Markets with random lifetimes and private values: mean-reversion and option to trade *

Jakša Cvitanić
Division of the Humanities and Social Sciences
California Institute of Technology, Pasadena, CA 91125
Email: cvitanic@hss.caltech.edu

Charles Plott
Division of the Humanities and Social Sciences
California Institute of Technology, Pasadena, CA 91125
Email: cplott@hss.caltech.edu

Chien-Yao Tseng
Division of the Humanities and Social Sciences
California Institute of Technology, Pasadena, CA 91125
Email: chienyao@caltech.edu

June 23, 2012

Abstract

We consider a market in which traders arrive at random times, with random private values for the single traded asset. A trader's optimal trading decision is formulated in terms of exercising the option to trade one unit of the asset at the optimal stopping time. We solve the optimal stopping problem under the assumption that the market price follows a mean-reverting diffusion process. The model is calibrated to experimental data taken from Alton and Plott (2010), resulting in a very good fit. In particular, the estimated long-term mean of the traded prices is close to the theoretical long-term mean at which the expected number of buys is equal to the expected number of sells. We call that value *Long-Term Competitive Equilibrium*, extending the concept of Flow Competitive Equilibrium (FCE) of Alton and Plott (2010).

Keywords: trading with private values, equilibrium price, optimal exercise of options, experimental markets, tick-by-tick trading.

*The research of J. Cvitanić is supported in part by NSF grant DMS 10-08219.

1 Introduction

This paper models optimal trading of individuals at microstructure level, by formulating the decision to trade as na optimal stopping problem. We adopt the setting of Alton and Plott (2010) of a market with a single asset, in which buyers and sellers arrive at random times, with random private values for one unit of the asset. A trader has an option to exchange the private value for the market price (or vice-versa), at time of her choosing, during a time interval of random length after which the option expires. Thus, assuming the traders are risk-neutral, the decision problem is the one of choosing the optimal stopping time at which the trade will take place, during a time interval of random (exponentially distributed) length. This makes it equivalent to the problem of optimally exercising an American option over a random horizon, with the market price as the underlying asset. The option is of the put type for the traders who consider buying the asset, and of the call type for the traders who consider selling the asset.

If the underlying asset follows the geometric Brownian motion (GBM) process, solving such problems is standard in the option pricing theory; see, for example, Carr (1998) and Shreve (2005). Extensions of the GBM model and/or different optimization objectives when looking for the optimal time to sell or buy a stock have been considered, among others, by Guo and Zhang (2005), Peskir and du Toit (2009), Shiryaev, Xu and Zhou (2008), and Zhang (2001). However, while exponential growth that the GBM process exhibits on average may be appropriate for long-term horizons, it is not realistic for very short-term, tick-by-tick trading. Moreover, it is not consistent with the market in which the private values of the traders are drawn in an iid manner from a fixed distribution. Instead, a process which exhibits mean-reversion is much more appropriate, and we model the market log-price as a mean-reverting Ornstein-Uhlenbeck process. This makes the optimal stopping problem harder than in the GBM case, but we are able to solve it semi-analytically, in terms of parabolic cylinder functions and up to one-dimensional integration.

Using the fact that we can compute the optimal trading strategies for the traders, in the second part of the paper we calibrate this model to the price data generated in trading experiments by Alton and Plott (2010). More precisely, we do the following: we estimate all the parameters of the mean-reverting process using (a part of) the price data, except for the long-term mean parameter. Then, we compute the value of the long-term mean for which the expected number of optimal buys is equal to the expected number of optimal sells in the model. We call this value *Long-Term Competitive Equilibrium*, or LTCE. We find that this value is close to the estimated value of the long-term mean, and that the price data is concentrated around it. In contrast, Alton and Plott (2010) introduce two notions of equilibrium price value: Temporal Equilibrium (TE) value at which there is an immediate, local (in time) clearing of the market, and Flow Competitive Equilibrium (FCE), at which the clearing is in expected value sense, over the whole time period of the experiment. Our LTCE value can be thought of as a generalization of FCE that accounts for the mean reversion feature of the price formation process.

While Alton and Plott (2010) find that the prices hover somewhere between TE and FCE, we find that they are mostly concentrated around LTCE, providing justification for defining LTCE as the equilibrium price. While LTCE seems to be a good measure for the equilibrium price, it has the disadvantage, relative to TE and FCE, that its value depends on how exactly one estimates the remaining parameters of the price process, and that the numerical procedure for computing it is much more involved than for computing TE and FCE. Nevertheless, we find it worthwhile developing a definition of a single equilibrium price that would fit well the average price pattern, as opposed to defining two equilibrium values between which the prices on average fluctuate; and even more so if the new definition arises from optimal behavior of the traders in a plausible model. Moreover, importantly, in the numerical examples the computed value of the LTCE *is very stable relative to the part of the data sample chosen to compute it.* More precisely, it does not make a big difference in the computed value of the LTCE whether we use a quarter, a half or the full sample of our data points. This is to be contrasted with the estimated value of the long-term mean, which is very sensitive to the chosen sample.

