LECTURE NOTES: OPTIMAL TRADING UNDER CONSTRAINTS*

by

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Abstract

These are lecture notes on the techniques and results of the theory of optimal trading for a single agent under convex constraints on his portfolio process, in a continuous-time model. A similar methodology is applied to the case of policy dependent prices, different interest rates for borrowing and lending and transaction costs problems. We study the hedging problem and the portfolio optimization problem for the investor in this market. Mathematical tools involved are those of continuous-time martingales, convex duality, forward-backward SDE's and PDE's.

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1. INTRODUCTION.

The main topic of these lecture notes is the problem of hedging (super-replication) and utility maximization for a single agent in a continuous-time financial market, under convex constraints on the proportions of wealth he/she invests in stocks. We present the model in Section 2; it is a standard, generalized Black-Scholes-Samuelson-Merton continuous-time diffusion model for several (d) risky assets (called "stocks") and one riskless "bank account". In Section 3 we introduce an agent who can trade in the assets, and describe his/her portfolio and wealth processes. We present the "equivalent martingale measure" approach to pricing financial contracts in the market in Section 4. The fair price of a contingent claim is obtained as its expectation under a change of probability measure that makes stocks prices martingales, after discounting by the interest rate of the bank account. In the special case of constant market parameters and the European call contract this leads to the famous Black-Scholes formula. Sections 5 and 6 generalize this approach to the case of a constrained market, in which the agent's hedging portfolio has to take values in a given closed convex set K. It is shown that the minimal hedging cost of a claim is now a supremum of Black-Scholes prices, taken over a family of auxiliary markets, parametrized by processes $\nu(\cdot)$, taking values in the domain of the support function of the set $-K$. These markets are chosen so that the wealth process becomes a supermartingale, under the appropriate change of measure. It is also shown that the supremum is attained if and only if the Black-Scholes (unconstrained) hedging portfolio happens to satisfy constraints. The latter result is used to prove, in the constant market parameters framework, that the minimal hedging cost under constraints can be calculated as the Black-Scholes price of an appropriately modified contingent claim, and that the corresponding hedging portfolio automatically satisfies the constraints. We end Section 6 by showing that there is no arbitrage in the constrained market if and only if a price of a claim is chosen in the interval determined by the minimal hedging price of the seller and the maximal hedging price of the buyer.

In Section 7 we show how the same methodology can be used to get analogous results in a market in which the drift of the wealth process is a concave function of the portfolio process. This can be regarded as a model in which the asset prices parameters depend on the trading strategy of the investor. More general model of this kind, in which both the drift and the diffusion terms of the prices depend on the portfolio and wealth of the investor is studied in Section 8, by different, Forward-Backward SDE's methods. An example is given in Section 9, providing a way of calculating the hedging price of options when the interest rate for borrowing is larger than the one for lending.

In Section 10 we introduce the concept of utility functions and prove existence of an optimal constrained portfolio strategy for maximizing expected utility from terminal wealth in Section 11. This is done indirectly, by first solving a dual problem, which is, loosely speaking, a problem to find an optimal change of probability measure associated to the constrained market. The optimal portfolio policy is the one that hedges the inverse of "marginal utility" (derivative of the utility function), evaluated at the Radon-Nikodym derivative corresponding to the optimal change of measure in the dual problem. Explicit solutions are provided for the case of logarithmic and power utilities. Next, in Section 12, we argue that it makes sense to price contingent claims in the constrained market by calculating the Black-Scholes price in the unconstrained auxiliary market that corresponds to the optimal dual change of measure. Although in general this price depends on the utility of the agent and his/her initial capital, in many cases it does not. In particular, if the contraints are given by a cone, and the market parameters are constant, the optimal dual process is independent of utility and initial capital.

In Sections 13-17 we study the hedging and utility maximization problem in the presence of proportional transaction costs. Similarly as in the case of constraints, we identify the family of (pairs of) changes of probability measure, under which the "wealth process" is a supermartingale, and the supremum over which gives the minimal hedging price of a claim in this market. In this case we do not know how to actually calculate this price, although it is known in some special cases. In particular, it is trivial for European call; namely, in order to hedge the call almost surely in the presence of positive transaction costs, one has to buy a whole share of the stock and hold it. Next, we consider the utility maximization problem in this setting, and its dual. We prove the existence in the primal problem, but, unfortunately, we do not know in general, whether an optimal solution exists for the dual problem. Under the assumption that the optimal dual solution does exist, the nature of the optimal terminal wealth in the primal problem is the same as in the case of constraints - it is equal to the inverse of the marginal utility evaluated at the optimal dual solution. This result is used to get sufficient conditions for the optimal policy to be the one of no trade at all - this is the case if the return rate of the stock is not very different from the interest rate of the bank account and the transaction costs are large relative to the time horizon.

In Section 18 we study the problem of maximizing the long-term growth rate of agent's wealth, under the constraint that the wealth never falls below a given fraction of its maximum up-to-date. A simple"trick" makes this problem equivalent to an unconstrained utility maximization problem, which is solved by the methods of previous sections.

We collect some basic results from stochastic calculus in Appendix.

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2. THE MODEL

We consider a financial market $\mathcal{M}$ which consists of one *bank account* and several (d) *stocks.* The prices $P_0(t), \{P_i(t)\}_{1\le i\le d}$ of these financial instruments evolve according to the equations

(2.1)
$$dP_0(t) = P_0(t)r(t)dt\ , \quad P_0(0)=1$$

(2.2)
$$dP_i(t) = P_i(t)[b_i(t)dt+\sum_{j=1}^{d}\sigma_{ij}(t)dW^{(j)}(t)]\ , \quad P_i(0)=p_i\in(0,\infty)\ ;\ \ i=1,\ldots,d\,.$$

Here $W=(W^{(1)},\ldots,W^{(d)})^*$ is a standard Brownian motion in $\mathbb{R}^d$, defined on a complete probability space $(\Omega,\mathcal{F},\mathbf{P})$, and we shall denote by $\{\mathcal{F}_t\}$ the $\mathbf{P}$-augmentation of the filtration $\mathcal{F}_t^W=\sigma(W(s);\ 0\le s\le t)$ generated by W. The *coefficients* (or parameters) of $\mathcal{M}$ - i.e., the processes $r(t)$ (scalar interest rate), $b(t)=(b_1(t),\ldots,b_d(t))^*$ (vector of appreciation rates) and $\sigma(t)=\{\sigma_{ij}(t)\}_{1\le i,j\le d}$ (volatility matrix) - are assumed to be progressively measurable with respect to $\{\mathcal{F}_t\}$ and *bounded* uniformly in $(t,\omega)\in[0,T]\times\Omega$. We shall also impose the following strong non-degeneracy condition on the matrix $a(t):=\sigma(t)\sigma^*(t)$:

(2.3)
$$\xi^*a(t)\xi\ge\varepsilon||\xi||^2,\ \ \forall\ (t,\xi)\ \in\ [0,T]\times\mathbb{R}^d$$

almost surely, for a given real constant $\varepsilon>0$.

We introduce also the "relative risk" process

(2.5)
$$\theta(t):=\sigma^{-1}(t)[b(t)-r(t)\mathbf{1}]\ ,$$

where $\mathbf{1}=(1,\ldots,1)^*$. The exponential martingale

(2.6)
$$Z_0(t):=\exp[-\int_0^t\theta^*(s)dW(s)-\frac{1}{2}\int_0^t||\theta(s)||^2ds]$$

and the discount process

(2.7)
$$\gamma_0(t):=\exp\{-\int_0^t r(s)ds\}$$

will be employed quite frequently.

2.1 Remark: It is a straightforward consequence of the strong non-degeneracy condition (2.3), that the matrices $\sigma(t),\sigma^*(t)$ are invertible, and that the norms of $(\sigma(t))^{-1},(\sigma^*(t))^{-1}$ are bounded above and below by δ and $1/\delta$, respectively, for some $\delta\ \in\ (1,\infty)$; see Karatzas & Shreve (1991), page 372. The boundedness of

$b(\cdot), r(\cdot)$ and $(\sigma(\cdot))^{-1}$ implies that of $\theta(\cdot)$, and thus also the martingale property of the process $Z_0(\cdot)$ in (2.6).

3. PORTFOLIO AND WEALTH PROCESSES

Consider now an economic agent whose actions cannot affect market prices, and who can decide, at any time $t \in [0,T]$, what proportion $\pi_i(t)$ of his wealth $X(t)$ to invest in the i^{th} stock $(1 \le i \le d)$. Of course these decisions can only be based on the current information $\mathcal{F}_t$, without anticipation of the future. With $\pi(t) = (\pi_1(t), \ldots, \pi_d(t))^*$ chosen, the amount $X(t)[1 - \sum_{i=1}^d \pi_i(t)]$ is invested in the bank. Thus, in accordance with the model set forth in (2.1), (2.2), the wealth process $X(t)$ satisfies the linear stochastic equation

$$\begin{aligned} dX_t &= \sum_{i=1}^d \pi_i(t) X_t \{b_i(t)dt + \sum_{j=1}^d \sigma_{ij}(t) dW^{(j)}(t)\} + \{1 - \sum_{i=1}^d \pi_i(t)\} X_t r_t dt \\ &= r(t)X(t)dt + X(t)\pi^*(t)\sigma(t)dW_0(t) \ ; \quad X(0) = x > 0 \ , \end{aligned} \tag{3.1}$$

where the real number $x > 0$ represents initial capital and

$$W_0(t) \ := \ W(t) + \int_0^t \theta(s)ds \ , \quad 0 \le t \le T \ . \tag{3.2}$$

We formalize the above discussion as follows.

3.1 Definition: (i) An $\mathbb{R}^d$ - valued, $\{\mathcal{F}_t\}$ - progressively measurable process $\pi = \{\pi(t),$
$0 \le t \le T\}$ with $\int_0^T X^2(t)||\pi(t)||^2 dt < \infty$, a.s., will be called a *portfolio process* (here, X is the corresponding wealth process defined in (ii) below).

(ii) Given the portfolio $\pi(\cdot)$ as above, the solution $X \equiv X^{x,\pi}$ of the equation (3.1) will be called the *wealth process* corresponding to the portfolio π and initial capital $x \in (0, \infty)$.

3.2 Definition: A portfolio process π is called *admissible* for the initial capital $x \in (0, \infty)$, if

$$X^{x,\pi}(t) \ge 0, \qquad \forall \ 0 \le t \le T \tag{3.3}$$

holds almost surely. The set of admissible portfolios π will be denoted by $\mathcal{A}_0(x)$. $\diamond$

In the notation of (2.6), (2.7), the equation (3.1) leads to

$$M_0(t) := \gamma_0(t)X(t) = x + \int_0^t \gamma_0(s)X(s)\pi^*(s)\sigma(s)dW_0(s). \tag{3.4}$$

In particular, the discounted wealth process $M_0(\cdot)$ of (3.4) is seen to be a continuous local martingale under the so-called "risk-neutral" probability measure

(or "equivalent martingale measure")

$$\mathbf{P}^0(A) := E[Z_0(T)1_A], \quad A \in \mathcal{F}_T \ . \tag{3.5}$$

If $\pi \in \mathcal{A}_0(x)$, the $\mathbf{P}^0$-local martingale $M_0(\cdot)$ of (3.4) is also nonnegative, thus a supermartingale. Consequently,

$$E^0[\gamma_0(T)X^{x,\pi,c}(T)] \le x, \quad \forall\ \pi \in \mathcal{A}_0(x) \ . \tag{3.6}$$

Here, E^0 denotes the expectation operator under the measure $\mathbf{P}^0$; under this measure, the process W_0 of (3.2) is standard Brownian motion by Girsanov theorem and the discounted stock prices $\gamma_0(\cdot)P_i(\cdot)$ are martingales, since

$$dP_i(t) = P_i(t)[r(t)dt + \sum_{j=1}^{d} \sigma_{ij}(t)dW_0^{(j)}(t)] \ , \ P_i(0) = p_i; \ i = 1, \dots, d \tag{3.7}$$

from (2.2) and (3.2).

3.3 Remark: For any given $\pi \in \mathcal{A}_0(x)$, let $X(\cdot) \equiv X^{x,\pi}(\cdot)$ and define the "bankruptcy time"

$$S := \inf\{t \in [0,T]; X(t) = 0\} \wedge T. \tag{3.8}$$

Because the continuous process $M_0(\cdot)$ of (3.4) is a $\mathbf{P}^0$-supermartingale, the same is true of $\gamma_0(\cdot)X(\cdot)$, and thus (see Karatzas & Shreve (1991), Problem 1.3.29) for a.e. $\omega \in \{S < T\}$:

$$X(t,\omega) = 0, \quad \forall\ t \in [S(\omega), T]. \tag{3.9}$$

In other words, if the wealth $X(\cdot)$ becomes equal to zero before the end T of the horizon, it stays there; further values of the portfolio $\pi(\cdot)$ become irrelevant. This remark seems to be unnecessary since the solution $X(\cdot)$ to the linear equation (3.1) is always positive. However, we shall, in fact, allow the possibility of bankruptcy; i.e., we shall allow continuous wealth processes modeled by (3.1) for $t < S$, where S is some stopping time, and $X(\cdot) \equiv 0$ for $S \le t \le T$.

3.4 Definition: We say that a portfolio strategy $\pi(\cdot)$ results in *arbitrage* if the initial investment $x = 0$, $X^{0,\pi}(T) \ge 0$, but $X^{0,\pi}(T) > 0$ with positive probability.

Notice that inequality (3.6) implies that an admissible portfolio $\pi \in \mathcal{A}_0(0)$ cannot result in arbitrage.

4. PRICING CONTINGENT CLAIMS IN A COMPLETE MARKET

Let us suppose now that an agent promises to pay a random amount $B(\omega) \ge 0$ at time $t = T$. *What is the value of this promise at time* $t = 0$? In other words, how much should the agent charge for selling a contractual obligation that entitles its holder to a payment of size $B(\omega)$ at $t = T$?

