-Milgrom model to allow for general utility functions. Sannikov (2008) was the first to consider a dynamic moral hazard model with continuous payment, in a model in which the agent's continuation value process is the unique state variable. Williams (2006) investigates a general version of the same problem. Demarzo and Sannikov (2006) analyze dynamic capital security design with hidden savings. Biais, Mariotti, Plantin, and Rochet (2007) consider a model in which the arrival rate of investment opportunities is controlled by the agent.

[2]He, Wei and Yu (2012) consider an infinite-horizon variation of the Holmstrom and Milgrom (1987) model and study optimal dynamic contracting with endogenous learning. Giat, Hackman, and Subramanian (2010) consider the model in which the project value is observed, but its "risk premium" (drift term) is not observed, and the principal and the agent may have different prior beliefs about it; this is a useful approach for modeling venture capital investments, for example. Similarly, Prat and Jovanovic (2010) extend Sannikov (2008) model to the case of unobserved drift; the problem becomes hard and requires use of the maximum principle from stochastic control theory. Unlike our paper, the settings of these papers do not include adverse selection.

models that combine dynamic moral hazard and adverse selection.

In our model the private shock is constant and it models agent's skill, type or preferences, that remain unchanged throughout the agent's lifetime. In addition, our model involves dynamic moral hazard, which, in combination with a constant private shock, makes the problem difficult. The reason for the difficulty is that the contract payment transferred to the agent not only has to provide instantaneous incentives for the agent to work, but also to provide aggregate incentives for the agent to report the true shock value at time zero. Dealing with both simultaneously is quite challenging.

The model is a generalization of Sannikov (2008) to the case of adverse selection. He develops a continuation value based approach to explore the dynamic moral hazard problem that is a continuous-time analogue of the model in Spear and Srivastava (1987). The agent's continuation value is the total future expected utility conditional on the past history. Sannikov (2008) manages to reduce the agent's incentive problem to instantaneous conditions involving the volatility of the continuation value process. In our model, with private information at time zero, it is not enough to consider the volatility of each agent's continuation value, because of the additional concern that the agent may not be truthful about her type. Hence, continuation value is not the only state variable. Rather, we need to consider also the continuation value if the agent untruthfully reports her type, that we call the "temptation value process". The temptation value process, implicitly determined by the payment stream offered to the honest agent, provides incentives to the dishonest agent to exert effort. Then, by restricting the initial value of the temptation value process, the principal can induce the agent to report truthfully. Hence, when the principal designs the contract, he not only needs to consider how to provide incentives for exerting effort from an honest agent, but also how to control the temptation value process of her dishonest counterpart. The continuation value and temptation value processes then both affect the optimal payment stream. That is, the optimal contract is based on two state variables.

The main difficulty relative to the pure moral hazard model, is that, with the continuation value and the temptation value processes being coupled, it is not straightforward to identify appropriate boundary conditions and the domain of the relevant value functions. This domain, called the "credible set", is the set of pair values that can be implemented as expected utility values by admissible payment streams. If a pair consisting of the initial values of the continuation value and the temptation value processes lies outside the credible set, then there exists no payment stream that implements the honest and dishonest agents' expected payoffs at time zero.

Motivated by Abreu, Pearce and Stacchetti (1990) and Sannikov (2007b), we construct a method for computing the credible set. It has two boundaries, that we call "stationary boundary" and "extreme boundary". When the state variable processes reach the stationary boundary of the credible set, the contract is terminated. In our version of the pure moral hazard problem, the contract is terminated (the agent retires) simply when the continuation value reaches the minimum or the maximum possible value. However, when the moral hazard is mixed with adverse selection, it is possible that the contract is terminated at any level of the continuation value process, depending on the temptation value process. The contract of the good agent is terminated when the temptation value process of the bad agent becomes large, as the good agent becomes too expensive to motivate. The contract of the bad agent is terminated when the temptation value process of the good agent becomes small, because the contract offers too few incentives. When the state variable processes reach the extreme boundary of the credible set, they continue moving along the tangential direction of the set. While the stationary boundary is a line, the extreme boundary is more complex, it is a solution to an ordinary differential equation (ODE).

We also consider shutdown contracts, that is, the contracts that the bad agent would not accept. We compare the efficiency of the optimal screening contract and the optimal shutdown contract for different utility reservation values of the bad agent. We find that, when the reservation value is high, it is more profitable for the principal to offer the shutdown contract. When the reservation value is low, it is better for the principal to offer the screening contract (that agents of both types will accept). In static models (see Laffont and Martimort 2002), a significant inefficiency is a feature of the shutdown contract, because the bad agent will not be producing. In our model, however, it may or may not be more efficient to offer the screening contract. Expanding on Sannikov (2008), who identifies the "income effect" inefficiency in dynamic moral hazard problems, we find that the good agent's incentives are affected not only by her own income effect, but also by the dishonest agent's income effect. When the reservation value is low, the shutdown contract leads to a low rent for the good agent and the expected utility of the dishonest agent has to be lower than the reservation value. However, low expected payoff for the dishonest agent may reduce incentives to the honest agent. By comparison, under the screening contract and with low reservation value, both the good and bad agents' expected payoffs are not binding at reservation utility, thus providing better incentives to the good agent. When the reservation utility is high and the income effect of the dishonest agent becomes less relevant, then the shutdown contract is better because it brings down the good agent's rent. Hence, we conclude that the screening contract may be better not only because the bad agent does not produce if not offered a contract, but also because screening provides strong incentives to the good agent if the reservation utility is low.

The remainder of this paper is organized as follows. In Section 2 we discuss the model and setup. In Section 3 we find the optimal contract with pure moral hazard. In Section 4 we investigate the optimal shutdown contract with both moral hazard and adverse selection. The optimal screening contract is presented in Section

5. We conclude the paper in Section 6. The proofs are presented in the appendix.

## 2 Model

Time is continuous. A standard Brownian motion $Z = \{Z(t), \mathcal{F}(t)\}_{t \geq 0}$ on $(\Omega, \mathcal{F}, P)$ drives the output process. The total output $Y(t)$ produced up to time $t$ evolves according to

$$dY(t) = a(t)dt + \sigma dZ(t),$$

where $a(t)$ is the manager's choice of effort level and $\sigma$ is a constant. The manager's effort is a stochastic process $a = \{a(t)\}_{t \geq 0}$ that is progressively measurable with respect to $\mathcal{F}(t)$, where the set of feasible effort levels $\mathcal{A}$ is binary: $\mathcal{A} = \{0, a_M\}$. The effort is costly with instantaneous cost $g(a)$ such that $g(a_M) > g(0) = 0$, and is measured in the same units as the utility of the agent's consumption.

A firm owner (the principal, he) signs a contract with a manager (the agent, she) at time 0. As determined by the contract, the principal makes an instantaneous payment $c = \{c(t)\}_{t \geq 0}$ to the manager, and the manager's utility of payment is $\theta u(c(t))$, where $u(\cdot)$ is increasing and concave. We normalize $u(0) = 0$ and denote the inverse function of $u(\cdot)$ by $v(\cdot)$. The instantaneous payment $c(t)$ can take values only in a compact set $\mathcal{C} = [0, c_M]$. Parameter $\theta$ stands for the manager's type. We assume that the managers in the labor market have only two types taking values in the set $\Theta = \{\theta_g, \theta_b\}$ with $\theta_b < \theta_g$. We call the manager good (bad) if her type is $\theta_g$ ($\theta_b$ ). Moreover, it is common knowledge that the proportion of the managers of type $\theta_i, i = g, b$, is $p_i$.

The output process $Y$ is publicly observable by both the firm owner and the manager. $\mathcal{F}^Y(t)$ is the information flow generated by $\{Y(s)\}_{s \leq t}$. The firm owner cannot observe the manager's effort $a$ or her type $\theta$, known only by the manager. Hence,we have a contracting problem with both adverse selection and moral hazard.

The firm owner offers a menu of contracts $\Psi_i = \{c_i, a_i\}$, $i = g, b$, that specifies a bounded flow of payments $c_i = \{c_i(t)\}_{t \geq 0}$ and desired effort $a_i = \{a_i(t)\}_{t \geq 0}$ based on his observations of output and the agent's reported type. The desirable level of effort is the level that the firm owner recommends to the manager.

Assume that both the firm owner and the manager discount the flow of profits and utility at a common rate $r$. If the manager's type is $\theta_i$, with payment $c_i$ and chosen effort level $a_i$, then her expected utility is given by

$$V(\theta_i, c_i, a_i) = rE\left\{\int_0^\infty e^{-rs}\left[\theta_i u(c_i(s)) - g(a_i(s))\right] ds\right\},$$

and the firm owner is risk neutral with expected profit

$$r\sum_{i=g,b} p_i E\left\{\int_0^\infty e^{-rs}\left[dX(s)-c_i(s)ds\right]\right\} = r\sum_{i=g,b} p_i E\left\{\int_0^\infty e^{-rs}\left[a_i(s)-c_i(s)\right]ds\right\}. \tag{2.1}$$

Factor $r$ in front of the integrals normalizes the cumulative payoffs to the same scale as the flow payoffs.

### 2.1 Formulation of firm owner's problem

Assume that the reservation utility for managers of both types is $R$. The owner's problem is then to offer a contract menu $\{\Psi_i\}_{i=g,b}$ that maximizes his profit (**??**) subject to delivering to the agent a required initial utility value of at least $R$. We write these "individual reservation" (IR) constraints as

$$V(\theta_i, c_i, a_i) \geq R,\ \ i = g, b. \tag{2.2}$$

There are also two incentive compatibility conditions:

$$V(\theta_i, c_i, a_i) = \max_{\hat{a}}\ V(\theta_i, c_i, \hat{a}), \tag{2.3}$$

$$V(\theta_i, c_i, a_i) \geq \max_{\hat{a}}\ V(\theta_i, c_j, \hat{a}), \tag{2.4}$$

where $i \neq j,\ i, j = g, b$. Such a contract is called a "screening contract." Condition (**??**) states that, given $c_i$, the principal's effort recommendation is the agent's best response when she truthfully reports her type. Condition (**??**) means that, if the agent adversely selects a contract, then her expected utility cannot be better than what it would be if she truthfully reported her type at time 0.

We first derive the optimal contract under pure moral hazard, without adverse selection, as the main benchmark. Next, we derive the "optimal shutdown contract", that is, the contract which deliberately excludes the bad type. Finally, we find the optimal screening contract.

## 3 Optimal Contract with Pure Moral Hazard

In this section, we assume that the manager's type is publicly known and discuss optimal contracting under pure moral hazard. This contract can be found by familiar methods that summarize the agent's incentives using her continuation value, i.e., her future expected payoff when she chooses the principal's desired effort,[3] that is

$$W_i(t) = rE_t\left\{\int_t^\infty e^{-r(s-t)}\left[\theta_i u(c_i(s)) - g(a_i(s))\right]ds\right\},\ \ i = g, b.$$

[3] Continuation value based methods were developed by Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990), and in continuous time by Sannikov (2008).

The optimal contract can then be derived using the dynamic programming approach. Denote by $F^i(W_i), i = g, b$, the principal's expected profit if the agent's type is $\theta_i$.

Note that the agents' continuation values are bounded. The following definition introduces the domain of their feasible payoffs.