Trading a single asset using market orders or limit orders has been modeled by a number of papers in the literature. Most of those construct equilibrium strategies when trading is performed by different types of traders, such as market makers, informed traders, noise traders, patient and impatient traders, and so on. This approach is taken in Parlour (1998), Foucault (1999), Biais, Martimor and Rochet (2000), Parlour and Seppi (2003), Foucault, Kadan and Kandel (2005), Goettler, Parlour, and Rajan (2005), Back and Baruch (2007), Rosu (2009), Biais and Weill (2009), and Biais, Foucault and Moinas (2010) among others. Then, there are partial equilibrium models like those of Avellaneda and Stoikov (2008), Kuhn and Stroh (2009), and Cont, Stoikov, and Talreja (2008), that, like this paper, take the price process as given, and find the best strategy for the single trader. Perhaps most similar in spirit to the theoretical part of our paper is Pagnotta (2010), who, in a different and more complex model, also assumes that there is a given price for the asset, and what the traders decide on is the frequency of their trades. However, in that model there is a "true asset value" about which the traders have asymmetric information, and they also decide whether to submit market or limit orders. In contrast, in our model the orders are essentially market orders, and there is no true asset value, rather, the traders differ by their private values, not by information they have. Our aim is different than that of the latter papers – we are not interested in providing sophisticated algorithms for a trader to follow, or in finding how the limit order book features depend on the level of information traders have. Rather, as mentioned above, we examine whether a relatively simple model with mean reversion and optimal timing of trade describes well, at least in aggregate, the price formation in the experimental market designed to mimic the model.[1]

We present the theoretical model in Section 2, compute the value of the option to trade and the probability of exercising that option in Section 3, calibrate the model in Section 4 and conclude in the last section.

[1]It should be noted that the reason why we opted for fitting the model to experimental data rather than real market data is that in the experiments the private values are known, in fact, chosen by the experimenter, while it would be hard or impossible to estimate what they are in real markets. However, see the conclusions section for a possible future research on reverse-engineering the distributions of private values from real market data.

2 Model

There is a single asset to be traded in single units, and buyers/sellers decide at what time t to buy/sell, submitting orders at the market price P_t. Let v_B^i denote the "private value" for buyer i, who lives during a random interval $I_i^B = [s_i^B, t_i^B]$, where s_i^B and $t_i^B - s_i^B$ are independent and exponentially distributed. Similarly for the sellers. It is assumed that v_B^i, v_S^i are two iid sequences, also mutually independent.

By private value, we mean that (risk-neutral) buyer i's problem is

$$\max_{\tau \in I_i^B} E\left[e^{-r_i^B \tau}\left(v_B^i - P_\tau\right)^+\right] \tag{2.1}$$

and (risk-neutral) seller j's problem is

$$\max_{\tau \in I_i^S} E\left[e^{-r_i^S \tau}\left(P_\tau - v_S^j\right)^+\right] \tag{2.2}$$

where r_j^B, r_j^S are the traders' discount rates, and τ is the time to maturity, which is modeled below as an exponentially distributed random variable.

The above optimization problems are equivalent to the problem of pricing American options with random maturity. Assuming v_k^i is a constant for each buyer/seller and that the price process P of the asset is mean-reverting, more precisely, that $\log P$ is an Ornstein-Uhlenbeck process, then this problem can be solved and we can compute the optimal time to trade for each buyer/seller, as we will see next. In order to calibrate the model to the experimental data of Alton and Plott (2010), in addition to optimal exercise levels, we also need to find the formula for the probability that the buyer/seller will make a transaction during his lifetime interval I_i^k given the initial price P_0.

3 Random maturity American options with mean-reverting underlying

We assume that the mean-reverting asset follows the exponential Ornstein-Uhlenbeck (OU) process. More precisely, let X_t be the log-price of this asset, $X_t = \log P_t$, then X_t is given as the solution to the stochastic differential equation

$$dX_t = \kappa(\theta - X_t)dt + \sigma dW_t, \tag{3.3}$$

where $\kappa > 0$ and W_t is a standard Brownian motion process. This is a well-known Gaussian process used in finance to model economic variables which tend to fluctuate around a long-term mean θ. Parameter κ measures the speed of mean-reversion, and σ is the variance parameter.

Before starting computations needed to solve the model, let us present a brief outline of the model's timeline and of what we want to do in the rest of the paper:

- A trader with a private value arrives.

- She computes the optimal level at which to trade, assuming OU price process.
- Based on these levels, we compute the expected numbers of buy transactions and of sell transactions.
- We find the long-term mean θ (LTCE) that makes those two numbers equal.

3.1 Put and call values

We need to introduce some notation first. Let $D_{-\nu}(z)$ denote the so-called parabolic cylinder function and

$$\phi(X) := e^{z^2/4} D_{-\nu}(z) \tag{3.4}$$

$$\psi(X) := e^{z^2/4} D_{-\nu}(-z) \tag{3.5}$$

Also introduce the *Wronskian* of ϕ and ψ:

$$W(x) := \phi(x)\psi'(x) - \phi'(x)\psi(x)$$

We need to compute the value of (2.1), which is the value of the American put with strike price

$$K = v_B^i$$

and for which the maturity date τ is random (and independent of everything else), with density

$$Pr\{\tau \in dt\} = \lambda e^{-\lambda t} dt. \tag{3.6}$$

Here, $\lambda = \dfrac{1}{T}$ and T is the mean maturity of the put. Let $P(X)$ denote the value of such an American put option, where X is the initial value of X_t. We have