For instance, suppose that this obligation stipulates selling one share of the first stock at a contractually specified price q. If at time $t = T$ the price $P_1(T,\omega)$

of the stock is below q, the contract is worthless to its holder; if not, the holder can purchase the stock at the price q per share and then sell it at price $P_1(T,\omega)$, thus making a profit of $P_1(T,\omega) - q$. In other words, this contract entitles its holder to a payment of $B(\omega) = (P_1(T,\omega) - q)^+$ at time $t = T$; it is called a (European) *call option* with "exercise price" q and "maturity date" T.

To answer the question of the first paragraph, one argues as follows. Suppose the agent sets aside an amount $x > 0$ at time $t = 0$; he/she invests in the market $\mathcal{M}$ according to some portfolio $\pi(\cdot)$, but wants to be certain that at time $t = T$ he/she will be able to *cover his/her obligation*, i.e., that $X^{x,\pi}(T) \geq B$ will hold almost surely. What is the smallest value of $x > 0$ for which such "hedging" is possible? This smallest value will then be the "price" of the contract at time $t = 0$.

4.1 Definition: A *Contingent Claim* is a nonnegative, $\mathcal{F}_T$-measurable random variable B that satisfies

$$0 < E^0[\gamma_0(T)B] < \infty \ . \tag{4.1}$$

The *hedging price* of this contingent claim is defined by

$$u_0 := \inf\{x > 0;\ \exists \pi \in \mathcal{A}_0(x) \ \ s.t. \ \ X^{x,\pi}(T) \geq B \ \ a.s.\} \ . \tag{4.2}$$

The following classical result identifies u_0 as the expectation, under the risk-neutral probability measure of (3.5), of the claim's discounted value (Harrison & Kreps (1979), Harrison & Pliska (1981, 83).

4.2 Proposition: *The infimum in (4.2) is attained, and we have*

$$u_0 = E^0[\gamma_0(T)B] \ . \tag{4.3}$$

Furthermore, there exists a portfolio $\pi_0(\cdot)$ such that $X_0(\cdot) \equiv X^{u_0,\pi_0}(\cdot)$ is given by

$$X_0(t) = \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B|\mathcal{F}_t] \ , \quad 0 \leq t \leq T \ . \tag{4.4}$$

Proof: Suppose $X^{x,\pi}(T) \geq B$ holds a.s. for some $x \in (0,\infty)$ and a suitable $\pi \in \mathcal{A}_0(x)$. Then from (3.6) we have $x \geq z := E^0[\gamma_0(T)B]$ and thus $u_0 \geq z$.

On the other hand, from the martingale representation theorem, the process

$$X_0(t) := \frac{1}{\gamma_0(t)} E^0[\gamma_0(T)B|\mathcal{F}_t] \ , \quad 0 \leq t \leq T$$

can be represented as

$$X_0(t) := \frac{1}{\gamma_0(t)}[z + \int_0^t \psi^*(s)dW_0(s)] \tag{4.5}$$

for a suitable $\{\mathcal{F}_t\}$-progressively measurable process $\psi(\cdot)$ with values in $\mathbb{R}^d$ and $\int_0^T ||\psi(t)||^2 dt < \infty$, a.s. Then $\pi_0(t) := \frac{1}{\gamma_0(t)X_0(t)}(\sigma^*(t))^{-1}\psi(t)$ is a well-defined, portfolio process (recall Remarks 2.1 and 3.3), and a comparison of (4.5) with (3.4) yields $X_0(\cdot) \equiv X^{z,\pi_0}(\cdot)$. Therefore, $z \geq u_0$. $\diamond$

Notice that

$$X_0(T) = X_0^{u_0,\pi_0}(T) = B \ , \quad a.s. \tag{4.6}$$

in Theorem 4.2; we express this by saying that the contingent claim is *attainable* (with initial capital u_0 and portfolio π_0).

4.3 Example. *Constant* $r(\cdot) \equiv r > 0, \sigma(\cdot) \equiv \sigma$ *nonsingular.* In this case, the solution $P(t) = (P_1(t), \ldots, P_d(t))^*$ is given by $P_i(t) = h_i(t-s, P(s), \sigma(W_0(t) - W_0(s)))$, $0 \leq s \leq t$ where $h : [0,\infty) \times \mathbb{R}_+^d \times \mathbb{R}^d \to \mathbb{R}_+^d$ is the function defined by

$$h_i(t,p,y;r) := p_i \exp[(r - \frac{1}{2}a_{ii})t + y_i] \ , \quad i = 1, \ldots, d. \tag{4.7}$$

Consider now a contingent claim of the type $B = \varphi(P(T))$, where $\varphi : \mathbb{R}_+^d \to [0,\infty)$ is a given continuous function, that satisfies polynomial growth conditions in both $||p||$ and $1/||p||$. Then the value process of this claim is given by

(4.8)
$$\begin{aligned} X_0(t) &= e^{-r(T-t)} E^0[\varphi(P(T))|\mathcal{F}_t] \\ &= e^{-r(T-t)} \int_{\mathcal{R}^d} \varphi(h(T-t, P(t), \sigma z)) \frac{1}{(2\pi(T-t))^{d/2}} \exp\{-\frac{\|z\|^2}{2(T-t)}\} dz \\ &= U(T-t, P(t)), \end{aligned}$$

where

$$U(t,p) := \left\{ \begin{array}{cl} e^{-rt} \int_{\mathcal{R}^d} \varphi(h(t,p,\sigma z; r)) \frac{e^{-||z||^2/2t}}{(2\pi t)^{d/2}} dz & ; \ t > 0, \ p \in \mathbb{R}_+^d \\ \varphi(p) & ; \ t = 0, \ p \in \mathbb{R}_+^d \end{array} \right\}. \tag{4.9}$$

In particular, the price u_0 of (4.3) is given, in terms of the function U of (4.9), by

$$u_0 = X_0(0) = U(T, P(0)) \ . \tag{4.10}$$

Moreover, function U is the unique solution to the Cauchy problem (by Feynman-Kac theorem)

$$\frac{1}{2} \sum_{n=1}^d \sum_{l=1}^d a_{nl} x_n x_l \frac{\partial^2 U}{\partial x_n \partial x_l} + \sum_{n=1}^d r(x_n \frac{\partial U}{\partial x_n} - U) = \frac{\partial U}{\partial t},$$

with the initial condition $U(0,x) = \varphi(x)$. Applying Ito's rule, we obtain

$$dU(T-t, P(t)) = rU(T-t, P(t)) + \sum_{n=1}^d \sum_{l=1}^d \sigma_{nl} P_n(t) \frac{\partial U}{\partial x_n}(T-t, P_n(t)) dW_0^{(l)}(t).$$

Comparing this with (3.1), we get that the hedging portfolio is given by

$$\pi_n(t)U(T-t,P(t)) = P_n(t)\frac{\partial U}{\partial x_n}(T-t,P(t)),\ n=1,\dots,d. \tag{4.11}$$

It should be noted that none of the above depends on vector $b(\cdot)$ of return rates.

A very explicit computation for the function U is possible for $d=1$ in the case $\varphi(p)=(p-q)^+$ of a *European call option*: with $\sigma=\sigma_{11}>0$, exercise price $q>0, \Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z}e^{-u^2/2}du$ and $\nu_\pm(t,p) := \frac{1}{\sigma\sqrt{t}}\left[\log(\frac{p}{q})+(r\pm\frac{\sigma^2}{2})t\right]$, we have the famous *Black & Scholes (1973) formula*

$$U(t,p) = \left\{\begin{array}{ll} p\Phi(\nu_+(t,p)) - qe^{-rt}\Phi(\nu_-(t,p)) & ;\ t>0,\ p\in(0,\infty) \\ (p-q)^+ & ;\ t=0,\ p\in(0,\infty) \end{array}\right\}. \tag{4.12}$$

5. CONVEX SETS AND CONSTRAINED PORTFOLIOS

We shall fix throughout a nonempty, closed, convex set K in $\mathbb{R}^d$, and denote by

$$\delta(x)\equiv\delta(x|K) := \sup_{\pi\in K}(-\pi^*x)\ :\ \mathbb{R}^d\to\mathbb{R}\cup\{+\infty\} \tag{5.1}$$

the support function of the convex set $-K$. This is a closed, positively homogeneous, proper convex function on $\mathbb{R}^d$ (Rockafellar (1970), p.114). It is finite on its *effective domain*

(5.2)
$$\tilde{K} := \{x\in\mathbb{R}^d;\ \delta(x|K)<\infty\} = \{x\in\mathbb{R}^d;\ \ \exists\ \beta\in\mathbb{R}\ s.t.\ -\pi^*x\le\beta,\ \ \forall\,\pi\in K\}\,,$$

which is a convex cone (called the "barrier cone" of $-K$). It will be assumed throughout this paper that the function

$$\delta(\cdot|K)\quad\text{is continuous on}\quad\tilde{K} \tag{5.3}$$

and that $0\in K$, so that:

$$\delta(x|K)\ge 0,\quad\forall\ x\in\mathbb{R}^d. \tag{5.4}$$

5.1 Remark: Theorem 10.2, p.84 in Rockafellar (1970) guarantees that (5.3) is satisfied, in particular, if K is locally simplicial.

5.2 Examples: The role of the closed, convex set K that we just introduced, is to model reasonable constraints on portfolio choice. One may, for instance, consider the following examples, all of which satisfy conditions (5.3) and (5.4).

(i) *Unconstrained case:* $K=\mathbb{R}^d$. Then $\tilde{K}=\{0\}$, and $\delta\equiv 0$ on $\tilde{K}$.

(ii) *Prohibition of short-selling:* $K=[0,\infty)^d$. Then $\tilde{K}=K$, and $\delta\equiv 0$ on $\tilde{K}$.

(iii) *Incomplete Market:* $K=\{\pi\in\mathbb{R}^d;\pi_i=0,\ \forall\ i=m+1,\dots,d\}$ for some fixed $m\in\{1,\dots,d-1\}$. Then $\tilde{K}=\{x\in\mathbb{R}^d;\ x_i=0,\ \forall\ i=1,\dots,m\}$ and $\delta\equiv 0$ on $\tilde{K}$.

(iv) *Incomplete Market with prohibition of short-selling:* $K = \{\pi \in \mathbb{R}^d;\ \pi_i \geq 0,\ \forall\, i = 1, \ldots, m\ \text{ and }\ \pi_i = 0,\ \forall\, i = m+1, \ldots, d\}$ with m as in (iii). Then $\tilde{K} = \{x \in \mathbb{R}^d;\ x_i \geq 0,\ \forall\ i = 1, \ldots, m\}$ and $\delta \equiv 0$ on $\tilde{K}$.

(v) *K is a closed, convex cone* in $\mathcal{R}^d$. Then $\tilde{K} = \{x \in \mathbb{R}^d;\ \pi^* x \geq 0,\ \forall\ \pi \in K\}$ is the polar cone of $-K$, and $\delta \equiv 0$ on $\tilde{K}$. This case obviously generalizes (i) - (iv).

(vi) *Prohibition of borrowing:* $K = \{\pi \in \mathbb{R}^d; \sum_{i=1}^d \pi_i \leq 1\}$. Then $\tilde{K} = \{x \in \mathbb{R}^d;\ x_1 = \ldots = x_d \leq 0\}$, and $\delta(x) = -x_1$ on $\tilde{K}$.

(vii) *Rectangular constraints:* $K = \times_{i=1}^d I_i,\ I_i = [\alpha_i, \beta_i]$ for some fixed numbers $-\infty \leq \alpha_i \leq 0 \leq \beta_i \leq \infty$, with the understanding that the interval I_i is open to the right (left) if $b_i = \infty$ (respectively, if $\alpha_i = -\infty$). Then $\delta(x) = \sum_{i=1}^d (\beta_i x_i^- - \alpha_i x_i^+)$ and $\tilde{K} = \mathbb{R}^d$ if all the $\alpha_i's$, $\beta_i's$ are real. In general, $\tilde{K} = \{x \in \mathbb{R}^d; x_i \geq 0,\ \forall\ i \in \mathcal{S}_+$ and $x_j \leq 0,\ \forall\ j \in \mathcal{S}_-\}$ where $\mathcal{S}_+ := \{i = 1, \ldots, d\ /\ \beta_i = \infty\}$, $\mathcal{S}_- := \{i = 1, \ldots, d\ /\ \alpha_i = -\infty\}$. ◇

From now on, we also allow our investor to spend some money for consumption. More precisely, we add the term $-dc(t)$ to the right-hand side of (3.1), where $c(\cdot)$ is a *cumulative consumption process*, a nondecreasing process, with $c(0) = 0$. The set of admissible policies $(\pi(\cdot), c(\cdot))$ is defined similarly as before, and still denoted by $\mathcal{A}_0(x)$. We consider only portfolios that take values in the given, convex, closed set $K \subset \mathbb{R}^d$, i.e., we replace the set of admissible policies $\mathcal{A}_0(x)$ with

$$\mathcal{A}'(x) := \{(\pi, c) \in \mathcal{A}_0(x);\ \pi(t, \omega) \in K \quad \text{for} \quad \ell \times \mathbf{P} - a.e.\ \ (t, \omega)\}\ . \tag{5.5}$$

Here, ℓ stands for Lebesgue measure on $[0, T]$. Consider the class $\mathcal{H}$ of $\tilde{K}$-valued, $\{\mathcal{F}_t\}$-progressively measurable processes $\nu = \{\nu(t), 0 \leq t \leq T\}$ which satisfy $E \int_0^T \|\nu(t)\|^2 dt + E \int_0^T \delta(\nu(t)) dt < \infty$, and introduce for every $\nu \in \mathcal{H}$ the analogues

$$\theta_\nu(t) := \theta(t) + \sigma^{-1}(t)\nu(t)\ , \tag{5.6}$$

$$\gamma_\nu(t) := \exp[-\int_0^t \{r(s) + \delta(\nu(s))\} ds]\ , \tag{5.7}$$

$$Z_\nu(t) := \exp[-\int_0^t \theta_\nu^*(s) dW(s) - \frac{1}{2} \int_0^t ||\theta_\nu(s)||^2 ds]\ , \tag{5.8}$$

$$W_\nu(t) := W(t) + \int_0^t \theta_\nu(s) ds\ , \tag{5.9}$$

of the processes in (2.5)-(2.7), (3.2), as well as the measure

$$\mathbf{P}^\nu(A) := E[Z_\nu(T) 1_A] = E^\nu[1_A], \quad A \in \mathcal{F}_T \tag{5.10}$$

by analogy with (3.5). Finally, denote by $\mathcal{D}$ the subset consisting of the processes $\nu \in \mathcal{H}$ which are bounded uniformly in (t, ω). Thus, for every $\nu \in \mathcal{D}$, the measure $\mathbf{P}^\nu$ of (5.10) is a probability measure and the process $W_\nu(\cdot)$ of (5.9) is a $\mathbf{P}^\nu-$Brownian motion.