**Definition 3.1** *Set*

$$\mathcal{V} = [0, \theta_b u(c_M)] \times [0, \theta_g u(c_M)] \subset \mathcal{R}^2$$

*is called the "feasible set" for the expected payoff pairs $[W_b, W_g]$ of bad and good managers.* [4]

It follows from the results below that the unique way of delivering $W_i(t) = 0$ is to offer zero payment after time $t$, in which case the agent's effort is zero. The corresponding principal's expected profit is $F^i(0) = 0$. Moreover, the unique way to deliver $W_i(t) = \theta_i u(c_M)$ is to make a constant payment $c_i(s) = c_M$ after time $t$, in which case the agent's optimal effort is $a_i(s) = 0$, for $s \geq t$, and so $F^i(\theta_i u(c_M)) = -c_M$. These are the boundary conditions needed to find the optimal contract in this setting. The method we apply is that of Sannikov (2008), and the proofs of Lemma **??** and Proposition **??** below, are the same as in that paper, although our pure moral hazard setting differs from Sannikov's (2008) in the boundary conditions.

The following result gives the instantaneous incentive compatibility conditions for the managers.

**Lemma 3.1** *Given a payment process $c_i$ and effort process $a_i$, there exists an adapted process $\beta^i$ such that the agent's continuation value evolves according to*

$$dW_i(t) = r\left[W_i(t) + g(a_i(t)) - \theta_i u(c_i(t))\right]dt + r\beta_i(t)(dY(t) - a_i(t)dt), \quad i = g, b. \tag{3.5}$$

*Moreover, the agent with type $\theta_i$ will optimally exert the recommended effort $a_i$ if and only if the following incentive compatibility condition holds:*

$$-g(a_i) + \beta_i a_i \geq -g(\hat{a}) + \beta_i \hat{a}, \ \textit{for all } \hat{a} \in \mathcal{A}. \tag{3.6}$$

For the sake of smoother terminology, we introduce the following definition.

**Definition 3.2** *If $(a_i, \beta_i)$ satisfies instantaneous incentive compatibility condition (**??**), then we say that $\beta_i$ enforces effort level $a_i$.*

The reason behind the incentive compatibility condition is that the drift of the agent's continuation value in (**??**) depends on $\beta_i a_i - g(a_i)$, so the best response for an agent of type $\theta_i$ is to maximizes $-g(a) + \beta_i a$.

[4] This domain is determined by the bounds on the instantaneous benefit $\theta_g u(c) - g(a)$.

Given the desired effort $a_i$, the principal will choose process $\beta_i$ that enforces $a_i$ and has the smallest absolute value among such process. The choice of the smallest absolute value process is due to the concavity of the principal's value function, as we will see below. We use $\gamma(a_i)$ to denote such $\beta_i$.

**Proposition 3.1** *The optimal (incentive compatible) contract* $\{c_i(W_i(t)), a_i(W_i(t))\}$ *is determined by the maximization in the optimality (HJB) equation for the principal's value function*

$$F^i(W) = \max_{a_i, c_i} \left\{ a_i - c_i + [W + g(a_i) - \theta_i u(c_i)] F^i_W(W) + \frac{r\sigma^2\gamma^2(a_i)}{2} F^i_{WW}(W) \right\},$$

*satisfying boundary conditions*

$$F^i(0) = 0 \ , \quad F^i(\theta_i u(c_M)) = -c_M. \tag{3.7}$$

*Here,* $W_i(t)$ *is the continuation value process of the agent with type* $\theta_i$ *following the dynamics (??), and it's initial value is any value such that* $W_i(0) \in arg\max_{\hat{w} \geq R, \in [0, \theta_i u(c_M)]} F^i(\hat{w})$.

An important finding in Sannikov (2008) is that the agent's initial expected payoff at time 0 may be strictly larger than the reservation utility $R$, if the reservation utility is low enough. A typical form of the value function $F^i(W)$, together with $c_i(W)$ and $a_i(W)$, is shown in Figure 1. Numerical results show that the optimal contract motivates the good manager to work throughout the contract period. However, it may be too costly for the principal to compensate the bad manager for her effort, and so the desired and optimal effort is 0 when her continuation value is sufficiently large. Moreover, consistent with the findings in Sannikov (2008), if the manager's continuation value is low enough, then even without being payed she still may have an incentive to work in order to move $W_i(t)$ away from the low retirement point (equal to zero).

Figure 1: Function $F^i(W)$ for $u(c) = \sqrt{c}$, $\theta_g = 2$, $\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$. Points $w_b^0$ and $w_g^0$ are the maximum of $F^b(W)$ and $F^g(W)$ respectively.

# 4 Optimal Shutdown Contract under Adverse Selection and Moral Hazard

Before discussing the optimal screening contract, it is helpful to investigate the shutdown contract first, in which the principal deliberately excludes the bad manager from hiring. Assume that the firm owner only wants to hire the good manager. He offers a contract $\Psi_g = \{c_g(s), a_g(s)\}_{s\geq 0}$, which only the good manager accepts, whereas the bad manager prefers to take outside opportunity $R$. The principal's problem is to choose $\Psi_g$ to maximize

$$rE\left\{\int_0^\infty e^{-rs}\left[a_g(s)-c_g(s)\right]ds\right\},$$

such that conditions (**??**) and (**??**) for the good manager hold, and

$$R \geq \max_{\hat{a}\in\mathcal{A}} V(\theta_b, c_g, \hat{a}). \tag{4.8}$$

Here, the right-hand side is the maximum expected utility that the bad manager can obtain if she takes the shutdown contract. Under constraint (**??**), she would not do it, rather, she would take the outside opportunity.

## 4.1 Credible set

### Definition of credible set

At time 0, the principal offers payment stream $c_g = \{c_g(t)\}_{t\geq 0}$, which is progressively measurable with respect to $\mathcal{F}^Y = \left\{\mathcal{F}_t^Y\right\}_{t\geq 0}$. Both the good and bad managers may choose to take it. When their best efforts are exerted by the managers, their continuation value processes in general will be different. We use $W_b^c = \{W_b^c(t)\}_{t\geq 0}$ to denote the bad manager's continuation value, if she takes the contract and behaves optimally. To distinguish it from the continuation value of the good manager, we call it the bad manager's "temptation value process." We use a superscript "c" in order to distinguish it from $W_b$ in the screening contract, which is the bad manager's continuation if she truthfully reports her type and obtains payment stream $c_b = \{c_b(t)\}_{t\geq 0}$ that is designed for the bad manager. It is crucial to distinguish between $W_b(t)$ and $W_b^c(t)$ in solving for the optimal contracts. Denote the bad manager's best effort choice by $a_b^c = \{a_b^c(t)\}_{t\geq 0}$. By Proposition **??**, we have

$$dW_g(t) = r\left[W_g(t) + g(a_g(t)) - \theta_g u(c_g(t))\right]dt + r\beta_g(t)[dY(t) - a_g(t)dt], \tag{4.9}$$

$$dW_b^c(t) = r\left[W_b^c(t) + g(a_b^c(t)) - \theta_b u(c_g(t))\right]dt + r\beta_b^c(t)[dY(t) - a_b^c(t)dt], \tag{4.10}$$

where the conditions of incentive compatibility are

$$a_g \in arg\max_{a\in\mathcal{A}} \; -g(a(t)) + \beta_g(t)a(t), \tag{4.11}$$

$$a_b^c \in arg\max_{a\in\mathcal{A}} \; -g(a(t)) + \beta_b^c(t)a(t). \tag{4.12}$$

If the good manager takes the contract, $W_g(t)$ is the continuation value process of the good manager and $\frac{1}{\sigma}\left[dY(t) - a_g(t)dt\right]$ is the increment of the Brownian motion process. If the bad manager takes the contract, $W_b^c(t)$ is the continuation value process, or "temptation value process," of the bad manager and $\frac{1}{\sigma}\left[dY(t) - a_b^c(t)dt\right]$ is the increment of the Brownian motion process.

When the principal designs the contract, he not only needs to consider the good manager's incentive, as he would in the pure moral hazard setting, but he also needs to identify the possible outcomes if the bad manager takes the contract, and process $W_b^c$ is the one which summarizes the bad manager's incentives. Hence, the optimal contract design should be based on two state variables, $W_g$ and $W_b^c$, which are fully coupled through $c_g$ and $Y$. The principal has to satisfy the following constraints: condition (**??**) for the good manager, equivalent to (**??**), condition (**??**), equivalent to (**??**), and the exclusion condition

$$W_b^c(0) \leq R. \tag{4.13}$$

Condition (**??**) states that if the bad manager pretends to be a good manager, then her expected utility at time 0 cannot be better than her reservation $R$. Hence, only the good manager will take the contract, assuming that

$$W_g(0) \geq R. \tag{4.14}$$

Thus, by utilizing the continuation value processes, we transform the global conditions into instantaneous conditions and initial value conditions, thereby greatly simplifying the contracting problem. It is to be noted that, although asymmetric information exists only at time 0, it has a long-term effect on contract design and the dynamics of optimal contracts.

In order to solve the problem, we will need to identify the right boundary conditions. From the previous section we know that $\{W_b^c(t), W_g(t)\}$ cannot move outside the feasible set $\mathcal{V}$. However, not every value pair in the feasible set can be implemented by incentive-compatible contracts. Motivated by Abreu, Pearce and Stacchetti (1990) and Sannikov (2007b), we define the "credible set" as follows.

**Definition 4.3** *Consider the set $\mathcal{E}$ of initial value pairs $(w_b^c, w_g)$ in $\mathcal{V}$ for which there exists a tuple $[c_g, a_g, a_b^c, \beta_g, \beta_b^c]$ such that the corresponding payoff processes pair $\{W_b^c(t), W_g(t)\}$, with dynamics (**??**) and (**??**), satisfies (**??**) and (**??**), and takes values in $\mathcal{V}$ for all $t$, almost surely. Set $\mathcal{E}$ is called the "credible set".*

In other words, given an initial value pair $(w_b^c, w_g)$ outside the credible set, there exists no payment stream $c = \{c(t)\}_{t\geq 0}$ taking values in $\mathcal{C}$, such that, if the bad (good) manager takes the contract, then her optimal expected utility at time 0 is $w_b^c$ ( $w_g$). That is, given such initial value pair $(w_b^c, w_g)$, with any payment stream the corresponding pair $(W_g^t, W_b^c(t))$ will move out of the feasible set with positive probability.

### Characterization of the credible set

Sannikov (2007b) developed a curvature-based approach for characterizing credible sets. Motivated by that work, we introduce a method useful in finding credible sets when the optimal contract is based on several coupled state processes.

We want to know, given $W_b^c(t)$, what the largest or the smallest expected utility is that the good manager can achieve at time $t$. We denote the largest utility by $U(W_b^c(t))$ and the smallest by $L(W_b^c(t))$. Define

$$\bar{\mathcal{E}} = \left\{ (w_b^c, w_g) \in \mathcal{V}, \text{ s.t. } w_b^c \in [0, \theta_b u(c_M)] \text{ and } L(w_b^c) \le w_g \le U(w_b^c) \right\}.$$

We will show that $\mathcal{E} = \bar{\mathcal{E}}$, so we call $U(w_b^c)$ ($L(w_b^c)$) the upper ( lower) boundary of the credible set. Figure **??** presents an example of the credible set.