Proposition 3.1. *The value $P(X)$ of the random maturity American put with mean-reverting underlying is given by*

$$P(X) = \begin{cases} C\phi(X), & \text{if } X \geq \underline{X_0} \equiv \log K \\ A\phi(X) + Q(X), & \text{if } X \in (\underline{X}, \underline{X_0}) \\ K - e^X, & \text{if } X \leq \underline{X} \end{cases} \tag{3.7}$$

where

$$Q(X) = \phi(X) \int_{\underline{X}}^{X} \frac{2\psi(s)}{\sigma^2 W(s)} \lambda(K - e^s) ds + \psi(X) \int_{X}^{\underline{X_0}} \frac{2\phi(s)}{\sigma^2 W(s)} \lambda(K - e^s) ds, \tag{3.8}$$

$$A = \frac{(K - e^{\underline{X}})\psi'(\underline{X}) + e^{\underline{X}}\psi(\underline{X})}{W(\underline{X})} \tag{3.9}$$

$$C = A + \int_{\underline{X}}^{\underline{X_0}} \frac{2\psi(s)}{\sigma^2 W(s)} \lambda(K - e^s) ds \tag{3.10}$$

and the critical value $\underline{X}$ satisfies

$$\frac{(K - e^{\underline{X}})\phi'(\underline{X}) + e^{\underline{X}}\phi(\underline{X})}{W(\underline{X})} = -\int_{\underline{X}}^{X_0} \frac{2\phi(s)}{\sigma^2 W(s)} \lambda(K - e^s) ds \tag{3.11}$$

Similarly, we have for the American call value in (2.2):

Proposition 3.2. *The value $C(X)$ of the random maturity American call with mean-reverting underlying is given by*

$$C(X) = \begin{cases} E\psi(X), & \text{if } X \leq \overline{X_0} \equiv \log K \\ B\psi(X) + Q(X), & \text{if } X \in (\overline{X_0}, \overline{X}) \\ e^X - K, & \text{if } X \geq \overline{X} \end{cases} \tag{3.12}$$

where

$$B = \frac{e^{\overline{X}}\phi(\overline{X}) - (e^{\overline{X}} - K)\phi'(\overline{X})}{W(\overline{X})} \tag{3.13}$$

$$E = B + \int_{\overline{X_0}}^{\overline{X}} \frac{2\phi(s)}{\sigma^2 W(s)} \lambda(e^s - K) ds \tag{3.14}$$

and the critical value $\overline{X}$ satisfies

$$\frac{(e^{\overline{X}} - K)\psi'(\overline{X}) - e^{\overline{X}}\psi(\overline{X})}{W(\overline{X})} = \int_{\overline{X_0}}^{\overline{X}} \frac{2\psi(s)}{\sigma^2 W(s)} \lambda(e^s - K) ds \tag{3.15}$$

3.2 Probability of exercising options to trade

In order to be able to compute the expected number of trades in a given interval, we need to compute the probability that a trader will submit an order during his lifetime. In this regard, consider a stochastic process X_t with $X_0 = x$ and a constant c. If $x \geq c$ we define the first passage time $T_c(x)$ as the random variable

$$T_c(x) = \inf\{t \geq 0 \mid X_t \leq c\}. \tag{3.16}$$

Similarly if $x > c$. Introduce the distribution function of X_t starting at $X(0) = x$,

$$P(x \mid y, t) = Pr\{X_t \leq y \mid X(0) = x\} \tag{3.17}$$

and the distribution function of T_c, $F_c(x \mid t) = Pr\{T_c(x) \leq t\}$. We assume P and F have densities p and f:

$$p(x \mid y, t) = \frac{\partial}{\partial y} P(x \mid y, t) \tag{3.18}$$

$$f_c(x \mid t) = \frac{\partial}{\partial t} F_c(x \mid t). \tag{3.19}$$

Introduce also the minimum and maximum values of X up-to-date,

$$m(x,t) := \inf_{0\le s\le t}\{X(s) \mid X(0)=x\} \tag{3.20}$$
$$M(x,t) := \sup_{0\le s\le t}\{X(s) \mid X(0)=x\} \tag{3.21}$$

and the following probabilities:

$$F_{\underline{X}}(x \mid t) = Pr\{m(x,t) \le \underline{X}\} = Pr\{T_{\underline{X}} \le t\}, \quad x \ge \underline{X} \tag{3.22}$$
$$F_{\overline{X}}(t \mid x) = Pr\{M(x,t) \ge \overline{X}\} = Pr\{T_{\overline{X}} \le t\}, \quad x \le \overline{X} \tag{3.23}$$

Also denote

$$\phi(x) = e^{z^2/4}D_{-\nu}(z) \tag{3.24}$$
$$\psi(x) = e^{z^2/4}D_{-\nu}(-z) \tag{3.25}$$

with

$$z = \frac{\sqrt{2\kappa}(x-\theta)}{\sigma}$$

and

$$\nu = \frac{\lambda}{\kappa}.$$

Denote by $\hat{f}$ the Laplace transform of f. We have the following extension of a classical result.