In general, there are several interpretations for the processes $\nu \in \mathcal{D}$: they are stochastic "Lagrange multipliers" associated with the portfolio constraints; in economics jargon, they correspond to the shadow prices relevant to the incompleteness of the market introduced by constraints; they can also be considered as the dual processes appearing in the stochastic maximum principle corresponding to the stochastic control problems we shall be considering.

5.3 Definition: A contingent claim B will be called *K-hedgeable*, if it satisfies

$$V(0) := \sup_{\nu \in \mathcal{D}} E^\nu[\gamma_\nu(T)B] < \infty. \tag{5.11}$$

This definition will be justified in the next section; more precisely, it will be shown there that for any K-hedgeable contingent claim B, there exists a pair $(\pi, c) \in \mathcal{A}'(V(0))$ such that $X^{V(0),\pi,c}(T) = B$, and that $V(0)$ is the minimal initial wealth for which this can be achieved.

5.4 Remark: In the unconstrained case $K = \mathcal{R}^d$ we have $\tilde{K} = \{0\}$, and $V(0) = E^0[\gamma_0(T)B]$ is then the unconstrained hedging price for the contingent claim B, as in Proposition 4.2. The number $u_\nu := E^\nu[\gamma_\nu(T)B] = E[\gamma_\nu(T)Z_\nu(T)B]$ is the unconstrained hedging price for B in an *auxiliary market* $\mathcal{M}_\nu$; this market consists of a bank account with interest rate $r^{(\nu)}(t) := r(t)+\delta(\nu(t))$ and d stocks, with the same volatility matrix $\{\sigma_{ij}(t)\}_{1\le i,j\le d}$ as before and appreciation rates $b_i^{(\nu)}(t) := b_i(t) + \nu_i(t) + \delta(\nu(t)),\ 1 \le i \le d$, for any given $\nu \in \mathcal{D}$. We shall show that the price for hedging B with a constrained portfolio in the market $\mathcal{M}$, is given by the supremum of the unconstrained hedging prices $u_\nu = E^\nu[\gamma_\nu(T)B]$ in these auxiliary markets $\mathcal{M}_\nu,\ \nu \in \mathcal{D}$.

5.5 Remark: In terms of the $\mathbf{P}^\nu$-Brownian motion $W_\nu(\cdot)$ of (5.9), the stock price equations (2.2) can be re-written as

$$dP_i(t) = P_i(t)\left[(r(t) - \nu_i(t))dt + \sum_{j=1}^d \sigma_{ij}(t)dW_\nu^{(j)}(t)\right], \quad i = 1, \dots, d \tag{5.12}$$

for any given $\nu \in \mathcal{D}$.

6. HEDGING WITH CONSTRAINED PORTFOLIOS

We introduce in this section the "hedging price" of a contingent claim B, with portfolios constrained to take values in the set K of section 5, and show that this price coincides with the number $V(0) = \sup_{\nu\in\mathcal{D}} E^\nu[\gamma_\nu(T)B]$ of (5.11).

6.1 Definition: The *hedging price with K-constrained portfolios* of a contingent claim B is defined by

$$(6.1)\quad h(0) := \left\{ \begin{array}{c} \inf\{x \in (0,\infty); \exists (\pi,c) \in \mathcal{A}'(x),\ s.t.\ X^{x,\pi,c}(T) \geq B\ a.s.\} \\ \infty \quad ,\ \text{if the above set is empty} \end{array} \right\}.$$

Let us denote by $\mathcal{S}$ the set of all $\{\mathcal{F}_t\}$-stopping times τ with values in $[0,T]$, and by $\mathcal{S}_{\rho,\sigma}$ the subset of $\mathcal{S}$ consisting of stopping times τ s.t. $\rho(\omega) \leq \tau(\omega) \leq \sigma(\omega)$, $\forall\ \omega \in \Omega$, for any two $\rho \in \mathcal{S}, \sigma \in \mathcal{S}$ such that $\rho \leq \sigma$, a.s. For every $\tau \in \mathcal{S}$ consider also the $\mathcal{F}_\tau$-measurable random variable

$$(6.2)\qquad V(\tau) := ess \sup_{\nu \in \mathcal{D}} E^\nu[B\gamma_0(T)\exp\{-\int_\tau^T \delta(\nu(s))ds\}|\mathcal{F}_\tau].$$

6.2 Proposition: *For any contingent claim that satisfies (5.11), the family (6.2) of random variables $\{V(\tau)\}_{\tau \in \mathcal{S}}$ satisfies the equation of Dynamic Programming*

$$(6.3)\quad V(\tau) = ess \sup_{\nu \in \mathcal{D}_{\tau,\theta}} E^\nu[V(\theta)\exp\{-\int_\tau^\theta \delta(\nu(u))du\}|\mathcal{F}_\tau]\ ; \quad \forall\ \ \theta \in \mathcal{S}_{\tau,T}\ ,$$

where $\mathcal{D}_{\tau,\theta}$ is the restriction of $\mathcal{D}$ to the stochastic interval $[\![\tau,\theta]\!]$.

6.3 Proposition: *The process $V = \{V(t), \mathcal{F}_t; 0 \leq t \leq T\}$ of Proposition 6.2 can be considered in its RCLL modification and, for every $\nu \in \mathcal{D}$,*

$$(6.4)\qquad \left\{ \begin{array}{c} Q_\nu(t) := V(t)e^{-\int_0^t \delta(\nu(u))du}, \mathcal{F}_t;\ 0 \leq t \leq T \\ \\ \textit{is a } \mathbf{P}^\nu\textit{-supermartingale with RCLL paths} \end{array} \right\}.$$

Furthermore, V is the smallest adapted, RCLL process that satisfies (6.4) as well as

$$(6.5)\qquad V(T) = B\gamma_0(T), \qquad a.s.$$

Proof of Proposition 6.2: Let us start by observing that, for any $\theta \in \mathcal{S}$, the random variable

$$\begin{aligned} J_\nu(\theta) &:= E^\nu[V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta] \\ &= \frac{E[Z_\nu(\theta)Z_\nu(\theta,T)V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta]}{E[Z_\nu(\theta)Z_\nu(\theta,T)|\mathcal{F}_\theta]} \\ &= E[Z_\nu(\theta,T)V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta] \end{aligned}$$

depends only on the restriction of ν to $[\![\theta,T]\!]$ (we have used the notation $Z_\nu(\theta,T) = \frac{Z_\nu(T)}{Z_\nu(\theta)}$). It is also easy to check that the family of random variables $\{J_\nu(\theta)\}_{\nu \in \mathcal{D}}$ is directed upwards; indeed, for any $\mu \in \mathcal{D}, \nu \in \mathcal{D}$ and with $A = \{(t,\omega);\ J_\mu(t,\omega) \geq J_\nu(t,\omega)\}$ the process $\lambda := \mu\mathbf{1}_A + \nu\mathbf{1}_{A^c}$ belongs to $\mathcal{D}$ and we have a.s. $J_\lambda(\theta)$

$= \min\{J_\mu(\theta), J_\nu(\theta)\}$; then from Neveu (1975), p.121, there exists a sequence $\{\nu_k\}_{k\in\mathbf{N}} \subseteq \mathcal{D}$ such that $\{J_{\nu_k}(\theta)\}_{k\in\mathbf{N}}$ is increasing and

$$(i) \qquad V(\theta) = \lim_{k\to\infty} \uparrow J_{\nu_k}(\theta), \quad a.s.$$

Returning to the proof itself, let us observe that

$$\begin{aligned} V(\tau) &= ess \sup_{\nu\in\mathcal{D}_{\tau,T}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds} E^\nu\{V(T)e^{-\int_\theta^T \delta(\nu(s))ds}|\mathcal{F}_\theta\}|\mathcal{F}_\tau] \\ &\le ess \sup_{\nu\in\mathcal{D}_{\tau,T}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds} V(\theta)|\mathcal{F}_\tau], \quad a.s. \end{aligned}$$

To establish the opposite inequality, it certainly suffices to pick $\mu \in \mathcal{D}$ and show that

$$(ii) \qquad V(\tau) \ge E^\mu[V(\theta)e^{-\int_\tau^\theta \delta(\mu(s))ds}|\mathcal{F}_\tau]$$

holds almost surely.

Let us denote by $M_{\tau,\theta}$ the class of processes $\nu \in \mathcal{D}$ which agree with μ on $[\![\tau,\theta]\!]$. We have

$$\begin{aligned} V(\tau) &\ge ess \sup_{\nu\in M_{\tau,\theta}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds - \int_\theta^T \delta(\nu(s))ds} V(T)|\mathcal{F}_\tau] \\ &= ess \sup_{\nu\in M_{\tau,\theta}} E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds} E^\nu\{e^{-\int_\theta^T \delta(\nu(s))ds} V(T)|\mathcal{F}_\theta\}|\mathcal{F}_\tau]. \end{aligned}$$

Thus, for every $\nu \in M_{\tau,\theta}$, we have

$$\begin{aligned} V(\tau) &\ge E^\nu[e^{-\int_\tau^\theta \delta(\nu(s))ds} J_\nu(\theta)|\mathcal{F}_\tau] \\ &= \frac{E[Z_\nu(\tau)Z_\nu(\tau,\theta).E\{Z_\nu(\theta,T)|\mathcal{F}_\theta\}.e^{-\int_\tau^\theta \delta(\nu(s))ds} J_\nu(\theta)|\mathcal{F}_\tau]}{E[Z_\nu(\tau)Z_\nu(\tau,\theta).E\{Z_\nu(\theta,T)|\mathcal{F}_\theta\}|\mathcal{F}_\tau]} \\ &= E[Z_\nu(\tau,\theta)e^{-\int_\tau^\theta \delta(\nu(s))ds} J_\nu(\theta)|\mathcal{F}_\tau] \\ &= E[Z_\mu(\tau,\theta)e^{-\int_\tau^\theta \delta(\mu(s))ds} J_\nu(\theta)|\mathcal{F}_\tau] \\ &= \ldots = E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds} J_\nu(\theta)|\mathcal{F}_\tau]. \end{aligned}$$

Now clearly we may take $\{\nu_k\}_{k\epsilon\mathbf{N}} \subseteq M_{\tau,\theta}$ in (i), as $J_\nu(\theta)$ depends only on the restriction of ν on $[\![\theta, T]\!]$; and from the above,

$$\begin{aligned} V(\tau) &\ge \lim_{k\to\infty} \uparrow E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds} J_{\nu_k}(\theta)|\mathcal{F}_\tau] \\ &= E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds}. \lim_{k\to\infty} \uparrow J_{\nu_k}(\theta)|\mathcal{F}_\tau] \\ &= E^\mu[e^{-\int_\tau^\theta \delta(\mu(s))ds} V(\theta)|\mathcal{F}_\tau], \; a.s. \end{aligned}$$

by Monotone Convergence. $\diamond$

It is an immediate consequence of this proposition that

$$(iii) \qquad V(\tau)e^{-\int_0^\tau \delta(\nu(u))du} \geq E^\nu[V(\theta)e^{-\int_0^\theta \delta(\nu(u))du}|\mathcal{F}_\tau], \quad a.s.$$

holds for any given $\tau \in \mathcal{S}, \theta \in \mathcal{S}_{\tau,T}$ and $\nu \in \mathcal{D}$.

Proof of Proposition 6.3: Let us consider the positive, adapted process $\{V(t,\omega), \mathcal{F}_t;\ t \in [0,T] \cap \mathcal{Q}\}$ for $\omega \in \Omega$. From (iii), the process

$$\{V(t,\omega)e^{-\int_0^t \delta(\nu(s,\omega))ds},\ \mathcal{F}_t;\ t \in [0,T] \cap \mathcal{Q}\} \text{ for } \quad \omega \in \Omega$$

is a $\mathbf{P}^\nu$- supermartingale on $[0,T] \cap \mathcal{Q}$, where $\mathcal{Q}$ is the set of rational numbers, and thus has a.s. finite limits from the right and from the left (recall Proposition 1.3.14 in Karatzas & Shreve (1991), as well as the right-continuity of the filtration $\{\mathcal{F}_t\}$). Therefore,

$$V(t+,\omega) := \left\{ \begin{array}{ccc} \lim_{\substack{s \downarrow t \\ s \in \mathcal{Q}}} V(s,\omega) & ; & 0 \leq t < T \\ V(T,\omega) & ; & t = T \end{array} \right\}$$

$$V(t-,\omega) := \left\{ \begin{array}{ccc} \lim_{\substack{s \uparrow t \\ s \in \mathcal{Q}}} V(s,\omega) & ; & 0 < t \leq T \\ V(0) & ; & t = 0 \end{array} \right\}$$

are well-defined and finite for every $\omega \in \Omega^*, P(\Omega^*) = 1$, and the resulting processes are adapted. Furthermore (loc.cit.), $\{V(t+)e^{-\int_0^t \delta(\nu(s))ds},\ \mathcal{F}_t;\ 0 \leq t \leq T\}$ is a RCLL, $\mathbf{P}^\nu$-supermartingale, for all $\nu \in \mathcal{D}$; in particular,

$$V(t+) \geq E^\nu[V(T)e^{-\int_t^T \delta(\nu(s))ds}|\mathcal{F}_t],\ a.s.$$

holds for every $\nu \in \mathcal{D}$, whence $V(t+) \geq V(t)$ a.s. On the other hand, from Fatou's lemma we have for any $\nu \in \mathcal{D}$:

$$\begin{aligned} V(t+) &= E^\nu[\lim_{n\to\infty} V(t+\frac{1}{n})\ e^{-\int_t^{t+1/n} \delta(\nu(u))du}|\mathcal{F}_t] \\ &\leq \lim_{n\to\infty} E^\nu[V(t+\frac{1}{n})\ e^{-\int_t^{t+1/n} \delta(\nu(u))du}|\mathcal{F}_t] \leq V(t),\ a.s. \end{aligned}$$

and thus $\{V(t+),\ \mathcal{F}_t;\ 0 \leq t \leq T\}, \{V(t),\ \mathcal{F}_t;\ 0 \leq t \leq T\}$ are modifications of one another.