Figure 2: **The credible set. The lower boundary is the line segment connecting $O = (0,0)$ and $M = (\theta_b u(c_M), \theta_g u(c_M))$. Vector $n_u = (1, -U_W(W_b^c))$ ($n_l = (\frac{\theta_g}{\theta_b}, -1)$) is the normal vector at the upper (lower) boundary (pointing outward). The parameters are $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.**

To derive $U(w_b^c)$, note that we have

$$e^{-rv} W_g(v) = rE_v \left\{ \int_v^\infty e^{-rs} \left[ \theta_g u(c_g(s)) - g(a_g(s)) \right] ds \right\}, \tag{4.15}$$

if the good manager takes the contract. The increment of Brownian motion is $dZ(t) = \frac{1}{\sigma}[dY(t) - a_g(t)\, dt]$, and the bad manager's temptation process from the good manager's perspective is

$$dW_b^c(t) = r \left( W_b^c - \theta_b u(c_g(t)) + g(a_b^c(t)) + \underbrace{\beta_b^c(t)(a_g(t) - a_b^c(t))}_{\textit{Private benefit of the bad manager}} \right) dt + r\sigma\beta_b^c(t)\, dZ(t), \tag{4.16}$$

with $(\beta_b^c, a_b^c)$ satisfying (**??**).

The term $\beta_b^c(t)(a_g(t) - a_b^c(t))$ is the bad manager's instantaneous benefit from the good manager's perspective. We consider the following formulation for $U(w_b^c)$. Given a fixed initial time $v$, the good manager chooses $[c_g(.), a_b^c(.), \beta_b^c(.)]$ to solve the problem

$$e^{-rv} U(w_b^c) = \max_{c_g, a_b^c, \beta_b^c} rE \left\{ \int_v^\infty e^{-rs} \left[ \theta_g u(c_g(s)) - g(a_g(s)) \right] ds | W_b^c(v) = w_b^c \right\}, \tag{4.17}$$

subject to dynamics (**??**), satisfying (**??**) and

$$a_g \in \arg\max_{\hat{a}_g} rE \left\{ \int_v^\infty e^{-rs} \left[ \theta_g u(c_g(s)) - g(\hat{a}_g(s)) \right] ds | W_b^c(v) = w_b^c \right\}, \tag{4.18}$$

As shown above, condition (**??**) implies condition (**??**).

**Remark 4.1** *The maximization problem (**??**) can be considered as the contracting problem in which the good manager hires the bad manager subject to the double-sided moral-hazard problem (as in Bhattacharyya and*

*Lafontaine 1996), where the bad manager and the good manager have heterogeneous beliefs about the expected payoff, and they agree to disagree. The bad manager evaluates his expected payoff under the measure in which the increment of the underlying Brownian motion is given by* $\frac{1}{\sigma}\left[dY(t) - a_b^c(t)dt\right]$.

To derive the optimality equation for $U(.)$, we need to show that the Dynamic Programming Principle, DPP, (or "recursive formulation") holds for the value function $U(.)$. Different from Spear and Srivastava (1987), the good manager's effort has to satisfy condition (**??**), and it is not obvious that the recursive formulation of Spear and Srivastava (1987) holds in our setting. Nevertheless, the DPP holds in the following form:

**Proposition 4.2** *For any stopping time* $\tau \geq v$*, we have*

$$U(w_b^c) = \max_{c_g, a_g, a_b^c, \beta_b^c} E\left\{\int_v^\tau re^{-r(s-v)}\left[\theta_g u(c_g(s)) - g(a_g(s))\right]ds + e^{-r(\tau-v)}U(W_b^c(\tau))|W_b^c(v) = w_b^c\right\} \tag{4.19}$$

*subject to* (**??**) *and (*__??__*).*

Note that the DPP of Proposition **??** does not require the good manager's incentive compatibility condition (**??**). Instead, the maximization is performed also over $a_g$, that is, by choosing a quadruple $[c_g(.), a_g(.), a_b^c(.), \beta_b^c(.)]$ in (**??**).

Applying the DPP of Proposition **??**, standard arguments imply that the following HJB equation for $U(W_b^c)$ holds:

$$U(W) = \max_{c_g, a_g, a_b^c, \beta_b^c}\left\{\theta_g u(c_g) - g(a_g) + U_W(W)\left[W - \theta_b u(c_g) + g(a_b^c) + \beta_b^c(a_g - a_b^c)\right] + \frac{r\sigma^2}{2}(\beta_b^c)^2 U_{WW}(W)\right\}, \tag{4.20}$$

such that $\beta_b^c$ enforces $a_b^c$, $U(0) = 0$ and $U(\theta_b u(c_M)) = \theta_g u(c_M)$. The last two conditions are boundary conditions. Obviously, if $W_b^c(t) = 0$ ($W_b^c(t) = \theta_b u(c_M)$), then the bad manager's expected utility at time $t$ if she takes the contract is 0 ($\theta_b u(c_M)$). The payment after time $t$ would be $\{c_g(s) = 0\}_{s\geq t}$ ($\{c_g(s) = c_M\}_{s\geq t}$). Then, the good manager's expected utility at time $t$ would be 0 ($\theta_g u(c_M)$).

While in (**??**) the incentive compatibility condition for the good manager is not explicit, it is implied. The optimal $a_g$ is computed by solving

$$\max_{a_g} -g(a_g) + \beta_b^c U_W(W_b^c)\, a_g$$

which means that $\beta_b^c U_W(W_b^c)$ enforces $a_g$. Moreover, the diffusion term of $W_g(t) = U(W_b^c(t))$ is

$$\sigma\beta_b^c(t)\, U_W(W_b^c(t))\, dZ(t) = \beta_b^c(t)\, U_W(W_b^c(t))\left[dY(t) - a_g(t)\, dt\right],$$

which implies that $\beta_g(t) = \beta_b^c U_W(W_b^c(t))$, hence (**??**) still holds on the upper boundary of the credible set.

Similarly, we can find the lower boundary of the credible set as a function $W_g = L(W_b^c)$. We summarize our main findings for the credible set as follows.

**Proposition 4.3** *Upper boundary $U(W)$ of the set $\bar{\mathcal{E}}$ is the unique solution of optimality equation (??), that is strictly increasing and strictly concave. The lower boundary is given by equation $W_g = L(W_b^c) = \frac{\theta_g}{\theta_b} W_b^c$ on $[0, \theta_b u(c_M)]$. Moreover, if the continuation value pair $(W_b^c(t), W_g(t))$ reaches the upper boundary, then it will move along the boundary following the strategy determined by equation (??) until it is absorbed by $(0,0)$ or $(\theta_b u(c_M), \theta_g u(c_M))$. If $(W_b^c(t), W_g(t))$ reaches the lower boundary at $P^* = (\theta_b w^*, \theta_g w^*)$ for any $w^* \in [0, u(c_M)]$ at time $t$, then it will stay at $P^*$ forever and the payment stream is a constant $c_g(s) = c^* = v(w^*)$ for $s \geq t$. That is, the contract is terminated at time $t$, and the agent is retired with a constant payment $c^*$ after time $t$.*

From Proposition **??** we see that the lower boundary is a "stationary boundary", in the sense that the continuation values do not change after hitting it. When a value pair reaches the stationary boundary the agent is retired and receives a constant payment after retirement. The upper boundary is an "extreme boundary" in the sense that the only way to implement an expected payoff pair on the extreme boundary is to make the continuation value and temptation value processes move along the tangent direction on the extreme boundary. Intuitively, if there exists $[c_g, a_g, a_b^c, \beta_g, \beta_b^c]$ such that the expected payoff pair moves inward, then the payoff pair should not be on the boundary of the credible set. The tangential movement on the extreme boundary can seen from optimality equation (**??**). Indeed, first we note that pair $(\beta_b^c, \beta_g)$ is in the tangent direction of the extreme boundary, because $\beta_g = \beta_b^c U_W(W_b^c)$, and vector $(-U_W(W_b^c), 1)$ is the normal vector of the extreme boundary. In other words, the volatility terms of the two value processes move on the tangent line. Moreover, denoting $l(W_b^c, W_g) = W_g - U(W_b^c)$ the level function, optimality equation (**??**) implies that $dl(W_b^c(t), W_g(t)) = 0$ after $(W_b^c, W_g)$ reaches the upper boundary. Hence, $(W_b^c, W_g)$ moves tangentially on the boundary of the zero level set of the level function $l(W_b^c, W_g)$, until it reaches $(0,0)$ or $(\theta_b u(c_M), \theta_g u(c_M))$. This tangential movement of $(W_b^c, W_g)$ on the upper boundary is consistent with the curvature characterization in Sannikov (2007b).

**Corollary 4.1** *There exists $w^*$ in $(0, \theta_b u(c_M))$ such that $U_W(w^*) = \frac{\theta_g}{\theta_b}$. On the extreme boundary, if $W_b^c < w^*$, the optimal compensation is 0, otherwise optimal compensation is $c_M$. Moreover, we have $a_g \geq a_b^c$ on the extreme boundary. For all $W \in (0, \theta_b u(c_M))$, we have $U_W(W) > 1$ and $\beta_g \geq \beta_b^c$.*

We have shown that all expected payoff pairs on the boundaries are all achievable. In order to show $\bar{\mathcal{E}} = \mathcal{E}$, it remains to show that all expected payoff pairs inside $\bar{\mathcal{E}}$ are achievable.

**Proposition 4.4** $\bar{\mathcal{E}} = \mathcal{E}$.

**Proof.** Given any pair $(W_b^c(0), W_g(0)) = (w_b^c, w_g)$ inside $\bar{\mathcal{E}}$, let $\beta_g(t) = \beta_b^c(t) = a_g(t) = a_b^c(t) = c_g(t) = 0$ for $t \leq \tau$, where $\tau$ is the first time

$$(W_b^c(t), W_g(t)) = (e^{rt} w_b^c, e^{rt} w_g)$$

hits the upper boundary of $\bar{\mathcal{E}}$, that is, $\tau$ is determined by

$$e^{r\tau} w_g = U(e^{r\tau} w_b^c).$$

For $t > \tau$, choose $[\beta_g(t), \beta_b^c(t), a_g(t), a_b^c(t), c_g(t)]_{t\geq\tau}$ as determined by the optimization in the HJB equation (**??**). Then, $(W_b^c(t), W_g(t))$ will remain in $\bar{\mathcal{E}}$, and thus $(w_b, w_g)$ is achievable. ■

Another natural question is if there is any payoff pair inside $\mathcal{E}$ which is stationary, that is, such that the only way to implement it is that it remains unchanged. And if there is any pair that is extreme, in the sense that there exists a unique way to achieve it. From the proof of Proposition **??**, we already know that no payoff pair inside $\mathcal{E}$ is stationary, because there is a path that leads it to the upper boundary. The following corollary implies that no payoff pair inside $\mathcal{E}$ is extreme either, because for any pair there is also a path that leads it to the lower boundary.[5]

**Corollary 4.2** *There exists a multiple $[c_g(t), a_g(t), \beta_g(t), a_b^c(t), \beta_b^c(t)]_{t\geq 0}$, such that $(W_b^c(t), W_g(t))$, starting from $(w_b^c, w_g) \in \mathcal{E}$ at time 0, ends at the lower boundary before time $T^*$ almost surely, where*

$$T^* = \frac{1}{r} log \left( \frac{\frac{\theta_g}{\theta_b} I(\frac{\theta_g}{\theta_b}) - U(I(\frac{\theta_g}{\theta_b}))}{w_g - U(w_b^c)} \right) > 0$$

*and $I(.)$ is the inverse function of $U_W(.)$.*

For concreteness, we described $\mathcal{E}$ as the credible set from the good manager's perspective. However, it is also the credible set from the bad manager's persepctive. This is because $\mathcal{E}$ depends on the dynamics of $(W_b^c(t), W_g(t))$, not on who takes the contract.