Proposition 3.3. **(Darling and Siegert (1953)).** *The probabilities that $m(x,t)$ ($M(x,t)$) is less than $\underline{X}$ ($\overline{X}$) during the exponentially distributed period with mean length $1/\lambda$ are given by, respectively,*

$$P_{min}(x \mid \underline{X}, \lambda) = \hat{f}_{\underline{X}}(x \mid \lambda) = \frac{\phi(x)}{\phi(\underline{X})}, \quad x \ge \underline{X} \tag{3.26}$$
$$P_{max}(x \mid \overline{X}, \lambda) = \hat{f}_{\overline{X}}(x \mid \lambda) = \frac{\psi(x)}{\psi(\overline{X})}, \quad x \le \overline{X} \tag{3.27}$$

3.3 Probability of exercising options to trade with random starting time

The above result is still not sufficient for computing the probability that a trader will submit an order during his lifetime that is assumed to be random. We now extend the result to random lifetimes.

Recall that $p(x \mid y,t)$ denotes the transition density of X. We have

Proposition 3.4. *The probability that the minimum of $X(t)$ is less than $\underline{X}$ during a buyer's lifetime is given by, in the notation of the previous sections, and given that $X_0 = x$,*

$$\begin{aligned}&\mathbf{P}_{min}(x \mid \underline{X}, \lambda_B, \rho_B)\\ &= \textstyle\int_{\underline{X}}^{\infty} \lambda_B P_{min}(y \mid \underline{X}, \rho_B)\hat{p}(x \mid y, \lambda_B)\ dy + \int_{-\infty}^{\underline{X}} \lambda_B \hat{p}(x \mid y, \lambda_B)\ dy,\end{aligned} \tag{3.28}$$

where

$$P_{min}(y \mid \underline{X}, \rho_B) = \frac{\phi(y, \rho_B)}{\phi(\underline{X}, \rho_B)}, \tag{3.29}$$

with

$$\phi(x, \lambda) = e^{z^2/4} D_{-\lambda/\kappa}(z), \qquad z \equiv \frac{\sqrt{2\kappa}(x-\theta)}{\sigma} \tag{3.30}$$

Similarly, the probability that the maximum of $X(t)$ is higher than $\overline{X}$ during a seller's lifetime is given by

$$\begin{aligned} &\mathbf{P}_{max}(x \mid \overline{X}, \lambda_S, \rho_S) \\ &= \textstyle\int_{-\infty}^{\overline{X}} \lambda_S P_{max}(y \mid \overline{X}, \lambda) \hat{p}(x \mid y, \lambda_S)\ dy + \int_{\overline{X}}^{\infty} \lambda_S \hat{p}(x \mid y, \lambda_S)\ dy, \end{aligned} \tag{3.31}$$

where

$$P_{max}(y \mid \overline{X}, \lambda) = \frac{\psi(y, \lambda)}{\psi(\overline{X}, \lambda)} \tag{3.32}$$

with

$$\psi(x, \lambda) = e^{z^2/4} D_{-\lambda/\kappa}(-z), \qquad z \equiv \frac{\sqrt{2\kappa}(x-\theta)}{\sigma} \tag{3.33}$$

Moreover, the Laplace transform in these expressions is given by

$$\begin{aligned} &\hat{p}(x \mid y, \lambda) \\ &= 0.398942\Gamma(\nu)\sqrt{\frac{2}{\kappa\sigma^2}} e^{\frac{z_x^2 - z_y^2}{4}} \left[\Theta(x-y) D_{-\nu}(z_x) D_{-\nu}(-z_y) + \Theta(y-x) D_{-\nu}(-z_x) D_{-\nu}(z_y)\right] \end{aligned}$$

where $\Theta(x)$ is Heaviside theta function.

We now have all the equations needed to compute the expected number of buys and sells during a given interval of time. We use those equations on experimental data in the following section.

4 Long Term Competitive Equilibrium and calibration to experimental data

In this section we first define the Long Term Competitive Equilibrium (LTCE) and then we calibrate our model to experimental data of Alton and Plott (2010).

Definition 4.1. *Given a fixed interval of time, the LTCE price is the value for which the expected number of buys is equal to the expected number of sells during that interval, if the traders submit their orders optimally according to the model of the previous sections.*

The LTCE price can be interpreted as an analogue of the Flow Competitive Equilibrium (FCE) price that Alton and Plott (2010) introduced as the value at which expected number of buys is equal to the expected number of sells in the market in which the traders do not behave strategically, but immediately submit their private values as buy and sell orders. Alton and Plott (2020) also define Temporal Equilibrium (TE) price, which is the price at which the market clears at the present time. The data from their experiments tends to move from one of these equilibria to the other. Our LTCE price is a more sophisticated notion of equilibrium, that takes into account the traders' optimization and the mean reverting nature of the price.

We now calibrate our model to the experimental data from an experiment in Alton and PLott (2010). In that paper the authors report on experiments in which participants (college students) receive random private values at random times, that last for a certain lifetime, after which the values are no longer available. During those lifetime intervals, if a buyer buys a unit of the asset she can sell it later to the experimenter at the guaranteed private value. Analogously for the seller. The participants trade in a standard limit order market, that is, using a continuous double auction mechanism. That part is not modeled in our optimization framework, that can be thought of as a stylized way to depict the actual experimental market.