The remaining claims are immediate. $\diamond$

6.4 Theorem: (Cvitanić & Karatzas (1993), El-Karoui & Quenez (1995)) *For an arbitrary contingent claim B, we have $h(0) = V(0)$. Furthermore, if $V(0) < \infty$, there exists a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ such that $X^{V(0),\hat{\pi},\hat{c}}(T) = B,\ a.s.$*

Proof: We first want to show $h(0) \leq V(0)$. Clearly, we may assume $V(0) < \infty$. From (6.4), the martingale representation theorem and the Doob-Meyer decomposition, we have for every $\nu \in \mathcal{D}$:

$$Q_\nu(t) = V(0) + \int_0^t \psi_\nu^*(s)dW_\nu(s) - A_\nu(t), \quad 0 \le t \le T \ , \tag{6.6}$$

where $\psi_\nu(\cdot)$ is an $\mathbb{R}^d$-valued, $\{\mathcal{F}_t\}$-progressively measurable and a.s. square-integrable process and $A_\nu(\cdot)$ is adapted with increasing, RCLL paths and $A_\nu(0) = 0, EA_\nu(T) < \infty$ a.s. The idea then is to consider the positive, adapted, RCLL process

$$\hat{X}(t) := \frac{V(t)}{\gamma_0(t)} = \frac{Q_\nu(t)}{\gamma_\nu(t)}, \quad 0 \le t \le T \quad (\forall\ \nu \in \mathcal{D}) \tag{6.7}$$

with $\hat{X}(0) = V(0), \hat{X}(T) = B$ a.s., and to find a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ such that $\hat{X}(\cdot) = X^{V(0),\hat{\pi},\hat{c}}(\cdot)$. This will prove that $h(0) \le V(0)$.

In order to do this, let us observe that for any $\mu \in \mathcal{D}, \nu \in \mathcal{D}$ we have from (6.4)

$$Q_\mu(t) = Q_\nu(t) \exp\left[\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds\right] \ ,$$

and from (6.6):

(6.8)

$$\begin{aligned} dQ_\mu(t) = \exp[&\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds] \cdot [Q_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}dt \\ &+ \psi_\nu^*(t)dW_\nu(t) - dA_\nu(t)] \\ = \exp[&\int_0^t \{\delta(\nu(s)) - \delta(\mu(s))\}ds] \cdot [\hat{X}(t)\gamma_\nu(t)\{\delta(\nu(t)) - \delta(\mu(t))\}dt \\ &- dA_\nu(t) + \psi_\nu^*(t)\sigma^{-1}(t)(\nu(t) - \mu(t))dt + \psi_\nu^*(t)dW_\mu(t)] \ . \end{aligned}$$

Comparing this decomposition with

$$dQ_\mu(t) = \psi_\mu^*(t)dW_\mu(t) - dA_\mu(t) \ , \tag{6.6$'$}$$

we conclude that

$$\psi_\nu^*(t)\ e^{\int_0^t \delta(\nu(s))ds} = \psi_\mu^*(t)\ e^{\int_0^t \delta(\mu(s))ds}$$

and hence that this expression is independent of $\nu \in \mathcal{D}$:

$$\psi_\nu^*(t)\ e^{\int_0^t \delta(\nu(s))ds} = \hat{X}(t)\gamma_0(t)\hat{\pi}^*(t)\sigma(t); \quad \forall\ 0 \le t \le T,\ \nu \in \mathcal{D}\ , \tag{6.9}$$

for some adapted, $\mathbb{R}^d$-valued, a.s. square integrable process $\hat{\pi}$ (we do not know yet that $\hat{\pi}$ takes values in K). If $X(t) = 0$, then $X(s) = 0$ for all $s \ge t$, and we can set, for example, $\pi(s) = 0,\ s \ge t$ (in fact, one can show that $\int_0^T \mathbf{1}_{\{\hat{X}(t)=0\}} \|\psi_\nu(t)\|^2 dt = 0$, a.s; see Karatzas & Kou (1994)).

Similarly, we conclude from (6.8),(6.9) and (6.6)$'$:

$$e^{\int_0^t \delta(\nu(s))ds} dA_\nu(t) - \gamma_0(t)\hat{X}(t)[\delta(\nu(t)) + \hat{\pi}^*(t)\nu(t)]dt$$

$$= e^{\int_0^t \delta(\mu(s))ds} dA_\mu(t) - \gamma_0(t)\hat{X}(t)[\delta(\mu(t)) + \hat{\pi}^*(t)\mu(t)]dt$$

and hence this expression is also independent of $\nu \in \mathcal{D}$:

$$\hat{c}(t) := \int_0^t \gamma_\nu^{-1}(s)dA_\nu(s) - \int_0^t \hat{X}(s)[\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)]ds \ , \tag{6.10}$$

for every $0 \leq t \leq T, \nu \in \mathcal{D}$. From (6.10) with $\nu \equiv 0$, we obtain $\hat{c}(t) = \int_0^t \gamma_0^{-1}(s)dA_0(s), 0 \leq t \leq T$ and hence

$$\left\{ \begin{array}{c} \hat{c}(\cdot) \text{ is an increasing, adapted, RCLL process} \\ \text{with } \hat{c}(0) = 0 \quad \text{and} \quad \hat{c}(T) < \infty, a.s. \end{array} \right\} . \tag{6.11}$$

Next, we claim that

$$\delta(\nu) + \nu^*\hat{\pi}(t,\omega) \geq 0, \quad \ell \otimes \mathbf{P} - a.e. \tag{6.12}$$

holds for every $\nu \in \tilde{K}$. Then Theorem 13.1 of Rockafellar (1970) (together with continuity of $\delta(\cdot)$ and closedness of K) leads to the fact that

$$\hat{\pi}(t,\omega) \in K \quad \text{holds} \quad \ell \otimes \mathbf{P} - a.e. \ \text{ on } \ [0,T] \times \Omega \tag{6.12$'$}$$

In order to verify (6.12), notice that from (6.10) we obtain

$$\int_0^t \gamma_\nu^{-1}(s)A_\nu(s)ds = \hat{c}(t) + \int_0^t \hat{X}(s)\{\delta(\nu_s) + \nu_s^*\hat{\pi}_s\}ds; \ 0 \leq t \leq T, \ \nu \in \mathcal{D}.$$

Fix $\nu \in \tilde{K}$ and define the set $F_\nu := \{(t,\omega) \in [0,T] \times \Omega; \ \delta(\nu)) + \nu^*\hat{\pi}(t,\omega) < 0\}$. Let $\mu(t) := [\nu 1_{F_\nu^c} + n\nu 1_{F_\nu}], \ n \in \mathbf{N}$; then $\mu \in \mathcal{D}$, and assuming that (6.12) does not hold, we get for n large enough

$$\begin{aligned} E[\int_0^T \gamma_\mu^{-1}(s)A_\mu(s)ds] = E\left[\hat{c}(T) + \int_0^T \hat{X}(t)1_{F_\nu^c}\{\delta(\nu) + \nu^*\hat{\pi}(t)\}dt\right] \\ + nE\left[\int_0^T \hat{X}(t)1_{F_\nu}\{\delta(\nu) + \nu^*\hat{\pi}(t)\}dt\right] < 0 \ , \end{aligned}$$

a contradiction.

Now we can put together (6.6)-(6.10) to deduce

$$\begin{aligned} d(\gamma_\nu(t)\hat{X}(t)) = dQ_\nu(t) = \psi_\nu^*(t)dW_\nu(t) - dA_\nu(t) \\ = \gamma_\nu(t)[-d\hat{c}(t) - \hat{X}(t)\{\delta(\nu(t)) + \nu^*(t)\hat{\pi}(t)\}dt \\ + \hat{X}(t)\hat{\pi}^*(t)\sigma(t)dW_\nu(t)] \ , \end{aligned} \tag{6.13}$$

for any given $\nu \in \mathcal{D}$. As a consequence, the process

(3.4)$'$

$$\begin{aligned} \hat{M}_\nu(t) := \gamma_\nu(t)\hat{X}(t) + \int_0^t \gamma_\nu(s)d\hat{c}(s) + \int_0^t \gamma_\nu(s)\hat{X}(s)[\delta(\nu(s)) + \nu^*(s)\hat{\pi}(s)]ds \\ = V(0) + \int_0^t \gamma_\nu(s)\hat{X}(s)\hat{\pi}^*(s)\sigma(s)dW_\nu(s) \ , \quad 0 \leq t \leq T \end{aligned}$$

is a nonnegative, $\mathbf{P}^\nu$-local martingale.

In particular, for $\nu \equiv 0$, (6.13) gives:

$$(6.13)' \qquad \begin{aligned} d(\gamma_0(t)\hat{X}(t)) &= -\gamma_0(t)d\hat{c}(t) + \gamma_0(t)\hat{X}(t)\hat{\pi}^*(t)\sigma(t)dW_0(t), \\ \hat{X}(0) &= V(0)\ , \quad \hat{X}(T) = B\ , \end{aligned}$$

which is equation (3.1) (plus the consumption term) for the process $\hat{X}(\cdot)$ of (6.7). This shows $\hat{X}(\cdot) \equiv X^{V(0),\hat{\pi},\hat{c}}(\cdot)$, and hence $h(0) \le V(0) < \infty$.

To complete the proof, it thus suffices to show $h(0) \ge V(0)$. Clearly, we may assume $h(0) < \infty$, and then there exists a number $x \in (0,\infty)$ such that $X^{x,\pi,c}(T) \ge B$, a.s., for some $(\pi, c) \in \mathcal{A}'(x)$. Then the analogue of (6.13) holds, and it follows from the supermartingale property that

$$(6.14) \qquad \begin{aligned} x &\ge E^\nu[\gamma_\nu(T)X^{x,\pi,c}(T) + \int_0^T \gamma_\nu(t)dc(t) \\ &\quad + \int_0^T \gamma_\nu(t)X^{x,\pi,c}(t)\{\delta(\nu(t)) + \nu^*(t)\pi(t)\}dt] \\ &\ge E^\nu[B\gamma_\nu(T)]\ , \end{aligned}$$

$\forall\ \nu \in \mathcal{D}$. Therefore, $x \ge V(0)$, and thus $h(0) \ge V(0)$. $\diamond$

6.5 Definition: We say that a K-hedgeable contingent claim B is $K-$*attainable* if there exists a portfolio process π with values in K such that $(\pi, 0) \in \mathcal{A}'(V(0))$ and $X^{V(0),\pi,0}(T) = B$, a.s.

6.6 Theorem: *For a given K-hedgeable contingent claim B, and any given $\lambda \in \mathcal{D}$, the conditions*

$$(6.15) \qquad \{Q_\lambda(t) = V(t)e^{-\int_0^t \delta(\lambda(u))du}, \mathcal{F}_t;\ 0 \le t \le T\} \quad \text{is a } \mathbf{P}^\lambda\text{-martingale}$$

$$(6.16) \qquad \lambda \ \text{ achieves the supremum in } \ V(0) = \sup_{\nu \epsilon \mathcal{D}} E^\nu[B\gamma_\nu(T)]$$

$$(6.17) \qquad \left\{ \begin{matrix} B \text{ is } K\text{-attainable (by a portfolio } \pi\text{), and the} \\ \text{corresponding } \gamma_\lambda(\cdot)X^{V(0),\pi,0}(\cdot) \text{ is a } \mathbf{P}^\lambda\text{-martingale} \end{matrix} \right\}$$

are equivalent, and imply

$$(6.18) \qquad \hat{c}(t,\omega) = 0,\ \delta(\lambda(t,\omega)) + \lambda^*(t,\omega)\hat{\pi}(t,\omega) = 0; \quad \ell \otimes P - a.e.$$

for the pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}'(V(0))$ of Theorem 6.4.

Proof: The $\mathbf{P}^\lambda$-supermartingale $Q_\lambda(\cdot)$ is a $\mathbf{P}^\lambda$-martingale, if and only if $Q_\lambda(0) = E^\lambda Q_\lambda(T) \Leftrightarrow V(0) = E^\lambda[B\gamma_\lambda(T)] \Leftrightarrow$ (6.16).

On the other hand, (6.15) implies $A_\lambda(\cdot) \equiv 0$, and so from (6.10): $\hat{c}(t) = -\int_0^t \hat{X}(s)[\delta(\lambda(s)) + \lambda^*(s)\hat{\pi}(s)]ds$. Now (6.18) follows from the increase of $\hat{c}(\cdot)$ and the nonnegativity of $\delta(\lambda) + \lambda^*\hat{\pi}$, since $\hat{\pi}$ takes values in K.

From (6.16) (and its consequences (6.15), (6.18)), the process $\hat{X}(\cdot)$ of (6.7) and (6.13) coincides with $X^{V(0),\hat{\pi},0}(\cdot)$, and we have: $\hat{X}(T) = B$ almost surely, $\gamma_\lambda(\cdot)\hat{X}(\cdot)$ is a $\mathbf{P}^\lambda$-martingale; thus (6.17) is satisfied with $\pi \equiv \hat{\pi}$. On the other hand, suppose that (6.17) holds; then $V(0) = E^\lambda[B\gamma_\lambda(T)]$, so (6.16) holds.

6.7 Theorem: *Let B be a K-hedgeable contingent claim. Suppose that, for any $\nu \in \mathcal{D}$ with $\delta(\nu) + \nu^*\hat{\pi} \equiv 0$,*

(6.19) $\qquad Q_\nu(\cdot)$ in (6.4) is of class $DL[0,T]$, under $\mathbf{P}^\nu$.