## 4.2 Contract design

We now discuss the principal's problem. Denote the principal's value function by $J^g(W_b^c, W_g)$ if the good manager takes the contract. It is dependent on two state variables: the good manager's continuation value process and the bad manager's temptation process. We denote the first-order derivatives with respect to $W_g$ and $W_b^c$ by $J_2^g$ and $J_1^g$, and the second-order derivatives by $J_{11}^g$, $J_{12}^g$ and $J_{22}^g$. Recalling (**??**), the optimality equation is:

$$J^g(W_b^c, W_g) = \max_{c_g, a_g, a_b^c, \beta_g, \beta_b^c} a_g - c_g + \frac{r\sigma^2}{2} \left[\beta_g^2 J_{22}^g + 2\beta_g \beta_b^c J_{12}^g + (\beta_b^c)^2 J_{11}^g\right] + [W_g - \theta_g u(c_g) + g(a_g)] J_2^g + [W_b^c - \theta_b u(c_g) + g(a_b^c) + \beta_b^c (a_g - a_b^c)] J_1^g. \tag{4.21}$$

[5]Thus, for any pair inside $\mathcal{E}$, there are at least two different paths that can achieve it. In fact, if the pair is inside $\mathcal{E}$, the choice of $[\beta_g(t), \beta_b^c(t), a_g(t), a_b^c(t), c_g(t)]_{t\geq\tau}$ is very flexible, subject only to (**??**) and (**??**).

such that $\beta_b^c$ enforces $a_b^c$ and $\beta_g$ enforces $a_g$.

Moreover, the principal's value function is defined on credible set $\mathcal{E}$, and the boundary conditions depend on the behavior of the optimal contract on the boundary. First, notice that condition (**??**) is reduced to choosing the optimal initial value for the bad manager's value process. Next, as stated in the above proposition, on the extreme boundary the terms of the optimal vector $(c_g, a_g, a_b^c, \beta_g, \beta_b^c)$ are determined by optimality equation (**??**) as deterministic functions of $(W_b^c, W_g)$. Moreover, note that we have $W_g = U(W_b^c)$ on the extreme boundary, hence those terms can be written as deterministic functions of $W_g$ only. We now state the boundary conditions for the principal's value function. By "boundary conditions" we mean the description of the credible set and the properties of the solution at its boundaries.

**Lemma 4.2** *On the stationary (extreme) boundary,* $J^g(W_b^c, W_g) = K^{L,g}(W_g)$ $(K^{U,g}(W_g))$*, where* $K^{L,g}(W_g) = -v\left(\frac{W_g}{\theta_g}\right) = -v\left(\frac{W_b^c}{\theta_b}\right)$*, and* $K^{U,g}(W_g)$ *is the solution to*

$$K^{U,g}(W_g) = a_g(W_g) - c_g(W_g) + K_W^{U,g}(W_g)\left[W_g - \theta_g u(c_g(W_g)) + g(a_g(W_g))\right] + \frac{r\sigma^2\beta_g^2(W_g)}{2}K_{WW}^U(W_g), \tag{4.22}$$

*satisfying* $K^{U,g}(0) = 0$ *and* $K^{U,g}(\theta_g u(c_M)) = -c_M$.

Here, as discussed above, vector $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$ is the optimal solution determined by optimality equation (**??**).

Figure 3: **The value function on the boundaries. The graph on the left shows the principal's value function on the upper and lower boundaries. On the right, the first graph shows the optimal payment on the upper (in red) and lower (in blue) boundaries. The second graph shows the optimal effort on the extreme boundary.** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

Figure **??** shows the principal's value function on the stationary and extreme boundaries, as well as the optimal payment and effort. The effort on the stationary boundary is zero, hence not shown in the figure. As stated in Proposition **??**, on the extreme boundary the optimal payment and effort are determined by optimality equation (**??**), and the manager will get zero payment if and only if $\theta_g - \theta_b U_W(W_b^c) \leq 0$. Also on the extreme boundary, the bad manager will not work if $W_g$ is small, but this does not mean $\beta_b^c$ is zero, because it still may be better to provide incentives to the good manager to work. Our numerical results also show that, with the fixed continuation value $W_g$, the principal's value on the stationary boundary (a larger temptation value) is larger than that on the extreme boundary (a smaller temptation value), because the cost to maintain the truthfulness of the bad manager dominates the profit realized by the manager's work on the upper boundary. The extreme and stationary boundaries are inefficient in the sense that the principal cannot generate positive profit at the boundaries.

Figure 4: **Surface maps of the principal's value function (right) and the good manager's payment (left).** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

In Figure **??**, we present surface maps of the principal's value function and the payment to the good manager. In Sannikov (2008), where only dynamic moral hazard is considered, the principal's value function is non-monotonic in the continuation value of the manager. From Figure **??**, we can see that the principal's value function is not only non-monotonic in the continuation value, but also non-monotonic in the temptation value. More precisely, given $W_g$, the principal's value is low if the temptation value is either very small or very large.

This non-monotonicity of the value function stems from the inefficiency at the boundary of the credible set and has a large impact on the optimal contract. More precisely, from optimality equation (**??**), we can see that the optimal choice of compensation maximizes

$$-c - [\theta_b J_1^g(W_b^c, W_g) + \theta_g J_2^g(W_b^c, W_g)]u(c). \tag{4.23}$$

Thus , the agent's compensation is zero when $\theta_b J_1^g(W_b^c, W_g) + \theta_g J_2^g(W_b^c, W_g) \geq 0$. We call such a region of points $(W_b^c, W_g)$ "the probationary domain." Quantity $-\theta_b J_1^g(W_b^c, W_g) - \theta_g J_2^g(W_b^c, W_g)$ is the weighted marginal cost of giving the agent value through the two managers' continuation payoffs. In the probationary domain, where $W_g$ and $W_b^c$ are small, there is no cost in increasing values of $W_g$ and $W_b^c$, and the principal benefits by doing so. On the other hand, when $W_g$ and $W_b^c$ are large, the managers' continuation value pair has a large likelihood of hitting the (inefficient) boundaries and the principal's value decreases with $W_g$ and $W_b^c$.

Moreover, the inefficiency of the credible set's boundaries is due to double-sided income effects. First, when the continuation value of the good manager is sufficiently large, it costs the principal too much to compensate the manager for her effort, which is the inefficiency of the extreme boundary. Second, if the bad manager's temptation value becomes larger whereas the continuation value of the good manager remains the same, it is costly to provide incentives to the bad manager, hence even more costly for the good manager. Thus, it is optimal for the principal to retire the manager if $W_b^c$ is sufficiently large, which is the inefficiency of the stationary boundary.

Because the principal's value function is non-monotonic in both $W_g$ and $W_b^c$, the shutdown contract may be suboptimal compared to the screening contract, if the reservation utility is small. The principal may prefer to (potentially) hire either manager, by raising the initial value of $W_b^c$ above $R$, to obtain a greater profit from the good manager's work.

**Optimal contract**

Having described the boundary conditions, we can now describe the optimal shutdown contract. The following definition adopts the jargon of the repeated games literature.

**Definition 4.4** *Define set $\mathcal{D}(R) = \{(w_b^c, w_g) \in \mathcal{E},$ such that $w_b^c \leq R$ and $w_g \geq R\}$. Set $\mathcal{D}(R)$ is called the "initially and individually rational set " when the reservation utility is $R$.*

Set $\mathcal{D}(R)$ is the set of expected payoff pairs at time zero, such that the good manager will take the contract, and the bad manager will not.

As in the rest of the paper, we assume that there exists a strictly concave solution for optimality equation (**??**). One numerical example is illustrated by Figure **??**[6].

Figure 5: **Mesh map of the principal's value function if a good manager is hired.** $u(c) = \sqrt{c}$, $\theta_g = 2, \theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

We have the following result.

**Proposition 4.5** *The optimal contract is given by $\Psi_g = \left\{\bar{c}_g(t), \bar{a}_g(t), \bar{a}_b^c(t), \bar{\beta}_g(t), \bar{\beta}_b^c(t)\right\}_{t\geq 0}$, determined by optimality equation (**??**) in terms of continuation value process $W_g(t)$ and temptation value process $W_b^c(t)$, which satisfies boundary conditions stated in Lemma **??**. The dynamics of $W_g(t)$ and $W_b^c(t)$ follow equations*

$$d\bar{W}_g(t) = r\left[\bar{W}_g(t) + g(\bar{a}_g(t)) - \theta_g u(\bar{c}_g(t))\right]dt + r\bar{\beta}_g(t)[dY(t) - \bar{a}_g(t)dt], \tag{4.24}$$

$$d\bar{W}_b^c(t) = r\left[\bar{W}_b^c(t) + g(\bar{a}_b^c(t)) - \theta_b u(\bar{c}_g(t))\right]dt + r\bar{\beta}_b^c(t)[dY(t) - \bar{a}_b^c(t)dt], \tag{4.25}$$

*with initial values $\bar{W}_g(0)$ and $\bar{W}_b^c(0)$ satisfying*

$$[\bar{W}_b^c(0), \bar{W}_g(0)] \in arg \max_{[w_b^c, w_g]\in\mathcal{D}(R)} J^g(w_g, w_b^c). \tag{4.26}$$

[6] All numerical results in this article are computed by the finite difference approach. For the principal's value function, it consists in solving a nonlinear PDE defined on an irregular domain (the credible set is not rectangular). We apply Method 1 in Kwak(2007), page 18, to compute the function's value on or near the boundary recursively.

*When* $[\bar{W}_b^c, \bar{W}_g]$ *reaches the stationary boundary at* $[\theta_b w^*, \theta_g w^*]$*, the agent is retired with constant payment* $[c_g(t) = v(w^*)]$ *thereafter. When* $[\bar{W}_b^c, \bar{W}_g]$ *reaches the extreme boundary at* $[w^*, K^{U,g}(w^*)]$*, pair* $[\bar{W}_b^c, \bar{W}_g]$ *moves thereafter along the upper boundary defined by* $W_g = U(W_b^c)$ *until it reaches the low retiring value pair* $(0, 0)$*, or the high retiring value pair* $(\theta_b u(c_M), \theta_g u(c_M))$*, in which case the agent is retired at zero payment or constant payment* $c_M$*, respectively.*

Figure 6: **Optimal initial values. The right graph shows how** $[W_b^c(0), W_g(0)]$ **changes when** $R$ **increases. The left panel shows a contour map of the principal's value function and shows how set** $\mathcal{D}(R)$ **changes when** $R$ **increases.** $\mathcal{D}(w_b^*)$ **(**$\mathcal{D}(w_g^*)$**) is the rectangle at the right top of point** $P_b$ **(** $P_g$**). The principal's value function achieves the maximum at** $(w_b^*, w_g^*)$**.** $u(c) = \sqrt{c}$**,** $\theta_g = 2$**,**$\theta_b = 1$**,** $c_M = 4$**,** $a_M = 2$**,** $g(a_M) = 1$**,** $r = 2$**,** $\sigma = 1$**.**

Figure **??** provides numerical results for optimal initial values $W_g(0)$ and $W_b^c(0)$ given different reservation utilities $R$. Note that when $w_b^* \leq R \leq w_g^*$, the initial values are unchanged, set at those levels. This is because $J^g(W_b^c, W_g)$ takes a maximum value at $P^* = (w_b^*, w_g^*)$, when $P^* \in \mathcal{D}(R)$, and $w_b^*$ $(w_b^*)$ is the smallest (largest) reservation value level such that $\mathcal{D}(w_b^*)$ $(\mathcal{D}(w_g^*))$ contains $P^*$. Another interesting observation is that, although the principal does not want to hire the bad manager, when $R \leq w_b^*$ her participation constraint (**??**) is as high as possible for the shutdown contract, that is, binding at $R$. The reason is that the principal value function is not monotonic in $W_b^c$, and can be increased by raising $W_b^c$ in that region. This implies that the principal can do better by offering the screening contract instead of the shutdown contract when the reservation value is sufficiently low.