Even though it is unlikely that individually the participants estimate the price process as a mean-reverting process and then try to find the optimal exercise time as in our model, our hope is that on average the result of their trading would not be far away from the aggregate theoretical predictions. [2]

The aim of our exercise is to compute the LTCE price, denoted $\tilde{\theta}$. The following are the steps we do for this computation.

- We set $r_i = 0$, since our time interval is short, two hours.

- We observe transacted prices. We use these observations to estimate the parameters κ, σ and θ of the Ornstein-Uhlenbeck process, to get estimates $\hat{\kappa}$, $\hat{\sigma}$ and $\hat{\theta}$, using the maximum likelihood procedure. [3]

- We pick an initial value $\tilde{\theta}_0$ for the LTCE.

- We discretize the range of the private values (whose distribution is uniform in those experiments), and use the discrete values as the strike prices.

- For each private value as the strike price, we compute critical exercise values $\underline{\mathrm{X}}$ and $\bar{X}$.

- Assuming the initial asset price is equal to the initially chosen $\tilde{\theta}_0$, we compute the probabilities of buys and sells for different private values, using the formulas from the previous section.

- We estimate the expected number of buys by the number of buyers multiplied by the average of buy probabilities. If the expected number of buys is not sufficiently close to that of sells, we change the value of $\tilde{\theta}_0$ in the appropriate direction and repeat the procedure, until those numbers become close to each other. The final value $\tilde{\theta}$ so obtained is our Long-Term Competitive Equilibrium price.

The results of this procedure are illustrated in Figure 1. The figure shows the data generated by an experiment from Alton and Plott (2010) in which the distribution of the orders

[2]The phenomenon that individually participants in experiments do not behave optimally, but in aggregate the price formation is not far away to what it would be if they did, has been found before in experimental asset pricing, see, e.g., Bossaerts, Plott and Zame (2007).

[3]In doing this, we discard initial data points which are far away from “equilibrium price", as this is a period in which the participants are basically learning. Moreover, we smooth out the price values grouped in narrow time intervals, because our diffusion process would not be a good fit for the big jumps in price that often occur during those intervals.

changed in the middle, which made the typical price values move up in the second part of the experiment. Also shown are the FCE and the LTCE for the two parts of the experiment. We see that those values, which make the expected number of buys and sells equal in the corresponding models, are not very different for this data set. That is, assuming no strategical behavior on the part of the participants (resulting in the FCE value) results in a similar competitive equilibrium value as assuming that they optimally time the exercise of the option to trade using the OU-model (resulting in the LTCE value).

When computing the LTCE computed in Figure 1 we have used all the data points from the experiment (except for thirty initial trades for each part). We have then computed the LTCE using only the first quarter of the data points, as well as using a half and three quarters of the data points. Remarkably, the LTCE value does not change much with the length of the sample, even though the statistical estimates of the parameters of the OU process change more significantly. For example, for the first part of the experiment (excluding the first thirty data points from the sample) the computed value of the LTCE is equal to 293 using all the data points, and equal to 290.7 using only the first quarter of the sample. However, the estimated value of θ is $\hat{\theta} = 292.09$ using all the data points, and equal to $\hat{\theta} = 284.41$ using the first quarter only. Other estimated parameters for the first part are, using all the data points, $\hat{\kappa} = 0.0880$, $\hat{\sigma} = 0.0319$, and for the second part $\hat{\kappa} = 0.0338$ $\hat{\sigma} = 0.0169$. Using only a quarter of the data points, for the first part we have $\hat{\kappa} = 2.1838$, $\hat{\sigma} = 0.1635$ and for the second part $\hat{\kappa} = 0.0287$, $\hat{\sigma} = 0.0226$.

To reiterate, even though statistical estimation of the OU process parameters is somewhat unstable, the resulting LTCE value, that depends on that estimation, is quite robust.

5 Conclusions

We propose a model for trading an asset in a market with private reservation values, in which the traders decide optimally on the trade execution time. Assuming the market price follows a mean-reverting diffusion process, we find the equation for the optimal buy and sell levels, and expressions for the corresponding execution probabilities during a random interval of time. We then define Long-Term Competitive Equilibrium, LTCE, to be the value of the long-term stationary mean that makes the expected number of buys equal to the expected number of sells. The model is then fitted to the experimental data of Alton and Plott (2010). The data calibration results in a very good fit of the model, with the prices in the experiment fluctuating around LTCE. Moreover, and somewhat surprisingly, LTCE value is not very sensitive to the fraction of the sample we use to compute it, unlike the statistically estimated long-term mean.

While it would be desirable also to test the model on real market data, we cannot do such a calibration, because of the dependence on unknown private values. Let us mention that Lo, MacKinlay, and Zhang (2002), while performing a statistical analysis of the timing of limit orders, show that modeling trade execution times as passage times of a GBM process at a fixed level does not fit the market data well. In contrast, our price process is not a GBM process, but a mean-reverting process, and the trades are executed at varying passage times that are optimally decided by individual traders depending on their private values. Thus, it is a significantly richer model, and might not be necessarily rejected by the actual market data.