Then, for any given $\lambda \in \mathcal{D}$, the conditions (6.15), (6.16), (6.18) are equivalent, and imply

(6.17)^o $$\left\{ \begin{array}{c} B \text{ is } K\text{-attainable (by a portfolio } \pi\text{), and the} \\ \text{corresponding } \gamma_0(\cdot)X^{V(0),\pi,0}(\cdot) \text{ is a } \mathbf{P}^0\text{-martingale} \end{array} \right\}.$$

Proof: We have already shown the implications (6.15) $\Leftrightarrow$ (6.16) $\Rightarrow$ (6.18). To prove that these three conditions are actually *equivalent* under (6.19), suppose that (6.18) holds; then from (6.10): $A_\lambda(\cdot) \equiv 0$, whence the $\mathbf{P}^\lambda$-local martingale $Q_\lambda(\cdot)$ is actually a $\mathbf{P}^\lambda$-martingale (from (6.6) and the assumption (6.19)); thus (6.15) is satisfied.

Clearly then, if (6.15), (6.16), (6.18) are satisfied for some $\lambda \in \mathcal{D}$, they are satisfied for $\lambda \equiv 0$ as well; and from Theorem 6.6, we know then that (6.17)^o (i.e., (6.17) with $\lambda \equiv 0$) holds.

6.8 Remark: (i) Roughly speaking, Theorems 6.6, 6.7 say that *the supremum in (6.16) is attained if and only if it is attained by $\lambda \equiv 0$, if and only if the Black-Scholes (unconstrained) portfolio happens to satisfy constraints.*

(ii) It can be shown that the conditions $V(0) < \infty$ and (6.19) are satisfied (the latter, in fact, for every $\nu \in \mathcal{D}$) in the case of the simple European call option $B = (P_1(T) - q)^+$, provided

(6.20) $\qquad$ the function $\quad x \mapsto \delta(x) + x_1 \quad$ is bounded from below on $\tilde{K}$.

The same is true for any contingent claim B that satisfies $B \le \alpha P_1(T)$ a.s., for some $\alpha \in (0,\infty)$.

6.9 Remark: Note that the condition (6.20) is indeed satisfied, if the convex set

(6.20)$'$ $\qquad$ *K contains both the origin and the point* $(1,0,\ldots,0)$

(and thus also the line-segment adjoining these points); for then $x_1 + \delta(x) \ge x_1 + \sup_{0\le\alpha\le 1}(-\alpha x_1) = x_1^+ \ge 0$, $\forall x \in \tilde{K}$. This is the case in the Examples 5.2 (i)-(iv), (vi), and (vii) with $1 \le \beta_1 \le \infty$.

6.10 Remark: *If the condition (6.20) is not satisfied, we have* $V(0) = \infty$ *for the European call option* $B = (P_1(T) - q)^+$ *with* $\delta(\cdot) \geq 0,\ r(\cdot) \geq 0$. In other words, such constraints make impossible the hedging of this contingent claim , starting with a finite initial capital.

6.11 Remark: A slight modification of the proof of Theorem 6.4 shows that

$$V(0) = \sup_{\nu \in \mathcal{D}} E^{\nu}[B\gamma_{\nu}(T)] = \sup_{\nu \in \mathcal{H}} E[B\gamma_{\nu}(T)Z_{\nu}(T)] \tag{6.23}$$

holds for an arbitrary contingent claim B. The straightforward details are left to the diligence of the reader.

We would like now to get a method for calculating the price $h(0)$. In order to do that, we assume constant market coefficients r, b, σ and consider only the claims of the form $B = b(P(T))$, for a given function b. Similarly as in the no-constraints case, the minimal hedging process will be given as $X(t) = U(t, P(t))$, for some function $U(t,p)$, depending on the constraints. Introduce also, for a given process $\nu(\cdot)$ in $\mathbb{R}^d$, the auxiliary, shadow economy vector of stock prices $P^{\nu}(\cdot)$ (in analogy with (5.12)) by

$$dP_i^{\nu}(t) = P_i^{\nu}(t)\left[(r - \nu_i(t))dt + \sum_{j=1}^{d} \sigma_{ij} dW^{(j)}(t)\right] \tag{6.24}$$

Then, by standard results from stochastic control theory, we can restate Theorem 6.4 as follows:
6.12 Theorem: *With the above notation and assumptions, we have*

$$U(t,p) = \sup_{\nu \in \mathcal{D}} E\left[b(P^{\nu}(T))e^{-\int_t^T (r+\delta(\nu(s)))ds} \,\middle|\, P^{\nu}(t) = p\right]. \tag{6.25}$$

We will show that this complex looking stochastic control problem has a simple solution. First, we modify the value of the claim by considering the following function:

$$\hat{b}(p) = \sup_{\nu \in \tilde{K}} b(pe^{-\nu})e^{-\delta(\nu)}. \tag{6.26}$$

Here, $pe^{-\nu} = (p_1 e^{-\nu_1}, \ldots, p_d e^{-\nu_d})^*$, and we use the same notation for the componentwise product of two vectors throughout. We use below the term Feynman-Kac assumptions, with the understanding that those are the assumptions under which relevant expected values satisfy corresponding PDE's. A set of such assumptions is given in Duffie (1992). Here is the result which gives a way of calculating $U(t,p)$:
6.13 Theorem: *(Broadie, Cvitanić & Soner (1996)) The minimal* K*-hedging price function* $U(t,p)$ *of the claim* $b(P(T))$ *is the Black-Scholes cost function for*

replicating $\hat{b}(P(T))$. In particular, under Feynman-Kac assumptions, it is the solution to the PDE

$$V_t + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}p_ip_jV_{p_ip_j} + r\left(\sum_{i=1}^{d} p_iV_{p_i} - V\right) = 0, \tag{6.27}$$

with the terminal condition

$$V(T,p) = \hat{b}(p), \quad p \in \mathbb{R}_+^d. \tag{6.28}$$

Moreover, the corresponding hedging strategy π satisfies the constraints. Under Feynman-Kac assumptions, it is given by

$$\pi_i(t) = P_i(t)V_{p_i}(t, P(t))/V(t, P(t)), \quad i = 1, \dots, d \tag{6.29}$$

and $\pi(\cdot)$ takes values in K.

Proof: (a) We first show that portfolio π satisfies the constraints. Let $\nu \in \mathcal{D}$ and observe that, from the properties of the support function and the cone property of $\tilde{K}$,

$$(i)\ \ \hat{\hat{b}} = \hat{b}$$

$$(ii) \int_t^T \delta(\nu_s)ds \geq \delta(\int_t^T \nu_s ds),$$

$$(iii) \int_t^T \nu_s ds \ \text{ is an element of } \ \tilde{K},$$

where $\int_t^T \nu(s)ds := (\int_t^T \nu_1(s)ds, ..., \int_t^T \nu_d(s)ds)^*$. Moreover, we have

$$(iv)\ P_i^{\nu}(t) = P_i^0(t)e^{-\int_0^t \nu_i(s)ds},$$

because the processes on the left-hand side and the right-hand side satisfy the same linear SDE. Then, for every $\nu \in \mathcal{D}$ we have

(6.30)

$$\begin{aligned} E[\hat{b}(P^{\nu}(T))e^{-\int_0^T(r+\delta(\nu(s)))ds}] &\leq E[\hat{b}(P^0(T)e^{-\int_0^T \nu(s)ds})e^{-\delta(\int_0^T \nu(s)ds)}e^{-rT}] \\ &\leq E[\sup_{\nu\in\tilde{K}} \hat{b}(P^0(T)e^{-\nu})e^{-\delta(\nu)}e^{-rT}] \\ &= E[\hat{\hat{b}}(P^0(T))e^{-rT}] = E[\hat{b}(P^0(T))e^{-rT}]. \end{aligned}$$

Therefore the supremum (over $\mathcal{D}$) of the initial expression is obtained for $\nu = 0$. Similarly for conditional expectations of (6.25). It follows then from Theorems 6.6, 6.7 that $\hat{b}(P(T))$ can be attained by a portfolio which satisfies the constraints. Moreover, under Feynman-Kac assumptions, its value function is the solution to (6.27)-(6.28), and the portfolio is given by (6.29).

(b) To conclude we have to show that to hedge $b(P(T))$ we have to hedge at least $\hat{b}(P(T))$. It is sufficient to prove that the left limit of $U(t,p)$ at $t = T$ is larger than $\hat{b}(p)$. For this, let $\{\nu^k\}$ be the maximizing sequence in the cone $\tilde{K}$ attaining $\hat{b}(p)$, i.e., such that $b(pe^{-\nu^k})e^{-\delta(\nu^k)}$ converges to $\hat{b}(p)$ as k goes to infinity. Then, using (for fixed $t < T$) constant deterministic controls $\nu^k/(T-t)$ in (6.25), we get

$$U(t,p) \geq E\big[b(P^0(T)e^{-\nu^k})e^{-\delta(\nu^k)}e^{-r(T-t)} \;\big|\; P^0(t) = p\big],$$

hence

$$\varliminf_{t\to T} U(t,p) \geq b(pe^{-\nu^k})e^{-\delta(\nu^k)}$$

and letting k to infinity, we finish the proof.

Here is a sketch of a PDE proof for part (a) in the proof above: Let V be the solution to (6.27)-(6.28). For a given $\nu \in \tilde{K}$, consider the function $W_\nu = (pV_p)^*\nu + \delta(\nu)V$, where V_p is the vector of partial derivatives of V with respect to p_i, $i = 1, \ldots, d$. By Theorem 13.1 in Rockafellar (1970), to prove that portfolio π of (6.29) takes values in K, it is sufficient to prove that W_ν is nonnegative, for all $\nu \in \tilde{K}$. It is not difficult to see (assuming enough smoothness) that W_ν solves PDE (6.27), too. Moreover, it is also straightforward to check that $W_\nu(p,T) \geq 0$. So, by the maximum principle, $W_\nu \geq 0$ everywhere.

6.14 Examples: We restrict ourselves to the case of only one stock, $d = 1$, and the constraints of the type

$$K = [-l, u], \tag{6.31}$$

with $0 \leq l, u \leq +\infty$, with the understanding that the interval K is open to the right (left) if $u = +\infty$ (respectively, if $l = +\infty$). It is straightforward to see that

$$\delta(\nu) = l\nu^+ + u\nu^-, \tag{3.1}$$

and $\tilde{K} = \mathbb{R}$ if both l and u are finite. In general,

$$\tilde{K} = \{x \in \mathbb{R} \;:\; x \geq 0 \text{ if } u = +\infty,\; x \leq 0 \text{ if } l = +\infty\}. \tag{6.32}$$

For the European call $b(p) = (p-k)^+$, one easily gets that $\hat{b}(p) \equiv \infty$, if $u < 1$, $\hat{b}(p) = p$ if $u = 1$ (no-borrowing) and $\hat{b}(p) = b(p)$ if $u = \infty$ (short-selling constraints don't matter). For $1 < u < \infty$ we have (by ordinary calculus)

$$\hat{b}(p) = \begin{cases} p - k & ;\ p \geq \frac{ku}{u-1} \\ \frac{k}{u-1}\left(\frac{(u-1)p}{ku}\right)^u & ;\ p < \frac{ku}{u-1} \end{cases}. \tag{6.33}$$

For the European put $b(p) = (k-p)^+$, one gets $\hat{b} = b$ if $l = \infty$ (borrowing constraints don't matter), $\hat{b} \equiv k$ if $l = 0$ (no short-selling), and otherwise

$$\hat{b}(p) = \begin{cases} k - p & ;\ p \leq \frac{kl}{l+1} \\ \frac{k}{l+1}\left(\frac{ku}{(l+1)p}\right)^l & ;\ p > \frac{kl}{l+1} \end{cases}. \tag{6.34}$$

We finish this section by a theorem on connections with arbitrage. First, let us remark that one can, analogously to $h(0)$, define the hedging price $\tilde{h}(0)$ for the buyer of the claim B, as the maximal amount of money the buyer can borrow at $t = 0$ and have more than $-B$ at time $t = T$. We have

Theorem 6.15: (Karatzas & Kou (1994)) *There is no arbitrage with constrained portfolios if the price $B(0)$ of B satisfies*

$$\tilde{h}(0) < B(0) < h(0). \tag{6.35}$$

Conversely, if $B(0)$ is strictly larger than $h(0)$ or strictly smaller then $\tilde{h}(0)$, then there is arbitrage.

Proof: First the converse: suppose, for example, that $B(0) > h(0)$. Then one can sell the claim for $B(0)$, put $B(0) - h(0) > 0$ in the bank, and replicate B with $h(0)$ - arbitrage!

Suppose now that (6.35) is satisfied, and suppose, for example, that arbitrage can be obtained by selling the claim for $B(0)$ and investing $x < B(0)$ in the market, using the policy (π, c) such that $X^{x,\pi,c}(T) \geq B$, a.s.. But this is in contradiction with $x < B(0) < h(0)$.

7. NONLINEAR PORTFOLIO DYNAMICS - LARGE INVESTOR.

We consider an investor whose investment policy influences the behavior of the prices $P_0, \{P_i\}_{1 \leq i \leq d}$ of the financial instruments. More precisely, these prices evolve according to the equations

$$dP_0(t) = P_0(t)[r(t) + f_0(\pi_t)]dt \ , \qquad P_0(0) = 1 \tag{7.1}$$

$$dP_i(t) = P_i(t)[(b_i(t)+f_i(\pi_t))dt+\sum_{j=1}^{d}\sigma_{ij}(t)dW^{(j)}(t)] \ , \ \ P_i(0) = p_i \in (0,\infty) \tag{7.2}$$

for $i = 1, \ldots, d$. Here $f_i : \mathbb{R}^d \mapsto \mathbb{R}$, $i = 0, \ldots, d$ are some given functions that describe the effect of the investor's strategy on the prices.

Let us remark that the interpretation of policy-dependent prices is not the only possible one; one could simply start with the economy in which the wealth process of the investor is the one whose dynamics are given in (7.4) below, and forget about the prices.