We conclude this section with Figure **??**, which describes how the principal's expected profit changes with respect to reservation value $R$. In the pure moral hazard model, if $R$ is less than the point denoted $w_g^0$, it is good for the principal to raise the agent's expected utility, otherwise the manager's continuation value has a large chance of hitting the low retiring value zero. However, when the moral hazard is mixed with adverse selection, the principal's value is also dependent on the temptation process, whose initial value cannot be greater than the reservation utility. Hence the shutdown contract is costly if the reservation value is low.

Figure 7: **Effect of reservation utility on value functions. The left graph shows how the principal's value s changes under pure moral hazard if the good manager is hired. The right panel shows how the principal's value changes under combined moral hazard and adverse selection, for the optimal shutdown contract.** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

# 5 Optimal Screening Contract under Adverse Selection and Moral Hazard

A significant feature of Sannikov's (2008) approach is that in the pure moral hazard setting the agent's continuation value is the only state variable. This is no longer true if the agent's type is private information, the continuation value is not sufficient in contract design when the moral hazard is mixed with adverse selection. In fact, Sannikov (2007a) takes a non-standard approach in solving a similar problem, but in which the manager only consumes at a finite horizon.

We indicate now what the optimal solutions should depend on in our framework. We define the continuation value process for the bad agent and the temptation value process for the good agent, as follows.

$$dW_b(t) = r\left[W_b(t) + g(a_b(t)) - \theta_b u(c_b(t))\right]dt + r\beta_b(t)[dY(t) - a_b(t)dt], \tag{5.27}$$

$$dW_g^c(t) = r\left[W_g^c(t) + g(a_g^c(t)) - \theta_g u(c_b(t))\right]dt + r\beta_g^c(t)[dY(t) - a_g^c(t)dt], \tag{5.28}$$

Here, $\beta_b(t)$ enforces $a_b(t)$ and $\beta_g^c(t)$ enforces $a_g^c(t)$. In addition to $J^g(W_b^c, W_g)$, defined previously, we introduce the optimal expected profit $J^b(W_b, W_g^c)$ of the principal when hiring the bad manager. Then, the principal's optimal profit from issuing a screening contract is obtained by maximizing

$$p_g J^g(W_b^c(0), W_g(0)) + p_b J^b(W_b(0), W_g^c(0)).$$

where $[W_b(0), W_g^c(0), W_b^c(0), W_g(0)]$ are initial values.

## 5.1 Optimality equation

We first need to identify the credible set of $(W_b(t), W_g^c(t))$. Note that the feasible set and the dynamic structure of $(W_b(t), W_g^c(t))$ are the same as those of $(W_b^c(t), W_g(t))$. Hence, the credible set of $(W_b(t), W_g^c(t))$ is also $\mathcal{E}$. Recall that if $(W_b^c, W_g)$ is on the extreme boundary at time $t$, the only implementable contract is defined by $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$, which are deterministic functions of $W_g$ and determined by optimality equation (**??**). In characterizing the contract for the bad manager, if the continuation value and temptation value processes reach the extreme boundary at $(W_b, W_g^c)$, then the unique contract that keeps the value pair in credible set is the same as that for the good manager with $W_g^c$ replacing $W_g$: $c_b(W_g^c) = c_g(W_g^c)$, $a_g^c(W_g^c) = a_g\left(W_g^c\right), a_b(W_g^c) = a_b^c\left(W_g^c\right), \beta_g^c(W_b^c) = \beta_g\left(W_g^c\right), \beta_b(W_g^c) = \beta_b^c\left(W_g^c\right)$ The difference relative to the shutdown case is that in the screening contract the initial conditions for $(W_b^c(0), W_g(0))$ and $(W_b(0), W_g^c(0))$ have to be such that the managers will only accept the contract designed for their type. In the bad manager's contract, the increment of Brownian motion is $dZ(t) = \frac{1}{\sigma}\left[dY(t) - a_b(t)dt\right]$, and the dynamics of $W_g^c(t)$ become

$$dW_g^c(t) = r\left[W_g^c(t) + g(a_g^c(t)) - \theta_g u(c_b(t)) + \beta_g^c(t)(a_b(t) - a_g^c(t))\right]dt + \sigma r \beta_g^c(t) dZ(t). \tag{5.29}$$

Similar to Lemma ??, on the extreme boundary the principal's value function is

$$J^b(W_b, W_g^c)|_{W_g = U(W_b)} = K^{U,b}\left(W_g^c\right), \tag{5.30}$$

where $K^{U,b}\left(W_g^c\right)$ is the solution to

$$\begin{aligned} K^{U,b}\left(W_g^c\right) &= a_g^c\left(W_g^c\right) - c_b\left(W_g^c\right) + K_W^{U,b}\left(W_g^c\right)\left[W_g^c - \theta_g u\left(c_b\left(W_g^c\right)\right) + g\left(a_g\left(W_g^c\right)\right)\right] \\ &+ \frac{r\sigma^2\left(\beta_g^c(W_g^c)\right)^2}{2} K_{WW}^U\left(W_g^c\right) + K_W^{U,b}(W_g^c)\beta_g^c(a_b(W_g^c) - a_g^c(W_g^c)) \end{aligned} \tag{5.31}$$

satisfying $K^{U,b}(0) = 0$ and $K^{U,b}\left(\theta_g u\left(c_M\right)\right) = -c_M$. On the stationary boundary, same as in the contract for the good manager, the agent will be offered a constant payment stream $v(\frac{W_b}{\theta_b}) = v(\frac{W_g^c}{\theta_g})$. Hence the principal's value function on the stationary boundary is

$$J^b(W_b, W_g^c)|_{W_g^c = L(W_b)} = K^{L,b}\left(W_g^c\right) = K^{L,g}\left(W_g^c\right). \tag{5.32}$$

The principal's value function on the extreme boundary when hiring a bad manager may be different from that when hiring a good manager. The difference (as shown in Figure ??) is due to the principal's instantaneous payoff being $a_b(t) - c_b(t)$ if a bad manager is hired, and we have $a_b(t) \le a_g^c(t)$ on the extreme boundary.

Figure 8: **Difference in the principal's value on the upper boundary when hiring a good manager and a bad manager.** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$.

Next, we note that the optimal value function of the principal if the bad manager is hired satisfies

$$\begin{aligned} J^b(W_b, W_g^c) = \max_{c_b, a_g^c, a_b, \beta_g^c, \beta_b} \Bigg\{ & a_b - c_b + \frac{r\sigma^2}{2}\left[(\beta_g^c)^2 J_{22}^b(W_b, W_g^c) + 2\beta_g^c\beta_b J_{12}^b(W_b, W_g^c) + (\beta_b)^2 J_{11}^b(W_b, W_g^c)\right] \\ & + \left[W_g^c - \theta_g u\left(c_b\right) + g\left(a_g^c\right) + \beta_g^c(a_b - a_g^c)\right] J_2^b(W_b, W_g^c) + \left[W_b - \theta_b u\left(c_b\right) + g\left(a_b\right)\right] J_1^b(W_b, W_g^c) \Bigg\} \end{aligned} \tag{5.33}$$

such that $\beta_b$ enforces $a_b$ and $\beta_g^c$ enforces $a_g^c$,

with boundary conditions (**??**) and (**??**).

**Definition 5.5** *Set* $\mathcal{D}^s(R) = \left\{(w_b^c, w_g), (w_b, w_g^c) \in \mathcal{E} \times \mathcal{E},\ \text{such that } w_b \geq R,\ w_i \geq w_i^c\right\}$ *is called "initially and individually rational set" for the screening contract when the reservation utility is* $R$.[7]

Let $[\bar{c}_b(t), \bar{a}_g^c(t), \bar{a}_b(t), \bar{\beta}_g^c(t), \bar{\beta}_b(t)]$ be the vector of optimal processes, determined by optimality equation (**??**) in terms of continuation value process $W_b(t)$ and temptation process $W_g^c(t)$. The following proposition summarizes our results for the screening contract.

**Proposition 5.6** *The optimal contract is* $\Psi_i = \{\bar{c}_i(t), \bar{a}_i(t)\}_{i=g,b}$ *in which* $\Psi_g$ *depends on the processes in (**??**) and (**??**), and* $\Psi_b$ *depends on the processes*

$$dW_b(t) = r\left[W_b(t) + g(\bar{a}_b(t)) - \theta_b u(\bar{c}_b(t))\right]dt + r\bar{\beta}_b(t)[dY(t) - \bar{a}_b(t)dt], \tag{5.34}$$

$$dW_g^c(t) = r\left[W_g^c(t) + g(\bar{a}_g^c(t)) - \theta_g u(\bar{c}_b(t))\right]dt + r\bar{\beta}_g^c(t)[dY(t) - \bar{a}_g^c(t)dt], \tag{5.35}$$

*with initial values* $\bar{P}(0) = (\bar{W}_b(0), \bar{W}_g^c(0)) \times (\bar{W}_b^c(0), \bar{W}_g(0))$ *satisfying*

$$\bar{P}(0) \ \in arg \max_{(w_b, w_g^c)\times(w_b^c, w_g)\in\mathcal{D}^s(R)} \ p_g J^g(w_b^c, w_g) + p_b J^b(w_b, w_g^c). \tag{5.36}$$

The proportions $p_b$ and $p_g$ of good and bad managers in the labor market have no impact on contract dynamics, but they affect the initial values of the continuation and temptation processes, as seen from the following result.

**Corollary 5.3** *If* $p_g J^g(w_b, w_g) + p_b J^b(w_b, w_g)$ *attains the maximum value at interior point* $(w_b^+, w_c^+)$ *in* $\mathcal{E}$, *and if* $J_1^g(w_b^+, w_g^+) \geq 0$ *and* $J_2^b(w_b^+, w_g^+) \geq 0$, *then the optimal initial values are* $W_b(0) = W_b^c(0) = w_b^+$ *and* $W_g(0) = W_g^c(0) = w_g^+$, *assuming that* $R \leq w_b^+$.

Figure 9: **Optimal initial values for the screening contract.** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$, $p_g = 0.3$, $p_b = 0.7$.

The foregoing corollary is illustrated by Figure **??**. If the reservation utility is small, then both managers' expected utilities at time zero are not binding at $R$. The principal is better off increasing the utilities to the level $(w_b^+, w_g^+)$. Meanwhile, the optimal screening contract represents a weakly separating equilibrium: both

[7] Note that we do not require $w_g \geq R$, because $w_g \geq w_b$ in $\mathcal{E}$.

managers are indifferent between truth-telling and lying. However, the contract is not a pooling one, the payments and efforts are different. The principal may obtain a strictly separating equilibrium by increasing $W_g(0)$ and $W_b(0)$ by a tiny value $\epsilon$. When the reservation utility is large, the initial value of the bad manager's continuation value $W_b(0)$ is binding at $R$. The initial value of the good manager's continuation process is equal to the initial value of her temptation process. The binding of $W_b(0)$ at $R$ implies that it is suboptimal for the principal to offer the screening contract when the reservation utility is large. Rather, the shutdown contract should be offered, as seen in Figure **??**.