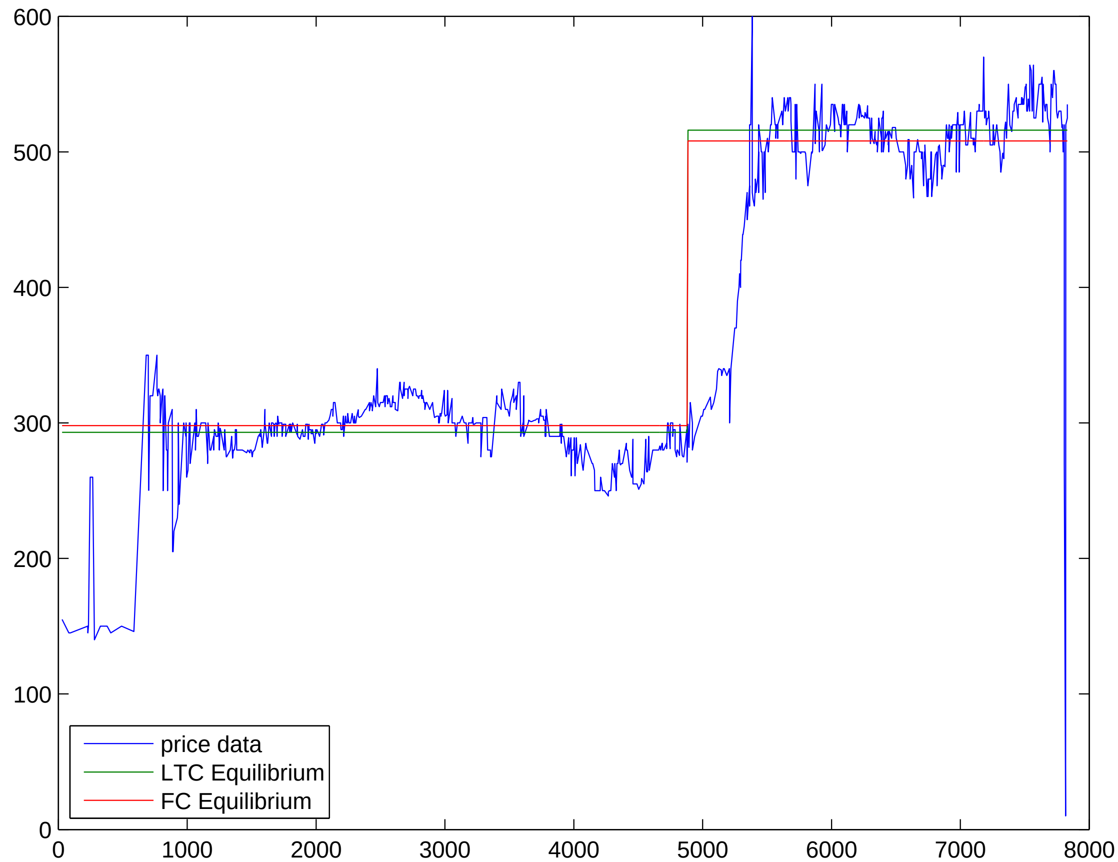

Figure 1: **Transaction prices from an Alton-Plott (2010) experiment.** Duration of the experiment is 2 hours, with first part lasting 71 minutes, and the second 49 minutes. Buyer and seller arrival rates are 4/min. Lifetime of private values is 6 minutes, and their distribution is $U(52, 451)$ for the first part and $U(273, 672)$ for the second part. The LTCE value for the first part is 293, while for the second part it is 516.

In future research, it would be of interest to do “reverse engineering" – taking the observed orders as given, and finding the implied distribution of private values. This would give a measure of the overall market sentiment during a chosen period of time, and this measure could be tested to see how well it depicts the actual mood changes over a sequence of time periods.

6 Appendix

Proof of Proposition 3.1: As in Carr (1998), $P(X)$ satisfies ordinary differential equation (ODE)

$$\frac{\sigma^2}{2}P''(X)+\kappa(\theta-X)P'(X)-rP(X)=\lambda\left[P(X)-\left(K-e^X\right)^+\right],\quad X>\underline{X} \tag{6.34}$$

subject to the boundary conditions

$$\lim_{X\to\infty}P(X)=0,\quad \lim_{X\to\underline{X}}P(X)=K-e^{\underline{X}},\quad \lim_{X\to\underline{X}}P'(X)=-e^{\underline{X}} \tag{6.35}$$

In the region $X>\log K\equiv\underline{X_0}$, the ODE is reduced to homogenous ODE

$$\frac{\sigma^2}{2}P''(X)+\kappa(\theta-X)P'(X)-(r+\lambda)P(X)=0,\quad X>\underline{X_0} \tag{6.36}$$

Introducing the change of variables

$$z=\frac{\sqrt{2\kappa}}{\sigma}(X-\theta)$$

and letting $P(X)=e^{z^2/4}\omega(z)$, equation (6.36) becomes

$$\omega''(z)+\left(\frac{1}{2}-\nu-\frac{z^2}{4}\right)\omega(z)=0 \tag{6.37}$$

with

$$\nu=(r+\lambda)/\kappa.$$

The general solution of (6.37) can be represented in the form

$$\omega(z)=CD_{-\nu}(z)+ED_{-\nu}(-z). \tag{6.38}$$

From $\lim_{X\to\infty}P(X)=0$ we get $E=0$. Therefore,

$$P(X)=Ce^{z^2/4}D_{-\nu}(z),\quad X>\underline{X_0}\equiv\log K \tag{6.39}$$

In the region $\underline{X}<X<\underline{X_0}$, the solution can be written as the general solution plus a particular solution,

$$P(X) = A\phi(X) + B\psi(X) + Q(X), \tag{6.40}$$

where $Q(X)$ is a particular solution that can be taken as in (3.8) (see, e.g., Johnson (2006)). From the boundary conditions Eq. (6.35) at $X = \underline{X}$ and using the continuity of $P(X)$ and $P'(X)$ at $X = \underline{X_0}$, it is not difficult to obtain $B = 0$, and A, C and $\underline{X}$ as in the statement of the proposition.