For a given $\{\mathcal{F}_t\}-$progressively measurable process $\nu(\cdot)$ with values in $\mathbb{R}^d$ and $\mu(\cdot)$ with values in $\mathbb{R}$, we introduce the discount process

$$\beta_\mu(t) := \beta_\mu(0,t), \quad \beta_\mu(u,t) := \exp\{-\int_u^t \mu(s)ds\}. \tag{7.3}$$

The wealth process $X(t)$ satisfies the stochastic differential equation

$$dX(t) = X(t)g(t,\pi_t)dt + X(t)\pi^*(t)\sigma(t)dW(t) - dc(t) \ , \quad X(0) = x > 0, \tag{7.4}$$

where

$$g(t,\pi) := r(t) + f_0(\pi) + \sum_{i=1}^{d} \pi_i[b_i(t) + f_i(\pi) - r(t) - f_0(\pi)] \ . \tag{7.5}$$

We will restrict ourselves by imposing the following assumption:

7.1 Assumption: The function $g(t,\cdot)$ is concave for all $t \in [0,T]$, and uniformly (with respect to t) Lipschitz:

$$|g(t,x) - g(t,y)| \leq k\|x-y\|, \quad \forall \ t \in [0,T]; \ x,y \in \mathbb{R}^d,$$

for some $0 < k < \infty$.

In analogy with the case of constraints we define the convex conjugate function $\tilde{g}$ of g by

$$\tilde{g}(t,\nu) := \sup_{\pi \in \mathbb{R}^d} \{g(t,\pi) + \pi^*\nu\}, \tag{7.5}$$

on its *effective domain* $\mathcal{D}_t := \{\nu : \tilde{g}(\nu,t) < \infty\}$. Introduce also the class $\mathcal{D}$ of processes $\nu(t)$ taking values in $\mathcal{D}_t$, for all t. We shall also assume that

$$\mathcal{D} \ \text{ is not empty;} \tag{7.6}$$

(7.7) The function $\tilde{g}(t,\cdot)$ is bounded on its effective domain, uniformly in t .

Denote

$$\gamma_\nu(t,u) := \exp\{-\int_t^u \tilde{g}(s,\nu_s)ds\}, \quad \gamma_\nu(t) := \gamma_\nu(0,t),$$

$$dZ_\nu(t) := -\sigma^{-1}(t)\nu(t)Z_\nu(t)dW(t), \ Z_\nu(0) = 1, \quad H_\nu(t) := Z_\nu(t)\gamma_\nu(t) \ . \tag{7.8}$$

For every $\nu \in \mathcal{D}$ we have (by Ito's rule)

$$\begin{aligned} H_\nu(t)X(t) + \int_0^t H_\nu(s)\left[X(s)(\tilde{g}(s,\nu_s) - g(s,\pi_s) - \pi^*(s)\nu(s))ds + dc(s)\right] \\ = x + \int_0^t H_\nu(s)X(s)\left[\pi^*(s)\sigma(s) + \sigma^{-1}(s)\nu(s)\right] dW(s). \end{aligned} \tag{7.9}$$

In particular, the process on the right-hand side is a nonnegative local martingale, hence a supermartingale. Therefore we get the following *necessary condition for π to be admissible*:

$$\sup_{\nu \in \mathcal{D}} E\left[H_\nu(T)X(T) + \int_0^T H_\nu(s)X(s)\{\tilde{g}(s,\nu_s) - g(s,\pi_s) - \pi^*(s)\nu(s)\}ds\right] \leq x \ . \tag{7.10}$$

7.2 Remark: The supermartingale property excludes arbitrage opportunities from this market: if $x = 0$, then necessarily $X(t) = 0$, $\forall \ 0 \leq t \leq T$, a.s.. If $f_i \equiv 0$, $i = 0,\ldots,d$, then $\mathcal{D}$ consists of only one process $\hat{\nu}(\cdot)$, given by

$\hat{\nu}_i(t) = r(t) - b_i(t)$, $i = 1, \ldots, d$, i.e., we are in the standard Black-Scholes-Merton complete market model with the unique "equivalent martingale risk neutral measure" $\mathbf{P}^{\hat{\nu}}$, defined below.

Next, for a given $\nu \in \mathcal{D}$, introduce the process

$$W_\nu(t) := W(t) - \int_0^t \sigma^{-1}(s)\nu(s)ds \ , \tag{7.11}$$

as well as the measure

$$\mathbf{P}^\nu(A) := E[Z_\nu(T)1_A] = E^\nu[1_A], \quad A \in \mathcal{F}_T \ . \tag{7.12}$$

Notice that Assumption 7.1 implies that the sets $\mathcal{D}_t$ are uniformly bounded. Therefore, if $\nu \in \mathcal{D}$, then Z_ν *is a martingale.* Thus, for every $\nu \in \mathcal{D}$, the measure $\mathbf{P}^\nu$ is a probability measure and the process $W_\nu(\cdot)$ is a $\mathbf{P}^\nu-$Brownian motion, by Girsanov theorem.

Given a contingent claim B, consider, for every stopping time τ, the $\mathcal{F}_\tau$-measurable random variable

$$V(\tau) := ess \sup_{\nu \in \mathcal{D}} E^\nu[B\gamma_\nu(\tau, T)|\mathcal{F}_\tau]. \tag{7.13}$$

The proof of the following theorem is similar to the corresponding theorem in the case of constraints.

7.3 Theorem: (El-Karoui, Peng & Quenez (1994)) *For an arbitrary contingent claim B, we have $h(0) = V(0)$. Furthermore, there exists a pair $(\hat{\pi}, \hat{c}) \in \mathcal{A}_0(V(0))$ such that $X^{V(0),\hat{\pi},\hat{c}}(\cdot) = V(\cdot)$.*

The theorem gives the minimal hedging price for a claim B; in fact, it is easy to see (using the same supermartingale argument) that the process $V(\cdot)$ is the minimal wealth process that hedges B. There remains the question of whether consumption is necessary. We show that, in fact, $\hat{c}(\cdot) \equiv 0$.

7.4 Theorem: *Every contingent claim B is attainable, namely the process $\hat{c}(\cdot)$ from Theorem 7.3 is a zero-process.*

Proof : Let $\{\nu_n; n \in \mathcal{N}\}$ be a maximizing sequence for achieving $V(0)$, i.e., $\lim_{n\to\infty} E^{\nu_n} B\gamma_{\nu_n}(T) = V(0)$. The necessary condition (7.11) (with $X(\cdot) = V(\cdot)$) implies $\lim_{n\to\infty} E^{\nu_n} \int_0^T \gamma_{\nu_n}(t)d\hat{c}(t) = 0$ and, since the processes $\gamma_{\nu_n}(\cdot)$ are bounded away from zero (uniformly in n), $\lim_{n\to\infty} E[Z_{\nu_n}(T)\hat{c}(T)] = 0$. Using weak compactness arguments as in Cvitanić & Karatzas (1993, Theorem 9.1) we can show that there exists $\nu \in \mathcal{D}$ such that $\lim_{n\to\infty} E[Z_{\nu_n}\hat{c}(T)] = E[Z_\nu(T)\hat{c}(T)] = 0$ (along a subsequence). It follows that $\hat{c}(\cdot) \equiv 0$. ◇

8. GENERAL NONLINEARITIES AND FORWARD-BACKWARD SDE's.

We denote by π the vector of amounts of money invested in stocks and change the model (2.1), (2.2) for the asset prices to

$$dP_0(t) = P_0(t) r(t, X_t, \pi_t) dt, \qquad P_0(0) = 1 \tag{8.1}$$

(8.2)

$$dP_i(t) = b_i(t, P_t, X_t, \pi_t) dt + \sum_{j=1}^{d} \sigma_{ij}(t, P_t, X_t, \pi_t) dW^{(j)}(t) \ , \quad P_i(0) = p_i \in (0, \infty)$$

for $i = 1, \ldots, d$. We require that the wealth replicates, at time T, the contingent claim with value $l(P(T))$, for a given function l, the assumptions on which are specified below. The wealth equation becomes

$$dX(t) = \hat{b}(t, P_t, X_t, \pi_t) dt + \hat{\sigma}(t, P_t, X_t, \pi_t) dW(t); \ X(T) = l(P(T)), \tag{8.3}$$

where

$$\begin{aligned} \hat{b}(t, p, x, \pi) &= (x - \sum_{i=1}^{d} \pi_i) r(t, x, \pi) + \sum_{i=1}^{d} \frac{\pi_i}{p_i} b_i(t, p, x, \pi); \\ \hat{\sigma}_j(t, p, x, \pi) &= \sum_{i=1}^{d} \frac{\pi_i}{p_i} \sigma_{ij}(t, p, x, \pi), \quad j = 1, \cdots, d, \end{aligned} \tag{8.4}$$

for $(t, p, x, \pi) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$. The system of SDE's (8.2) and (8.3) is called a *Forward-Backward* SDE; the forward component is the price process, having been assigned an initial value, whereas the backward component is the wealth process, having been assigned the terminal value $X(T) = l(P(T))$. An existence theory for such equations has been developed by Ma, Protter and Yong (1994).

The main differences compared to the model of Sections 2 and 7 are: (i) more general, nonconvex nonlinearities, including the volatility term; (ii) the contingent claim value $l(P(T))$ is not given in advance, but depends on the portfolio strategy and wealth of the investor through P; (iii) Markovian structure of the model.

In this section we shall use the following notations throughout: we denote $\mathbb{R}_+^d = \{(x_1, \cdots, x_d) \in \mathbb{R}^d | x_i > 0, i = 1, \cdots, d\}$; the inner product in $\mathbb{R}^d$ by $\langle \cdot, \cdot \rangle$; the norm in $\mathbb{R}^d$ by $|\cdot|$ and that of $\mathbb{R}^{d \times d}$, the space of all $d \times d$ matrices, by $\|\cdot\|$ and the transpose of a matrix $A \in \mathbb{R}^{d \times d}$ (resp. a vector $x \in \mathbb{R}^d$) by A^T (resp. x^T). We also denote $\underline{\mathbf{1}}$ to be the vector $\underline{\mathbf{1}} := (1, \cdots, 1) \in \mathbb{R}^d$, and define a (diagonal) matrix-valued function $\Lambda : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ by

$$\Lambda(x) := \begin{bmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_d \end{bmatrix}, \qquad x = (x_1, \cdots, x_d) \in \mathbb{R}^d. \tag{8.5}$$

It is obvious that $\|\Lambda(x)\| = |x|$ for any $x \in \mathbb{R}^d$, and whenever $x \notin \partial\mathbb{R}^d_+$, $\Lambda(x)$ is invertible and $[\Lambda(x)]^{-1}$ is of the same form as $\Lambda(x)$ with $x_1, \cdots, x_d$ being replaced by $x_1^{-1}, \cdots, x_d^{-1}$. We can then rewrite functions $\widehat{b}$ and $\widehat{\sigma}$ in (8.4) as

$$\begin{aligned} &\widehat{b}(t,p,x,\pi) = xr(t,x,\pi) + \langle \pi, b^1(t,p,x,\pi) - r(t,x,\pi)\underline{\mathbf{1}} \rangle; \\ &\widehat{\sigma}(t,p,x,\pi) = \langle \pi, \sigma^1(t,p,x,\pi) \rangle, \end{aligned} \tag{8.6}$$

where

$$\begin{aligned} &b^1(t,p,x,\pi) := [\Lambda(p)]^{-1} b(t,p,x,\pi) = \Big(\frac{b_1}{p_1}, \cdots, \frac{b_d}{p_d}\Big)(t,p,x,\pi); \\ &\sigma^1(t,p,x,\pi) := [\Lambda(p)]^{-1}\sigma(t,p,x,\pi) = \Big\{\frac{\sigma_{ij}}{p_i}\Big\}_{i,j=1}^d (t,p,x,\pi). \end{aligned} \tag{8.7}$$

To be consistent with the standard model, we henceforth call b^1 the *appreciation rate* and σ^1 the *volatility matrix* of the stock market. We restrict ourselves to the portfolios for which $E\int_0^T |\pi(t)|^2 dt < \infty$ and $X(t) \geq 0$, $\forall t \in [0,T]$, a.s.. Let us impose the following *Standing Assumptions*:

(A1) The functions b, $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ and $l : \mathbb{R}^d \mapsto \mathbb{R}$ are twice continuously differentiable, such that $b(t,0,x,\pi) = \sigma(t,0,x,\pi) = 0$, for all $(t,x,\pi) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$. The functions b^1 and σ^1, together with their first order partial derivatives in p, x and π are bounded, uniformly in (t,p,x,π). Further, we assume that partial derivatives of b^1 and σ^1 in p satisfy

$$\sup_{(t,p,x,\pi)} \Big\{ \Big| p_k \frac{\partial b^1}{\partial p_k} \Big|, \Big| p_k \frac{\partial \sigma^1_{ij}}{\partial p_k} \Big| \Big\} < \infty, \qquad i,j,k = 1, \cdots, d. \tag{8.8}$$

(A2) The function σ satisfies $\sigma\sigma^T(t,p,x,\pi) > 0$, for all (t,p,x,π) with $p \notin \partial\mathbb{R}^d_+$; and there exists a positive constant $\mu > 0$, such that

$$a^1(t,p,x,\pi) \geq \mu I, \qquad \text{for all} \quad (p,t,x,\pi), \tag{8.9}$$

where $a^1 = \sigma^1(\sigma^1)^T$.

(A3) The function r is twice continuously differentiable and such that the following conditions are satisfied:

(a) For $(t,x,\pi) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d$, $0 < r(t,x,\pi) \leq K$, for some constant $K > 0$.