Figure 10: **Comparison of shutdown and screening contracts.** $u(c) = \sqrt{c}$, $\theta_g = 2$,$\theta_b = 1$, $c_M = 4$, $a_M = 2$, $g(a_M) = 1$, $r = 2$, $\sigma = 1$, $p_g = 0.3$, $p_b = 0.7$.

Most of the arguments in favor of the screening contracts in the static models literature is based on the assumption that the labor is in short supply and the principal will suffer a loss if he hires the good manager only. If the market has a sufficient supply of both types of managers, then this argument is no longer valid. Our model shows that if the common reservation utility is low, then it is too costly and inefficient to hire only good managers, because the optimal shutdown contract needs to ensure that the bad manager's initial temptation value is no larger than the reservation utility, which damages the good manager's incentives. By increasing the bad manager's initial temptation value, the principal's expected profit may increase. In this case, the screening contract is better, with optimally chosen initial value that is not binding at the common reservation value. However, if the reservation utility is high, it becomes too expensive to hire a bad manager. The bad manager's expected utility in the screening contract is binding at the reservation value, which implies that the principal would prefer the bad manager to have a low reservation value. Then, the shutdown contract should be offered, because it specifies the initial value of the temptation process for the bad manager that is lower than the reservation value. In practice, screening contracts are not used for top management positions such as CEO's, who have high reservation utility values. However, these contracts may be optimal for positions with low reservation utility values.

## 5.2 Optimal screening contract: a simulation exercise

In this section, we illustrate the features of the optimal screening contract by a simulation of one particular event history. Figure **??** presents the movement of continuation value pairs $(W_b^c(t), W_g(t))$ and $(W_b(t), W_g^c(t))$

Figure 11: **Movement of Continuation Value Pairs**

inside the credible set, starting at the same initial value pair. For this event history, the contracts for both managers are terminated with early retirement (hitting the lower boundary), denoted by the crossed points. To see which contract is terminated earlier and compare the instantaneous payments, we provide Figure **??**, that describes the change of continuation values and payments of managers. The good manager is offered a higher retirement salary, but his contract is terminated earlier. Moreover, most of the time the good manager's payment is higher than the bad manager's payment, but with higher variation, implying the different risk levels of the payment stream are utilized to provide the incentives.

Figure 12: **Paths of Continuation Values and Payment Rates**

## 6 Conclusion

This paper considers a dynamic principal-agent problem with moral hazard that is present continuously, and adverse selection that occurs only at time zero. We derive the optimal contracts for good and bad managers, each of which is based on the honest manager's continuation value and the dishonest manager's temptation value. We find that it may be optimal for the agent to retire early, at varying levels of the manager's continuation value. Different from Sannikov (2008), in which the manager is retired either with zero or the highest payment, in our model retirement may occur at different levels of payment. Another finding is that the principal's value function is a function of two state variables, and is not only non-monotonic in the continuation values, but is also non-monotonic in the temptation values, due to the inefficiency of the credible set's boundary, caused by the double-sided income effects of the managers. We have shown that, when the common reservation utility is high, it is better for the principal to offer the shutdown contract to lower the information rent paid to the good manager. When the reservation utility is low, it is better to offer the screening contract, and raise the expected payoff for the bad manager at time zero so that the good manager can be offered better incentives.

Our model also could, in principle, be applied to investigate financial contracts and capital security design subject to constant private shocks. That is, one could extend the model of DeMarzo and Sannikov (2006) by allowing the manager to have private knowledge of the constant quality of the project [8]. Based on the results of this paper, we conjecture that the credible set would consist of two boundaries, a stationary boundary on which the financial contract is terminated, and a reflective boundary, on which the agent is paid. Moreover, our approach could be generalized to the case of effort taking values in a continuous range. It would be of

[8]This would be similar to the model of Sannikov(2007a), but with infinite horizon and instantaneous payment.

interest also to extend it to the case in which the agent's type is being exposed to repeated persistent shocks and dynamic moral hazard. [9]

## 7 Electronic Companion

An electronic companion to this paper is attached providing proofs for our main results.

[9] Wan (2011) provides a continuous-time model with i.i.d private shocks and dynamic moral hazard.

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# Electronic Companion for "Dynamics of Contract Design with Screening"

## A Proof of Proposition ??

To show that the recursive, DPP formulation (**??**) holds, and without the need of imposing condition (**??**), we need to make explicit the weak formulation of the model, that is, that the agent is controlling the probability measure by his actions.[10] More precisely, we work on a probability space $(\Omega, \mathcal{F}, P^0)$, on which $Z^0(t)$ is a standard Brownian motion such that $\frac{1}{\sigma}dY(t) = dZ^0(t)$. Hence, the information flow generated by $Z^0(t)$ is equivalent to $\left\{\mathcal{F}^Y(t)\right\}_{t\geq 0}$. Girsanov theorem implies that there is a measure $P^{a_g}$ such that $Z_t^{a_g}(t) = Z^0(t) - \int_0^t \frac{a_g(s)}{\sigma}ds$ is also a standard Brownian motion, where

$$\frac{dP^{a_g}}{dP^0}|\mathcal{F}^Y(t) = \mathcal{M}^{a_g}(t) = exp\left\{-\frac{1}{2}\int_0^t \left(\frac{a_g(s)}{\sigma}\right)^2 ds + \int_0^t \left(\frac{a_g(s)}{\sigma}\right) dZ^0(s)\right\}.$$

Then,

$$dY(t) = a_g(t)dt + \sigma dZ^{a_g}(t).$$

Moreover, the expected values in the main body of the paper, related to the good agent, are (implicitly assumed to be) taken under the measure $P^{a_g}$, that is, $E(.) = E^{a_g}(.)$ Switching to the original measure $P^0$, the upper boundary problem is equivalent to

$$U\left(w_b^c\right) = \max_{c_g, a_b^c, \beta_b^c} E^0\left\{\int_v^\infty re^{-r(s-v)}\frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)}\left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right]ds|W_b^c\left(v\right) = w_b^c\right\}, \tag{A.37}$$

subject to

$$dW_b^c\left(t\right) = r\left(W_b^c - \theta_b u\left(c_g\left(t\right)\right) + g\left(a_b^c\left(t\right)\right) - \beta_b^c\left(t\right)a_b^c\left(t\right)\right)dt + r\sigma\beta_b^c\left(t\right)dZ^0\left(t\right), \tag{A.38}$$

satisfying (**??**) and

$$a_g \in \arg\max_{\hat{a}_g} E^0\left\{\int_v^\infty re^{-r(s-v)}\frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)}\left[\theta_g u(c_g(s)) - g(\hat{a}_g\left(s\right))\right]ds|W_b^c\left(v\right) = w_b^c\right\}. \tag{A.39}$$

Here, $E^0(.)$ is the expectation under measure $P^0$. From the dynamics of the bad manager's continuation value $W_b^c(t)$ under $P^0$, we see that it does not depend on the choice of $a^g(.)$. Hence, the problem (**??**)- (**??**) and (**??**) can be written as

$$U\left(w_b^c\right) = \max_{c_g, a_g, a_b^c, \beta_b^c} E^0\left\{\int_v^\infty re^{-r(s-v)}\frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)}\left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right]ds|W_b^c\left(v\right) = w_b^c\right\}, \tag{A.40}$$

subject to (**??**) and (**??**). This is because, given an optimal solution $(c_g, a_g, a_b^c, \beta_b^c)$ to the problem (**??**), $a_g$ has to satisfy (**??**). Otherwise, if that is not the case, then there exists a process $\hat{a}_g$ such that $(c_g, \hat{a}_g, a_b^c, \beta_b^c)$

[10] The weak formulation was already implicitly present in Holmstrom and Milgrom (1987), and explicitly in Sung (1995).

makes the good manager better off and does not change the process $W_b^c(t)$, which contradicts the optimality of $(c_g, a_g, a_b^c, \beta_b^c)$ .

It is well known that the DPP holds for stochastic control problems of this type; see, e.g., Cvitanić and Karatzas (1993), Proposition 6.2. More precisely, we have, for $\tau > v$,

$$U\left(w_b^c\right) = \max_{c_g, a_g, a_b^c, \beta_b^c} E^0 \left\{ \int_v^\tau r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds + e^{-r(\tau-v)} \frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)} U(W_b^c(\tau)) | W_b^c\left(v\right) = w_b^c \right\} \tag{A.41}$$

Changing back to the expectation under $P^{a_g}$, we get our result:

$$U\left(w_b^c\right) = \max_{c_g, a_g, a_b^c, \beta_b^c} E^{a_g} \left\{ \int_v^\tau r e^{-r(s-v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds + e^{-r(\tau-v)} U(W_b^c(\tau)) | W_b^c\left(v\right) = w_b^c \right\}, \tag{A.42}$$

subject to

$$dW_b^c\left(t\right) = r\left(W_b^c - \theta_b u\left(c_g\left(t\right)\right) + g\left(a_b^c\left(t\right)\right) + \beta_b^c\left(t\right)\left(a_g\left(t\right) - a_b^c\left(t\right)\right)\right) dt + r\sigma\beta_b^c\left(t\right) dZ^{a_g}\left(t\right), \tag{A.43}$$

and (**??**) for $t \in (v, \tau]$.

For sake of completeness we provide here a sketch of the proof. For

$$\pi = (c_g, a_g, a_b^c, \beta_b^c)$$

we first show

$$U\left(w_b^c\right) \leq \max_\pi E^0 \left\{ \int_v^\tau r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds + e^{-r(\tau-v)} \frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)} U(W_b^c(\tau)) | W_b^c\left(v\right) = w_b^c \right\}. \tag{A.44}$$

Indeed, define

$$\mathcal{J}(w_b^c; \pi) = E^0 \left\{ \int_v^\infty r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds | W_b^c\left(v\right) = w_b^c \right\}.$$

and note that we have

$$\begin{aligned}
\mathcal{J}(w_b^c; \pi) &= E^0 \left\{ \int_v^\tau r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds | W_b^c\left(v\right) = w_b^c \right\} \\
&+ E^0 \left\{ E^0 \left\{ \int_\tau^\infty r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds | W_b^c(\tau) \right\} | W_b^c\left(v\right) = w_b^c \right\} \\
&= E^0 \left\{ \int_v^\tau r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds + e^{-r(\tau-v)} \frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)} \mathcal{J}(W_b^c(\tau); \pi) | W_b^c\left(v\right) = w_b^c \right\} \\
&\leq E^0 \left\{ \int_v^\tau r e^{-r(s-v)} \frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)} \left[\theta_g u(c_g(s)) - g(a_g\left(s\right))\right] ds + e^{-r(\tau-v)} \frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)} U(W_b^c(\tau); \pi) | W_b^c\left(v\right) = w_b^c \right\}.
\end{aligned}$$

Taking maximum on both sides we prove the inequality.