Proof of Proposition 3.3: Because the maturity date is exponential and independent of process X, we have

$$\begin{aligned} P_{min}(x \mid \underline{X}, \lambda) &= \lambda \int_0^\infty e^{-\lambda t} F_{\underline{X}}(x \mid t)\, dt & (6.41)\\ &= -\left[F_{\underline{X}}(x \mid t) e^{-\lambda t}|_0^\infty - \int_0^\infty e^{-\lambda t} f_{\underline{X}}(x \mid t)\, dt\right] & (6.42)\\ &= \int_0^\infty e^{-\lambda t} f_{\underline{X}}(x \mid t)\, dt & (6.43)\\ &= \hat{f}_{\underline{X}}(x \mid \lambda) & (6.44) \end{aligned}$$

Similarly,

$$P_{max}(x \mid \overline{X}, \lambda) = \hat{f}_{\overline{X}}(x \mid \lambda). \tag{6.45}$$

For our Ornstein-Uhlenbeck process X, function $p(x \mid y, t)$ satisfies the PDE

$$\frac{\partial p}{\partial t} = \kappa(\theta - x)\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2} \tag{6.46}$$

with initial and boundary conditions $p(\infty \mid y, t) = p(-\infty \mid y, t) = 0$, $p(x \mid y, 0) = \delta(x - y)$. Taking the Laplace transform of equation (6.46), we get

$$\lambda\hat{p} = \kappa(\theta - x)\frac{d\hat{p}}{dx} + \frac{\sigma^2}{2}\frac{d^2\hat{p}}{dx^2} \tag{6.47}$$

Therefore, we have

$$\hat{p}(x \mid y, \lambda) = \begin{cases} \psi(x)\phi(y), & y \geq x \\ \phi(x)\psi(y), & y \leq x \end{cases} \tag{6.48}$$

up to a constant factor. The result follows now from Theorem 3.1 in Darling and Siegert (1953).

Proof of Proposition 3.4: Note that we can write our Ornstein-Uhlenbeck process X in the form

$$X(t) = xe^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \sigma \int_0^t e^{-\kappa(t-u)}\, dW(u) \tag{6.49}$$

and that there is a Brownian motion $B(t)$ such that

$$\int_0^t e^{\kappa u} dW(u) = \frac{1}{\sqrt{2\kappa}} B(e^{2\kappa t} - 1). \tag{6.50}$$

Therefore, we have

$$X(t) = xe^{-\kappa t} + \theta(1 - e^{-\kappa t}) + \frac{\sigma}{\sqrt{2\kappa}} e^{-\kappa t} B(e^{2\kappa t} - 1), \quad X(0) = x \tag{6.51}$$

It is then not difficult to show that the transition density is given by

$$p(x \mid y, t) = \sqrt{\frac{\kappa}{\pi\sigma^2} \frac{1}{(1 - e^{-2\kappa t)}}} \exp\left\{ -\frac{\kappa}{\sigma^2} \frac{\left[y - xe^{-\kappa t} - \theta(1 - e^{-\kappa t})\right]^2}{1 - e^{-2\kappa t}} \right\} \tag{6.52}$$

Buyer i lives during random interval $[\tau_i^B, \tau_i^B + \Delta\tau_i^B]$ with

$$Pr\{\tau_i^B \in dt\} = \lambda_B e^{-\lambda_B t} dt, \qquad Pr\{\Delta\tau_i^B \in dt\} = \rho_B e^{-\rho t} dt \tag{6.53}$$

Then, the probability that the minimum of $X(t)$ is less than $\underline{X}$ during a buyer's lifetime is

$$\begin{aligned} &\mathbf{P}_{min}(x \mid \underline{X}, \lambda_B, \rho_B) \\ &= \textstyle\int_0^\infty \lambda_B e^{-\lambda_B \tau} \, d\tau \left[\int_{\underline{X}}^\infty P_{min}(y \mid \underline{X}, \rho_B) p(x \mid y, \tau) \, dy + \int_{-\infty}^{\underline{X}} p(x \mid y, \tau) \, dy \right] \\ &= \textstyle\int_{\underline{X}}^\infty \lambda_B P_{min}(y \mid \underline{X}, \rho_B) \hat{p}(x \mid y, \lambda_B) \, dy + \int_{-\infty}^{\underline{X}} \lambda_B \hat{p}(x \mid y, \lambda_B) \, dy, \end{aligned} \tag{6.54}$$

where we use the fact that if $X(\tau_i^B) \leq \underline{X}$, the buyer will make a transaction immediately after she enters the market, and if $X(\tau_i^B) \geq \underline{X}$, there is probability $P_{min}(y \mid \underline{X}, \rho_B)$ that $X(t)$ will hit $\underline{X}$ during the random period. The expression for $P_{min}(y \mid \underline{X}, \rho_B)$ follows from the previous section. The corresponding expression for the seller follows using the same method.