(b) The partial derivatives of r in x and π, denoted by a generic function ψ, satisfy

$$\overline{\lim_{|x|,|\pi|\to\infty,}} (|x| + |\pi|)^2 |\psi(t,x,\pi)| < \infty. \tag{8.10}$$

Either

(A4.a) The function l is bounded, C^2 and nonnegative; Its partial derivatives up to second order are all bounded;

or

(A4.b) The function l is nonnegative and $\lim_{|p|\to\infty} l(p) = \infty$; moreover, l has bounded, continuous partial derivatives up to third order, and there exist constants $K, M > 0$ such that

$$\begin{cases} |\Lambda(p) l_p(p)| \le K(1 + l(p)); \\ \sup\limits_{p\in\mathbb{R}^d_+} \|\Lambda^2(p) l_{pp}\| = M < \infty. \end{cases}$$

(A5) The partial derivatives of σ^1 in x and π satisfy

$$\sup_{(t,p,x,\pi)} \left\{ \left| x \frac{\partial \sigma^1_{ij}}{\partial x} \right| + \left| x \frac{\partial \sigma^1_{ij}}{\partial \pi_k} \right| \right\} < \infty, \qquad i, j, k = 1, \cdots, d. \tag{8.11}$$

Remark 8.1. The conditions here are quite restrictive, which is largely due to the generality of the setting of this section. The PDE method described below often works even if the assumptions are far from being satisfied. In particular, it works in the case of the model used in the previous section, if we restrict ourselves to the Markovian setting and to standard European options. We note that the assumptions (A1) and (A2) obviously contain those cases in which $b(t, p, x, \pi) = \Lambda(p) b_1(t, x, \pi)$ and $\sigma(t, p, x, \pi) = \Lambda(p)\sigma_1(t, x, \pi)$ where b_1 and σ_1 are bounded, continuously differentiable functions with bounded first order partial derivatives; and $\sigma_1 \sigma_1^T$ is positive definite and bounded away from zero, as is the case in the standard model. The second conditions on l restricts it to have at most quadratic growth. An example of a function σ satisfying (A1), (A2) and (A4.b) could be $\sigma(t, p, x, \pi) = p(\sigma(t) + \arctan(x^2 + |\pi|^2))$ with $\sigma(\cdot)$ satisfying (A2).

All the results below are proved in Cvitanić & Ma (1996):

Lemma 8.2. *Suppose that (A1), (A2) hold. Then for any portfolio π and initial wealth x, the price process P satisfies: $P_i(t) > 0$, $i = 1, \cdots, d$ for all $t \in [0, T]$, almost surely, provided the initial prices $p_1, \cdots, p_d$ are positive.*

The Four Step Scheme of Ma, Protter & Yong (1994), in our setting, consists of the following (and consists of three steps only):

Step 1: Solve the Black-Scholes type (but nonlinear) PDE

$$\begin{cases} 0 = \theta_t + \frac{1}{2}\mathrm{tr}\{\sigma\sigma^T(t, p, \theta, \Lambda(p)\theta_p)\theta_{pp}\} + (\langle p, \theta_p \rangle - \theta) r(t, \theta, \Lambda(p)\theta_p), \\ \theta(T, p) = l(p), \qquad p \in \mathbb{R}^d_+. \end{cases} \tag{8.12}$$

Step 2: Setting

$$\begin{cases} \tilde{b}(t, p) = b(t, p, \theta(t, p), \Lambda(p)\theta_p(t, p)) \\ \tilde{\sigma}(t, p) = \sigma(t, p, \theta(t, p), \Lambda(p)\theta_p(t, p)), \end{cases} \tag{8.13}$$

solve the forward SDE

$$P(t) = p + \int_0^t \tilde{b}(s, P(s))ds + \int_0^t \tilde{\sigma}(s, P(s))dW(s). \tag{8.14}$$

Step 3: Set

$$\begin{cases} X(t) = \theta(t, P(t)) \\ \pi(t) = \Lambda(P(t))\theta_p(t, P(t)), \end{cases} \tag{8.15}$$

Theorem 8.3. *Suppose that the standing assumptions (A1)—(A3), (A4.b) and (A5) hold. Then the PDE (8.12) and the SDE (8.14) admit unique solutions. Moreover, for any given $p \in \mathbb{R}_+^d$, the FBSDE (8.2), (8.3) admits a unique adapted solution (P, X, π), given by (8.15) with θ being the solution of (8.12).*

The theorem implies that the initial value $X(0)$ of the backward process provides an upper bound for the minimal hedging price $h(0)$ of the contingent claim $B = l(P(T))$, since the claim can indeed be hedged starting with $X(0)$. The next result shows that $X(0) = h(0)$, and that π given by (8.15) is the least expensive hedging portfolio.

Theorem 8.4. *(Comparison Theorem)*: *Suppose that (A1) - (A5) hold. Let initial prices $p \in \mathbb{R}_+^d$ be given, and let π be any admissible portfolio such that the corresponding price/wealth process (P, Y) satisfies $Y(T) \geq l(P(T))$. Then $Y(\cdot) \geq \theta(\cdot, P(\cdot))$, where θ is the solution to (8.12). In particular, $Y(0) \geq \theta(0, p) = X(0)$, where X is the solution to the FBSDE (8.2), (8.3), starting from $p \in \mathbb{R}_+^d$, constructed by the Four-Step Scheme.*

We conclude by examples in which a model like the one of this section would be appropriate:

(i) *Large investor.* If the investor keeps too much capital in the bank, the government (or the market) decides to decrease the bank interest rate. For example, we can assume that $r(t, x, \pi)$ is a decreasing function of $x - \pi$, for $x - \pi$ large.

(ii) Borrowing rate could be decreasing in wealth.

(iii) *Several agents - equilibrium model.* In Platen & Schweizer (1994), an SDE for the stock price is obtained from equilibrium considerations; both its drift and volatility coefficients depend on the hedging strategy of the agents in the market in a rather complex fashion. As the authors mention, " it is not clear at all how one should compute option prices in an economy where agents' strategies affect the underlying stock price process". Our results provide the price that would enable the seller to hedge against all the risk, i.e., the upper bound for the price.

9. EXAMPLE: HEDGING CLAIMS WITH HIGHER INTEREST RATE FOR BORROWING

We have studied so far a model in which one is allowed to borrow money, at an interest rate $R(\cdot)$ equal to the bank rate $r(\cdot)$. In this section we consider the more general case of a financial market $\mathcal{M}^*$ in which $R(\cdot) \geq r(\cdot)$, without constraints on portfolio choice. We assume that the progressively measurable process $R(\cdot)$ is also bounded.

In this market $\mathcal{M}^*$ it is not reasonable to borrow money and to invest money in the bank at the same time. Therefore, we restrict ourselves to policies for which the relative amount borrowed at time t is equal to $\left(1-\sum_{i=1}^d \pi_i(t)\right)^-$. Then, the wealth process $X = X^{x,\pi,c}$ corresponding to initial capital $x > 0$ and portfolio/cumulative consumption pair (π, c), satisfies

$$\begin{aligned} dX(t) =& r(t)X(t)dt - dc(t) \\ &+ X(t)\left[\pi^*(t)\sigma(t)dW_0(t) - (R(t)-r(t))\left(1-\sum_{i=1}^d \pi_i(t)\right)^- dt\right]. \end{aligned} \tag{9.1}$$

In the notation of Section 7, we get $\tilde{g}(\nu(t)) = r(t) - \nu_1(t)$ for $\nu \in \mathcal{D}$, where

$$\begin{aligned} \mathcal{D} := \{\nu;\ \nu \text{ progressively measurable, } \mathcal{R}^d - \text{valued process with} \\ r - R \le \nu_1 = \ldots = \nu_d \le 0,\quad \ell \otimes \mathbf{P} - a.e.\}. \end{aligned} \tag{9.2}$$

We also have

$$\begin{aligned} \tilde{g}(\nu(t)) - g(t,\pi(t)) - \pi^*(t)\nu(t) =& [R(t) - r(t) + \nu_1(t)]\left(1-\sum_{i=1}^d \pi_i(t)\right)^- \\ & - \nu_1(t)\left(1-\sum_{i=1}^d \pi_i(t)\right)^+, \end{aligned} \tag{9.3}$$

for $0 \le t \le T$. It can be shown, in analogy to the case of constraints, that the optimal dual process $\hat{\lambda}(\cdot) \in \mathcal{D}$ can be taken as the one that attains zero in (9.3), namely as

$$\hat{\lambda}(t) = \hat{\lambda}_1(t)\mathbf{1},\quad \hat{\lambda}_1(t) := [r(t) - R(t)]\ 1_{\{\sum_{i=1}^d \hat{\pi}_i(t) > 1\}}. \qquad \diamond \tag{9.4}$$

Consider the case $d = 1$, $B = \varphi(P_1(T))$ with $\varphi : \mathbb{R}_+ \to [0,\infty)$, and with constant $R > r$, If $p\varphi'(p) \ge \varphi(p)$ holds everywhere on $\mathbb{R}_+$ and strictly on a set of positive measure, then we may take $\hat{\lambda} \equiv r - R$, and the Black-Scholes formulae, remain valid if we replace in them r by R. This follows as in the following example.

9.1 Example: Let us consider the case of constant coefficients $r, R, \{\sigma_{ij}\} = \sigma$. Then the vector $P(t) = (P_1(t), \ldots, P_d(t))$ of stock price processes satisfies the equations

$$\begin{aligned} dP_i(t) &= P_i(t)[b_i(t)dt + \sum_{i=1}^d \sigma_{ij} dW^{(j)}(t)] \\ &= P_i(t)[(r - \nu_1(t))dt + \sum_{i=1}^d \sigma_{ij} dW_\nu^{(j)}(t)],\quad 1 \le i \le d, \end{aligned} \tag{9.5}$$

for every $\nu \in \mathcal{D}$. Consider now a contingent claim of the form $B = \varphi(P(T))$, for a given continuous function $\varphi : \mathbb{R}_+^d \to [0, \infty)$ that satisfies a polynomial growth condition, as well as the *value function*

$$Q(t,p) := \sup_{\nu \in \mathcal{D}} E^\nu[\varphi(P(T))e^{-\int_t^T (r-\nu_1(s))ds} | P(t) = p] \tag{9.6}$$

on $[0,T] \times \mathbb{R}_+^d$. Clearly, the processes $\hat{X}, V$ are given as

$$\hat{X}(t) = Q(t, P(t)), \quad V(t) = e^{-rt}\hat{X}(t) \; ; \quad 0 \le t \le T,$$

where Q solves the semilinear parabolic partial differential equation of Hamilton-Jacobi-Bellman (HJB) type

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j a_{ij} p_i p_j \frac{\partial^2 Q}{\partial p_i \partial p_j} + \max_{r-R \le \nu_1 \le 0} \left[(r - \nu_1)\{\sum_i p_i \frac{\partial Q}{\partial p_i} - Q\} \right] = 0, \tag{9.7}$$

for $0 \le t < T, \; p \in \mathbb{R}_+^d$,

$$Q(T,p) = \varphi(p); \; p \in \mathbb{R}_+^d$$

associated with the control problem of (9.6) and the dynamics (9.5) (cf. Ladyženskaja, Solonnikov & Ural'tseva (1968) for the basic theory of such equations, and Fleming & Rishel (1975) for the connections with stochastic control). Clearly, the maximization in (9.7) is achieved by $\nu_1^* = -(R-r).1_{\{\sum_i p_i \frac{\partial Q}{\partial p_i} \ge Q\}}$; the portfolio $\hat{\pi}(\cdot)$ and the process $\hat{\lambda}_1(\cdot)$ are then given, respectively, by

$$\hat{\pi}_i(t) = \frac{P_i(t) \cdot \frac{\partial}{\partial p_i} Q(t, P(t))}{Q(t, P(t))}, \qquad i = 1, \ldots, d \tag{9.8}$$

and

$$\hat{\lambda}_1(t) = (r - R)1_{\{\sum_i \hat{\pi}_i(t) \ge 1\}}. \tag{9.9}$$

Suppose now that the function φ satisfies $\sum_i p_i \frac{\partial \varphi(p)}{\partial p_i} \ge \varphi(p), \; \forall \; p \in \mathbb{R}_+^d$. Then the solution Q also satisfies this inequality:

$$\sum_i p_i \frac{\partial Q(t,p)}{\partial p_i} \ge Q(t,p), \qquad 0 \le t \le T \tag{9.10}$$

for all $p \in \mathbb{R}_+^d$, and is actually given explicitly as

(9.11)

$$\begin{aligned} Q(t,p) &= E^{(r-R)\mathbf{1}}[e^{-R(T-t)}\varphi(P(T)) | P(t) = p] \\ &= \begin{cases} e^{-R(T-t)} \int_{R^d} \varphi(h(T-t, p, \sigma z; R))(2\pi t)^{-d/2} e^{-\frac{\|z\|^2}{2t}} dz & ; \quad t < T, p > 0 \\ \varphi(p) & ; \quad t = T, p > 0 \end{cases} \end{aligned}$$

in the notation of previous sections. This is because, in this case, the PDE (9.7) becomes the Black-Scholes PDE

$$\frac{\partial Q}{\partial t} + \frac{1}{2}\sum_i \sum_j p_i p_j a_{ij} \frac{\partial^2 Q}{\partial p_i \partial p_j} + R\left(\sum_i p_i \frac{\partial Q}{\partial p_i} - Q\right) = 0; t < T, p > 0$$

$$Q(T,p) = \varphi(p); p > 0$$

In this case the portfolio $\hat{\pi}(\cdot)$ always borrows: $\sum_{i=1}^d \hat{\pi}_i(t) \geq 1,\ 0 \leq t \leq T$ (a.s.), and thus $\hat{\lambda}_1(t) = r - R,\ 0 \leq t \leq T$.