Finally, we prove

$$U\left(w_b^c\right) \geq \max_\pi E^0\left\{\int_v^\tau re^{-r(s-v)}\frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)}\left[\theta_g u(c_g(s)) - g(a_g(s))\right]ds + e^{-r(\tau-v)}\frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)}U(W_b^c(\tau))|W_b^c(v) = w_b^c\right\}. \tag{A.45}$$

A full proof can be done following Cvitanić and Karatzas (1993), Proposition 6.2. Here, we show it under the assumption that the maximum in (**??**), with $v$ replaced by $\tau$, is attained with $\pi = \pi^\tau$. For $t < \tau$ let $\pi(t) = \pi^\tau(t)$ be arbitrary.

Then[11], for the quadruple $\pi(t) = \pi^\tau(t) = [c_g(t), a_g(t), a_b^c(t), \beta_b^c(t)]$, we have

$$\begin{aligned} & E^0\left\{\int_v^\tau re^{-r(s-v)}\frac{\mathcal{M}^{a_g}(s)}{\mathcal{M}^{a_g}(v)}\left[\theta_g u(c_g(s)) - g(a_g(s))\right]ds + e^{-r(\tau-v)}\frac{\mathcal{M}^{a_g}(\tau)}{\mathcal{M}^{a_g}(v)}U(W_b^c(\tau))|W_b^c(v) = w_b^c\right\} \\ = \; & E^0\left\{\int_v^\infty re^{-r(s-v)}\frac{\mathcal{M}^{\hat{a}_g}(s)}{\mathcal{M}^{\hat{a}_g}(v)}\left[\theta_g u(\hat{c}_g(s)) - g(\hat{a}_g(s))\right]ds|W_b^c(v) = w_b^c\right\} \leq U(w_b^c) \end{aligned} \tag{A.46}$$

Because of the arbitrary choice of $\pi(t)$ for $v \leq t \leq \tau$, we get (**??**).

**Remark A.2** *If we didn't use the weak formulation for our model, the maximum of the right hand side of (**??**) would be subject to the good manager's incentive compatibility conditions on $[v, \tau]$, conditional on $U(W_b^c(\tau))$, where $W_b^c(\tau)$ would be affected by the choice of $a_g(t)$ for $t \in [v, \tau]$. Thus, the maximization of $\mathcal{J}(w_b^c; \pi)$ would be subject to the good manager's incentive compatibility condition on $[v, \infty]$, and the above argument would not work.*

# B Proof of Proposition ??

We first investigate the lower boundary. At any time $t \geq 0$, given payment stream $\{c_g(s)\}_{s \geq t}$ defined on $(W_g, W_b^c)$ as in (**??** ) and (**??**) with $W_b^c(t) > 0$, denoting the bad manager's optimal effort by $\{a_b^c(s)\}_{s \geq t}$, the conditional expected utility at time $t$ is given by

$$e^{-rt}W_b^c(t) = E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_b u(c_g(s)) - g(a_b^c(s))\right]ds\right\}.$$

Hence, the Brownian motion increment $dZ(t)$ is $\frac{1}{\sigma}\left[dY(t) - a_b^c(t)dt\right]$. The value of the good manager if she exerts the same effort as the bad manager is denoted $\hat{W}_g(t)$, and the value of the good manager if she exerts the best effort is denoted $W_g(t)$. Then, we have $W_g(t) \geq \hat{W}_g(t)$ and

$$e^{-rt}W_g(t) \geq e^{-rt}\hat{W}_g(t) = E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_g u(c_g(s)) - g(a_b^c(s))\right]ds\right\} \geq e^{-rt}W_b^c(t) > 0.$$

[11] $W_b^c$ starts from $w_b^c$ at time $v$ and is driven by $\pi^\tau(t)$ for $v \leq t \leq \tau$.

Moreover,

$$\frac{W_g(t)}{W_b^c(t)} \geq \frac{E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_g u(c_g(s)) - g(a_b^c(s))\right]ds\right\}}{E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_b u(c_g(s)) - g(a_b^c(s))\right]ds\right\}} \geq \frac{\theta_g}{\theta_b}.$$

Thus, in order to show that the lower boundary is given by $L(W_b^c) = \frac{\theta_g}{\theta_b}W_b^c$ on $[0, \theta_b u(c_M)]$ we need to find, for all payoff pairs on this line segment, a strategy that does not drive $(W_b^c(t), W_g(t))$ outside of $\mathcal{E}$. Such a strategy is constructed next.

Suppose $(W_b^c(t), W_g(t)) = (\theta_b w^*, \theta_g w^*)$. If $w^* = 0$, then the contract will be terminated and both the good and bad managers will retire with zero payment. If $w^* > 0$, then we have

$$\frac{\theta_g}{\theta_b} = \frac{W_g(t)}{W_b^c(t)} \geq \frac{E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_g u(c_g(s)) - g(a_b^c(s))\right]ds\right\}}{E_t^{a_b^c}\left\{\int_t^\infty e^{-rs}\left[\theta_b u(c_g(s)) - g(a_b^c(s))\right]ds\right\}} \geq \frac{\theta_g}{\theta_b},$$

which implies that the two inequalities hold with equality. The second equality means that the bad manager's effort has to be $a_b^c(s) = 0$ for $s \geq t$. The first inequality then implies that the good manager's best response cannot generate more expected payoff than that if she chooses $a_g(s) = 0$ for $s \geq t$. Moreover, given $c_g(s)$, zero effort is, in fact, the best effort for the good manager's for $s \geq t$, because it is least costly. Thus, we have

$$e^{-rt}W_g(t) = rE_t^0\left\{\int_t^\infty e^{-rs}\theta_g u(c_g(s))ds\right\}$$

and the principal's expected profit at time $t$ is

$$-rE_t^0\left\{\int_t^\infty e^{-rs}c_g(s)ds\right\}$$

It is easily verified that this profit, under the constraint that $W_g(t) = \theta_g w^* \geq R$, cannot be higher then if the principal offers $c^*(s) = v(w^*)$, $s \geq t$. Hence, $(W_b^c(s), W_g(s))$ will remain at $(\theta_b w^*, \theta_g w^*)$ for $s \geq t$, and at time $t$ the agent is retired at the constant retirement salary. Thus, we have proved the stated properties on the stationary boundary.

Now we consider the properties of the optimality equation (**??**) at the extreme boundary. Note that (**??**) is equivalent to

$$U_{WW}(W) = \min_{\beta_b^c, c_g, a_g, a_b^c} \frac{U(W) - \theta_g u(c_g) + g(a_g) - U_W(W)\left[W - \theta_b u(c_g) + g(a_b^c) + \beta_b^c(a_g - a_b^c)\right]}{\left(\beta_b^c\right)^2 r\sigma^2/2}, \quad \text{(B.47)}$$

such that $\beta_b^c$ enforces $a_b^c$ and boundary conditions are satisfied. The proofs of uniqueness, existence and concavity are the same as those in Lemma 1– Lemma 3 in Sannikov (2008) and Lemma 3 in Fong (2009). We now prove the strict monotonicity. Suppose that $\Phi(w)$ is the solution of (**??**). We want to show that $\Phi'(w) > 0$ for all $w \in [0, \theta_b u(c_M)]$. From the concavity of the solution, we only need to show that $\Phi'(\theta_b u(c_M)) > 0$. Suppose not, then $\Phi'(\theta_b u(c_M)) = \phi_0 \leq 0$. Note that $\Phi(w)$ is also the unique solution of (**??**) with initial conditions $\Phi(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\Phi'(\theta_b u(c_M)) = \phi_0$. We now consider the solution $\hat{\Phi}(w)$ of (**??**) with initial conditions $\hat{\Phi}(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\hat{\Phi}'(\theta_b u(c_M)) = 0$. Then we have $\hat{\Phi}(w) = \theta_g u(c_M)$. Note that $\hat{\Phi}'(\theta_b u(c_M)) \geq \phi_0$. Then, it follows from Lemma 2 in Sannikov (2008) that $\Phi(w)$ dominates $\hat{\Phi}$ for any $w \leq \theta_b u(c_M)$, which contradicts $\Phi(0) = 0$. Hence, $\phi_0 > 0$. and $\Phi(w)$ is strictly increasing.

# C Proof of Corollary ??

Recall that the extreme and stationary boundaries connect at $(0,0)$ and $(\theta_b u(c_M), \theta_g u(c_M))$. Hence, there exists a value pair $(w^*, U(w^*))$ on the extreme boundary at which the tangent line is parallel to the stationary boundary, that is $U_W(w^*) = \frac{\theta_g}{\theta_b}$. Moreover, that the optimal payment $c$ on the boundary is either zero or $c_m$ follows then from from optimality equation (**??**). Next, consider function $\hat{U}(W) = W + (\theta_g - \theta_b)u(c_M)$, which is a solution of (**??**) with initial condition $\hat{U}(\theta_b u(c_M)) = \theta_g u(c_M)$ and $\hat{U}_W(\theta_b u(c_M))) = 1$. Note that $\hat{U}(0) > U(0)$, hence from Lemma 2 of Sannikov (2008), we have $\hat{U}_W(\theta_b u(c_M))) = 1 < U_W(\theta_b u(c_M)))$. Then, we have $U_W(W) > 1$ and $\beta_g = U_W(W_b^c)\beta_b^c \geq \beta_b^c$. From the definition of enforcement, we have $a_g \geq a_b^c$.

# D Proof of Corollary ??

Figure 13: **Movement of Continuation Value Pair**

Consider a level function $l(W_b^c, W_g) = W_g - U(W_b^c)$ which is jointly convex in $(W_b^c, W_g)$. The upper boundary is given by $l(W_b^c, W_g) = 0$. At time 0, $(W_b^c(t), W_g(t))$ starts from a pair $(w_b^c, w_g)$ inside $\mathcal{E}$, so that $l(w_b^c, w_g) < 0$. As in the proof of Proposition **??**, we switch to the measure $P^0$, under which we have

$$dW_g(t) = r\left[W_g(t) + g(a_g(t)) - \beta_g(t)a_g(t) - \theta_g u(c_g(t))\right]dt + r\sigma\beta_g(t)dZ^0(t), \tag{D.48}$$

$$dW_b^c(t) = r\left[W_b^c(t) + g(a_b^c(t)) - \beta_b^c(t)a_b^c(t) - \theta_b u(c_g(t))\right]dt + r\sigma\beta_b^c(t)dZ^0(t), \tag{D.49}$$

subject to (**??**) and (**??**), $W_g(0) = w_g$ and $W_b^c(0) = w_b^c$. Then

$$\begin{aligned} dl(W_b^c(t), W_g(t)) &= r\left[W_g(t) + g(a_g(t)) - \beta_g(t)a_g(t) - \theta_g u(c_g(t))\right]dt \qquad \text{(D.50)} \\ &- r\left\{U_W(W_b^c(t))\left[W_b^c(t) + g(a_b^c(t)) - \beta_b^c(t)a_b^c(t) - \theta_b u(c_g(t))\right]\frac{r\sigma^2}{2}(\beta_b^c(t))^2 U_{WW}(W_b^c(t))\right\}dt \\ &+ r\sigma\left[\beta_g(t) - U_W(W_b^c(t))\beta_b^c(t)\right]dZ^0(t) \end{aligned}$$

Next, we take the contract obtained by maximization in the HJB equation for the upper boundary, which implies

$$\beta_g(t) = U_W(W_b^c(t))\beta_b^c(t) \text{ and } dl(W_b^c(t), W_g(t)) = rl(W_b^c(t), W_g(t))dt\ . \tag{D.51}$$