Next, we calculate the Laplace Transform of $p(x \mid y, t)$,

$$\hat{p}(x \mid y, \lambda) = \int_0^\infty e^{-\lambda t} p(x \mid y, t) \, dt \tag{6.55}$$

We know that $p(x \mid y, t)$ satisfies Kolmogorov equation

$$\frac{\partial f}{\partial t} = \kappa(\theta - x)\frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2} \tag{6.56}$$

subject to $f(\infty \mid y, t) = f(-\infty \mid y, t) = 0$ and $f(x \mid y, 0) = \delta(x - y)$. Taking Laplace transform on both sides of equation (6.56), we get

$$\lambda\hat{p} - \delta(x - y) = \kappa(\theta - x)\frac{d\hat{p}}{dx} + \frac{2}{\sigma^2}\frac{d^2\hat{p}}{dx^2} \tag{6.57}$$

Letting $z \equiv \dfrac{\sqrt{2\kappa}(x - \theta)}{\sigma}$ and $z_y \equiv \dfrac{\sqrt{2\kappa}(y - \theta)}{\sigma}$, equation (6.57) becomes

$$\frac{d^2\hat{p}}{dz^2} - z\frac{d\hat{p}}{dz} - \frac{\lambda}{k}\hat{p} = -\sqrt{\frac{2}{\kappa\sigma^2}} \, \delta(z - z_y) \tag{6.58}$$

Imposing the boundary conditions, we get

$$\hat{p} = \begin{cases} Ae^{z^2/4} D_{-\nu}(z), & z \geq z_y \\ Be^{z^2/4} D_{-\nu}(-z), & z \leq z_y \end{cases} \tag{6.59}$$

From equation (6.58), we know that $\frac{d\hat{p}}{dz}$ cannot be continuous. Integrating both sides of equation (6.58) from z_y^- to z_y^+, and because $\hat{p}$ is continuous, it is straightforward to get

$$\frac{d\hat{p}}{dz}(z_y^+) - \frac{d\hat{p}}{dz}(z_y^-) = -\sqrt{\frac{2}{\kappa\sigma^2}} \tag{6.60}$$

With equation (6.60) and $\hat{p}$ continuous, we get

$$\hat{p}(x \mid y, \lambda) = \begin{cases} \sqrt{\frac{2}{\kappa\sigma^2}} e^{-z_y^2/4} \frac{D_{-\nu}(-z_y)}{D_{1-\nu}(z_y)D_{-\nu}(-z_y)+D_{1-\nu}(-z_y)D_{-\nu}(z_y)} e^{z^2/4} D_{-\nu}(z), & z \geq z_y \\ \sqrt{\frac{2}{\kappa\sigma^2}} e^{-z_y^2/4} \frac{D_{-\nu}(z_y)}{D_{1-\nu}(z_y)D_{-\nu}(-z_y)+D_{1-\nu}(-z_y)D_{-\nu}(z_y)} e^{z^2/4} D_{-\nu}(-z), & z \leq z_y \end{cases} \tag{6.61}$$

with $\nu = \lambda/\kappa$. We can further simplify the answer by calculating $T(\nu, z) \equiv D_{1-\nu}(z)D_{-\nu}(-z) + D_{1-\nu}(-z)D_{-\nu}(z)$. First, we prove $T(\nu, z)$ is independent of z:

$$\begin{aligned} \frac{dT(\nu, z)}{dz} &= D'_{1-\nu}(z)D_{-\nu}(-z) - D_{1-\nu}(z)D'_{-\nu}(-z) - D'_{1-\nu}(-z)D_{-\nu}(z) + D_{1-\nu}(-z)D'_{-\nu}(z) \\ &= D_{1-\nu}(z)\left[-zD_{-\nu}(-z) + \nu D_{-\nu-1}(-z)\right] - D_{1-\nu}(-z)\left[zD_{-\nu}(z) + \nu D_{-\nu-1}(z)\right] \\ &= D_{1-\nu}(z)D_{1-\nu}(-z) - D_{1-\nu}(-z)D_{1-\nu}(z) \\ &= 0 \end{aligned} \tag{6.62}$$

Here, we use the recursion relation for parabolic cylinder functions,

$$D_{\nu+1}(z) - zD_\nu(z) + \nu D_{\nu-1}(z) = 0 \tag{6.63}$$

$$D'_\nu(z) + \frac{1}{2} z D_\nu(z) - \nu D_{\nu-1}(z) = 0 \tag{6.64}$$

From these, it is also not difficult to get

$$T(\nu) = \nu T(\nu + 1) \tag{6.65}$$

(Note we dropped dependence on z here.) Then, we have

$$T(\nu) = \frac{T(1)}{\Gamma(\nu)}, \quad \nu > 0, \tag{6.66}$$

where $T(1) = 2.50663$. Plugging this result into Eq. (6.61), we get the stated expression for $\hat{p}$.

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