Because $(w_b^c, w_g)$ is inside $\mathcal{E}$, we have $l(W_b^c(0), W_g(0)) < 0$ and $l(W_b^c(t), W_g(t))$ is decreasing for $t > 0$. Thus, the continuation value pair $(W_b^c(t), W_g(t))$ moves from a boundary of one level set to another, lower one, in the direction from $L_1$ to $L_2$ (for example) as shown in Figure **??**. We don't know when the first time is at which $(W_b^c(t), W_g(t))$ hits the lower boundary, but we can find a deterministic time $T^*$, that depends on $(w_b^c, w_g)$, such that $(W_b^c(t), W_g(t))$ will end on the lower boundary before time $T^*$, almost surely. Define $H(t) = l(W_b^c(t), W_g(t))$, with $H(0) = w_g - U(w_b^c) < 0$. As time passes, eventually the boundary of the level set on which $(W_b^c(t), W_g(t))$ finds itself, will be tangent to the lower boundary of the feasible set, as shown by point B in Figure **??**, with $B = (\bar{w}_b^c, \bar{w}_g)$. Then, we have

$$\bar{w}_g = \frac{\theta_g}{\theta_b}\bar{w}_b^c \; and \; -U_W(\bar{w}_b^c)\theta_b + \theta_g = 0.$$

That is, $\bar{w}_b^c = I(\frac{\theta_g}{\theta_b})$ and $\bar{w}_g = \frac{\theta_g}{\theta_b}I(\frac{\theta_g}{\theta_b})$, where $I(.)$ is the inverse function of $U_W(.)$. Then, we define $T^*$ by $H(T^*) = \frac{\theta_g}{\theta_b}I(\frac{\theta_g}{\theta_b}) - U(I(\frac{\theta_g}{\theta_b}))$. Hence

$$T^* = \frac{1}{r}log(\frac{\frac{\theta_g}{\theta_b}I(\frac{\theta_g}{\theta_b}) - U(I(\frac{\theta_g}{\theta_b}))}{w_g - U(w_b^c)})$$

Note that since $(w_b^c, w_g)$ is inside $\mathcal{E}$, we must have $\frac{\theta_g}{\theta_b}I(\frac{\theta_g}{\theta_b}) - U(I(\frac{\theta_g}{\theta_b})) < w_g - U(w_b^c) < 0$ and $T^* > 0$ is well-defined, and the statement of the corollary holds for this value of $T^*$.

# E Proof of Lemma ??

First, from Proposition **??** we know that when $(W_b^c(t), W_g(t))$ reaches the stationary (lower) boundary, the good manager is paid a constant $v\left(\frac{W_g}{\theta_g}\right)$, and the principal's value is $K^{L,g}(W_g) = -v\left(\frac{W_g}{\theta_g}\right)$. That is,

$$J^g(W_b^c, W_g)|_{W_g = L(W_b^c)} = K^{L,g}(W_g).$$

On the extreme (upper) boundary, the optimal $(c_g, a_g, a_b^c, \beta_g, \beta_b^c)$ is given by optimality equation (**??**). Also, because $W_g = U(W_b^c)$, the terms of the contract determined by (**??**) are deterministic functions of $W_g$, which we denote $(c_g(W_g), a_g(W_g), a_b^c(W_g), \beta_g(W_g), \beta_b^c(W_g))$. We denote the principal's conditional expected utility on the upper boundary by $K^{U,g}(W_g(t))$:

$$e^{-rt}K^{U,g}(W_g(t)) = E_t\left\{r\int_t^\infty e^{-rs}\left[a_g(W_g(s)) - c_g(W_g(s))\right]ds\right\},$$

subject to $dW_g(t) = r(W_g(t) - \theta_g u(c_g(W_g(t))) + g(a_g(t)))dt + \beta_g(W_g(t))\sigma dZ(t)$. Hence $e^{-rt}K^{U,g}(W_g(t)) + \int_0^t e^{-rs}\left[a_g(W_g(s)) - c_g(W_g(s))\right]ds$ is a martingale. By Feynman-Kac theorem, the principal's value function at the upper boundary satisfies

$$K^{U,g}(W_g) = a_g(W_g) - c_g(W_g) + K_W^{U,g}(W_g)\left[W_g - \theta_g u(c_g(W_g)) + g(a_g(W_g))\right] + \frac{r\sigma^2\beta_g^2(W_g)}{2}K_{WW}^U(W_g),$$

with $K^{U,g}(0) = 0$ and $K^{U,g}(\theta_g u(c_M)) = -c_M$, which was to be shown.

## F Proof of Proposition ??

Let $[\bar{c}_g(t), \bar{a}_g(t), \bar{a}_b^c(t), \bar{\beta}_g(t), \bar{\beta}_b^c(t)]$ be the vector of processes determined by optimality equation (??), contingent on $\bar{W}_g$ and $\bar{W}_b^c$, with dynamics (??) and (??). The corresponding principal's expected utility is denoted $J^g(\bar{W}_b^c(0), \bar{W}_g(0))$, and the initial conditions are chosen so that

$$(W_b^c(0), W_g(0)) \in arg \max_{(w_b^c, w_g) \in \mathcal{D}(R)} J^g(w_b^c, w_g). \tag{F.52}$$

with $W_b^c(0) = w_b^c$ and $W_g(t) = w_g$ and $(w_b^c, w_g)$ is inside of $\mathcal{E}$. Suppose $c_{g(t)}$ takes constant value in $[0, c_M]$, then it will reach the upper or lower boundary in deterministic time and hence $(w_b^c, w_g)$ is achievable.

Consider now an arbitrary payment stream $\hat{c} = \{\hat{c}(t)\}_{t \geq 0}$ that satisfies (??) and (??) for the good manager, and (??). Let $\hat{W}_g(t)$ ( $\hat{W}_b^c(t)$) be the corresponding continuation value if the good (bad) manager takes the contract, and $(\hat{a}_g(t), \hat{a}_b^c(t), \hat{\beta}_g(t), \hat{\beta}_b^c(t))$ be the corresponding optimal efforts and volatilities of the continuation values, in which $\hat{\beta}_g$ ($\hat{\beta}_b^c$) enforces $\hat{a}_g$ ($\hat{a}_b^c$). To prevent misreporting, we need to have $(\hat{W}_b^c(0), \hat{W}_b^c(0)) \in \mathcal{D}(R)$. We claim that this arbitrary payment stream cannot be better for the principal than $\bar{c}$. Let us introduce the gains process by

$$\hat{G}(t) = \int_0^{t \wedge \tau} e^{-rs} [\hat{a}_g(s) - \hat{c}_g(s)] \, ds + e^{-rt \wedge \tau} J^g(\hat{W}_b^c(t \wedge \tau), \hat{W}_g(t \wedge \tau)),$$

where $\tau$ is the first hitting time of pair $(\hat{W}_b^c(t), \hat{W}_g(t))$ of the boundaries of the credible set. If we can show that $\hat{G}(t)$ is a supermartingale, then, because $\hat{G}(t)$ is bounded, it is also a supermartingale all the way to $t = \infty$, and the principal's expected utility satisfies

$$E\left[\int_0^\infty e^{-rs} [\hat{a}_g(s) - \hat{c}_g(s)] \, ds\right] \leq J^g(\hat{W}_b^c(0), \hat{W}_g(0)).$$

Furthermore,

$$J^g(\hat{W}_b^c(0), \hat{W}_g(0)) \leq J^g(\bar{W}_b^c(0), \bar{W}_g(0)), \quad in\ \mathcal{D}(R)$$

from the definition of $(\bar{W}_b^c(0), \bar{W}_g(0))$ in Proposition ??. The supermartingale property of $\hat{G}(t)$ is straightforward to verify by applying Ito's Lemma on $e^{-rt} J^g(\hat{W}_b^c(t), \hat{W}_g(t))$ and recalling optimality equation (??).

## G Proof of Proposition ??

Consider an alternative contract $(\hat{W}_b^c(0), \hat{W}_g(0), \hat{a}_g, \hat{a}_b^c, \hat{\beta}_g, \hat{\beta}_b^c)$ for the good manager, and $(\hat{W}_b(0), \hat{W}_g^c(0), \hat{a}_g, \hat{a}_b, \hat{\beta}_g^c, \hat{\beta}_b)$ for the bad manager such that

(a) $(\hat{W}_b^c(0), \hat{W}_g(0)) \times (\hat{W}_b(0), \hat{W}_g^c(0)) \in \mathcal{D}^s(R)$,

(b) $(\hat{\beta}_g, \hat{\beta}_b^c)$ enforces $(\hat{a}_g, \hat{a}_b^c)$, $(\hat{\beta}_g^c, \hat{\beta}_b)$ enforces $(\hat{a}_g^c, \hat{a}_b)$,

(c) Once $(\hat{W}_b^c(t), \hat{W}_g(t))$ or $(\hat{W}_b(t), \hat{W}_g^c(t))$ reaches the boundary of the credible set, the alternative contract follows the unique contract on the boundary.

If (a) is not satisfied, the managers will lie or not take the contract. Item (b) is the necessary and sufficient conditions for the agent's incentive compatibility. Item (c) is a necessary property of any incentive compatible contract. Then, as in the proof of Proposition **??**, the principal's expected profit when hiring the good (bad) manager cannot be more than $J^g(\hat{W}_b^c(0), \hat{W}_b^c(0))$ $(J^b(\hat{W}_b(0), \hat{W}_g^c(0)))$. Hence, the principal's total expected payoff cannot be more than $p_g J^g(\hat{W}_b^c(0), \hat{W}_b^c(0)) + p_b J^b(\hat{W}_b(0), \hat{W}_g^c(0))$, which is not larger than $p_g J^g(\bar{W}_b^c(0), \bar{W}_g(0)) + p_b J^b(\bar{W}_b(0), \bar{W}_g^c(0))$.

# H Proof of Corollary ??

When $(IR_b)$ is not binding, the initial value problem is reduced to

$$\max_{w_g, w_b, \delta_g, \delta_b} p_g J^g(w_b - \delta_b, w_g) + p_b J^b(w_b, w_g - \delta_g),$$

subject to $\delta_b, \delta_g \geq 0$. The Lagrangian for this problem is $p_g J^g(w_b - \delta_b, w_g) + p_b J^b(w_b, w_g - \delta_g) + \lambda_b \delta_b + \lambda_g \delta_g$, with $\lambda_i \geq 0$. First-order conditions are

$$p_g J_1^g(w_b - \delta_b, w_g) + p_b J_1^b(w_b, w_g - \delta_g) = 0, \quad p_g J_2^g(w_b - \delta_b, w_g) + p_b J_2^b(w_b, w_g - \delta_g) = 0,$$

$$-p_g J_1^g(w_b - \delta_b, w_g) + \lambda_b = 0, \quad -p_b J_2^b(w_b, w_g - \delta_g) + \lambda_g = 0.$$

Because $p_g J^g(w_b, w_g) + p_b J^b(w_b, w_g)$ attains the maximum value at interior point $(w_b^+, w_c^+)$, and $J_1^g(w_b^+, w_g^+) \geq 0$, $J_2^b(w_b^+, w_g^+) \geq 0$, the solution is $\delta_b = \delta_g = 0$, $w_b = w_b^+$ and $w_g = w_g^+$, with $p_g J_1^g(w_b^+, w_g^+) = \lambda_b$, $p_b J_2^b(w_b^+, w_g^+) = \lambda_g$.