10. UTILITY FUNCTIONS.

A function $U : (0,\infty) \to \mathbb{R}$ will be called a *utility function* if it is strictly increasing, strictly concave, of class C^1, and satisfies

(10.1) $$U'(0+) := \lim_{x\downarrow 0} U'(x) = \infty\ ,\quad U'(\infty) := \lim_{x\to\infty} U'(x) = 0\ .$$

We shall denote by I the (continuous, strictly decreasing) inverse of the function U'; this function maps $(0,\infty)$ onto itself, and satisfies $I(0+) = \infty, I(\infty) = 0$. We also introduce the Legendre-Fenchel transform

(10.2) $$\tilde{U}(y) := \max_{x>0}[U(x) - xy] = U(I(y)) - yI(y),\quad 0 < y < \infty$$

of $-U(-x)$; this function $\tilde{U}$ is strictly decreasing and strictly convex, and satisfies

(10.3) $$\tilde{U}'(y) = -I(y),\quad 0 < y < \infty,$$

(10.4) $$U(x) = \min_{y>0}[\tilde{U}(y) + xy] = \tilde{U}(U'(x)) + xU'(x)\ ,\quad 0 < x < \infty\ .$$

The useful inequalities

(10.5) $$U(I(y)) \geq U(x) + y[I(y) - x]$$

(10.6) $$\tilde{U}(U'(x)) + x[U'(x) - y] \leq \tilde{U}(y)\ ,$$

valid for all $x > 0, y > 0$, are direct consequences of (10.2), (10.4). It is also easy to check that

(10.7) $$\tilde{U}(\infty) = U(0+),\quad \tilde{U}(0+) = U(\infty)$$

hold; cf. KLSX (1991), Lemma 4.2.

10.1 Remark: We shall have occasion, in the sequel, to impose the following conditions on our utility functions:

(10.8) $$c \mapsto cU'(c)\ \text{ is nondecreasing on }\ (0,\infty)\ ,$$

(10.9)
for some $\alpha \,\epsilon\, (0,1), \gamma \,\epsilon\, (1,\infty)$ we have : $\alpha U'(x) \geq U'(\gamma x), \quad \forall\, x \,\epsilon\, (0,\infty)$.

10.2 Remark: Condition (4.8) is equivalent to

$$(10.8)' \qquad y \mapsto yI(y) \text{ is nonincreasing on } (0,\infty) ,$$

and implies that

$$(10.8)'' \qquad x \mapsto \tilde{U}(e^x) \text{ is convex on } \mathcal{R} .$$

(If U is of class C^2, then condition (10.8) amounts to the statement that $\frac{-cU''(c)}{U'(c)}$, the so-called "Arrow-Pratt measure of relative risk - aversion", does not exceed 1.)

Similarly, condition (10.9) is equivalent to having

$$(10.9)' \qquad I(\alpha y) \leq \gamma I(y), \quad \forall\, y \,\epsilon\, (0,\infty) \quad \text{for some } \alpha \,\epsilon\, (0,1), \quad \gamma > 1 .$$

Iterating $(10.9)'$, we obtain the apparently stronger statement

$$(10.9)'' \quad \forall\, \alpha \,\epsilon\, (0,1),\ \exists\, \gamma \,\epsilon\, (1,\infty) \text{ such that } \quad I(\alpha y) \leq \gamma I(y), \quad \forall\, y \,\epsilon\, (0,\infty) .$$

11. PORTFOLIO OPTIMIZATION PROBLEM.

In this section we consider the optimization problem of maximizing utility from terminal wealth for our investor, i.e., we want to maximize

$$(11.1) \qquad J(x;\pi) := EU(X^{x,\pi}(T)) ,$$

over the constrained portfolios $\pi \in \mathcal{A}'(x)$, provided that the expectation is well-defined. More precisely, we have

11.1 Definition: The *utility maximization problem* is to maximize the expression of (11.1) over the class $\mathcal{A}'(x)$ of processes $\pi \in \mathcal{A}'(x)$ that satisfy

$$(11.2) \qquad EU^-(X^{x,\pi}(T)) < \infty.$$

(x^- denotes the negative part of x : $x^- = max\{-x, 0\}$.) The value function of this problem will be denoted by

$$(11.3) \qquad V(x) := \sup_{\pi \in \mathcal{A}'_0(x)} J(x;\pi) , \quad x \in (0,\infty) .$$

11.2 Assumption: $V(x) < \infty, \quad \forall\, x \in (0,\infty)$.

It is fairly straightforward that the function $V(\cdot)$ is increasing and concave on $(0,\infty)$.

11.3 Remark: It can be checked that the Assumption 11.2 is satisfied if the function U is nonnegative and satisfies the growth condition

$$(11.7) \qquad 0 \leq U(x) \leq \kappa(1+x^{\alpha}) \; ; \quad \forall \; (x) \in (0,\infty)$$

for some constants $\kappa \in (0,\infty)$ and $\alpha \in (0,1)$ - cf. KLSX (1991) for details.

Denote

$$H_\nu(t) = \gamma_\nu(t) Z_\nu(t),$$

in the notation of (5.7) and (5.8).

11.4 Definition: We introduce the function

$$(11.9) \qquad \mathcal{X}_\nu(y) := E\Big[H_\nu(T) I(yH_\nu(T))\Big] \; , \quad 0 < y < \infty$$

and consider the subclass D' of $\mathcal{H}$ (in the notation of Section 5) given by

$$(11.10) \qquad \mathcal{D}' := \{\nu \in \mathcal{H}; \;\; \mathcal{X}_\nu(y) < \infty, \quad \forall \; y \in (0,\infty)\} \; .$$

For every $\nu \in \mathcal{D}'$, the function $\mathcal{X}_\nu(\cdot)$ of (11.9) is continuous and strictly decreasing, with $\mathcal{X}_\nu(0+) = \infty$ and $\mathcal{X}_\nu(\infty) = 0$; we denote its inverse by $\mathcal{Y}_\nu(\cdot)$.

11.5 Remark: Suppose that $U(\cdot)$ satisfies condition (11.9). It is then easy to see, using $(11.9)''$, that $\mathcal{X}_\nu(y) < \infty$ for some $y \in (0,\infty)$ implies: $\nu \in \mathcal{D}'$.

Next, we prove a crucial lemma, which provides sufficient conditions for optimality in the problem of (11.1). The duality approach of the lemma and subsequent analysis was implicitly used in Pliska (1986), Karatzas, Lehoczky & Shreve (1987), Cox & Huang (1989) in the case of no constraints, and explicitly in He & Pearson (1991), Karatzas et al. (1991), Xu & Shreve (1992) for special types of constraints.

11.6 Lemma: For any given $x > 0, \; y > 0$ and $\pi \in \mathcal{A}'(x)$, we have

$$(11.11) \qquad EU(X^{x,\pi}(T)) \leq E\tilde{U}(yH_\nu(T)) + yx, \quad \forall \; \nu \in \mathcal{H}.$$

In particular, if $\hat{\pi} \in \mathcal{A}'(x)$ is such that *equality* holds in (11.11), for some $\lambda \in \mathcal{H}$ and $\hat{y} > 0$, then $\hat{\pi}$ is optimal for our (primal) optimization problem, while λ is optimal for the *dual problem*

$$(11.12) \qquad \tilde{V}(\hat{y}) = \inf_{\nu \in \mathcal{H}} E\tilde{U}(\hat{y}H_\nu(T)).$$

Furthermore, equality holds in (11.11) if and only if

$$(11.13) \qquad X^{x,\pi}(T) = I(yH_\nu(T)) \; a.s.,$$

$$(11.14) \qquad \delta(\nu_t) = -\nu^*(t)\pi(t) \; a.e.,$$

$$(11.15) \qquad E[H_\nu(T) X^{x,\pi}(T)] = x$$

(the latter being equivalent to $\nu \in \mathcal{D}'$ and $y = \mathcal{Y}_\nu(x)$, if (11.13) holds).

Proof: By definitions of $\tilde{U}, \delta$ we get
(11.16)
$$U(X(T)) \leq \tilde{U}(yH_\nu(T)) + yH_\nu(T)X(T) + \int_0^T H_\nu(t)X(t)[\delta(\nu_t) + \nu^*(t)\pi(t)]dt.$$

The upper bound of (11.11) follows from the supermartingale property (6.14); condition (11.13) follows from (10.2), condition (11.14) is obvious, and condition (11.15) corresponds to equality holding in (6.14). $\diamond$

Remark 11.7: Lemma 11.6 suggests the following strategy for solving the optimization problem:

(i) show that the dual problem (11.12) has an optimal solution $\lambda_y \in \mathcal{D}'$ for all $y > 0$;

(ii) using Theorem 6.4, find the minimal hedging price $h_y(0)$ and a corresponding portfolio $\hat{\pi}_y$ for hedging $B := I(yH_{\lambda_y}(T))$;

(iii) prove (11.14) for the pair $(\hat{\pi}_y, \lambda_y)$;

(iv) show that, for every $x > 0$, you can find $y = y_x > 0$ such that $x = h_y(0) = E[H_{\lambda_y}(T)I(yH_{\lambda_y}(T))]$.

Then (i)-(iv) would imply that $\hat{\pi}_{y_x}$ is the optimal portfolio process for the utility maximization problem of an investor starting with initial capital equal to x.

To verify that step (i) can be accomplished, we impose the following condition:

(11.17) $\quad \forall\ y \ \in\ (0,\infty), \ \ \exists\ \nu \in \mathcal{H} \ \text{ such that } \ \tilde{J}(y;\nu) := E\tilde{U}(yH_\nu(T)) < \infty$

We shall also impose the assumption

(11.18) $\quad U(0+) > -\infty \quad , \quad U(\infty) = \infty\ .$

11.8 Remark: Under the conditions of Remark 11.3, the requirement (11.17) is satisfied. Indeed, the condition (11.7) leads to

$$0 \leq\ \tilde{U}(y) \leq \tilde{\kappa}(1 + y^{-\rho})\ ; \quad \forall\ \ y\ \in\ (0,\infty)$$

for some $\tilde{\kappa}\ \in\ (0,\infty)$ and $\rho = \frac{\alpha}{1-\alpha}$.

Even though the log function does not satisfy (11.18), we solve that case directly in examples below.

11.9 Theorem: (Cvitanić & Karatzas (1992)) *Assume that (10.8), (10.9), (11.17) and (11.18) are satisfied. Then condition (i) of Remark 11.7 is true, i.e. the dual problem admits a solution in the set $\mathcal{D}'$, for every $y > 0$.*

The fact that the dual problem admits a solution under the conditions of Theorem 11.9 follows almost immediately (by standard weak compactness arguments) from Proposition 11.10 below. The details, as well as a relatively straightforward proof of Proposition 11.10, can be found in CK(1992). Denote by $\mathcal{H}'$ the Hilbert space of progressively measurable processes ν with norm $[\![\nu]\!] = E\int_0^T \nu^2(s)ds < \infty$.

11.10 Proposition: *Under the assumptions of Theorem 11.9, the functional* $\tilde{J}(y;\cdot) : \mathcal{H}' \to \mathbb{R}\cup\{+\infty\}$ *of (11.12) is* (i) *convex,* (ii) *coercive:* $\lim_{[\![\nu]\!]\to\infty} \tilde{J}(y;\nu) = \infty$, and (iii) *lower-semicontinuous:* for every $\nu\epsilon\mathcal{H}'$ and $\{\nu_n\}_{n\epsilon\mathbf{N}} \subseteq \mathcal{H}'$ with $[\![\nu_n - \nu]\!] \to 0$ as $n \to \infty$, we have

$$\tilde{J}(y;\nu) \leq \lim_{n\to\infty} \tilde{J}(y;\nu_n) \; . \tag{11.19}$$

11.11 Remark: It can be shown that the optimal dual process λ_y satisfies $\lambda_y \in \mathcal{D}'$; see, for example, Karatzas, Lehoczky, Shreve & Xu (1991), proof of Theorem 12.3.

We move now to step (ii) of Remark 11.7. We have the following useful fact:

11.12 Lemma: For every $\nu \in \mathcal{H}$, $0 < y < \infty$, we have

$$E[H_\nu(T)B_{\lambda_y}] \leq E[H_{\lambda_y}(T)B_{\lambda_y}] \; . \tag{11.20}$$

11.13 Remark: In fact, (11.20) is equivalent to λ_y being optimal for the dual problem, but we shall not need that result; its proof is quite lengthy and technical (see CK (1992, Theorem 10.1). We are going to provide a simpler proof for Lemma 11.12, but under the additional assumption that

$$E[H_{\lambda_y}(T)I(yH_\nu(T))] < \infty, \; \forall \nu \in \mathcal{H}, y > 0. \tag{11.21}$$

Proof of Lemma 11.12: Fix $\varepsilon \in (0,1), \nu \in \mathcal{H}$ and define (supressing dependence on t)

$$\begin{aligned} &G_\varepsilon := (1-\varepsilon)H_{\lambda_y} + \varepsilon H_\nu, \quad \mu_\varepsilon := G_\varepsilon^{-1}((1-\varepsilon)H_{\lambda_y}\lambda_y + \varepsilon H_\nu \nu), \\ &\tilde{\mu}_\varepsilon := G_\varepsilon^{-1}((1-\varepsilon)H_{\lambda_y}\delta(\lambda_y) + \varepsilon H_\nu \delta(\nu)). \end{aligned} \tag{11.22}$$

Then $\mu_\varepsilon \in \mathcal{H}$, because of the convexity of $\tilde{K}$. Moreover, we have

$$dG_\varepsilon = (\theta + \sigma^{-1}\mu_\varepsilon)G_\varepsilon dW - \tilde{\mu}_\varepsilon G_\varepsilon dt,$$

and convexity of δ implies $\delta(\mu_\varepsilon) \leq \tilde{\mu}_\varepsilon$, and therefore, comparing the solutions to the respective (linear) SDE's, we get

$$G_\varepsilon(\cdot) \leq H_{\mu_\varepsilon}(\cdot), \; a.s.. \tag{11.23}$$

Since λ_y is optimal and $\tilde{U}$ is decreasing, (11.23) implies

$$\varepsilon^{-1}\left(E[\tilde{U}(yH_{\lambda_y}(T)) - \tilde{U}(yG_\varepsilon(T))]\right) \leq 0. \tag{11.24}$$

Next, recall that $I = -\tilde{U}'$ and denote by V_ε the random variable inside the expectation operator in (11.24). Fix $\omega \in \Omega$, and assume, supressing the dependence on ω and T, that $H_\nu \geq H_{\lambda_y}$. Then $\varepsilon^{-1}V_\varepsilon = I(F)y(H_\nu - H_{\lambda_y})$, where $yH_{\lambda_y} \leq F \leq yH_{\lambda_y} + \varepsilon y(H_\nu - H_{\lambda_y})$. Since I is decreasing we get $\varepsilon^{-1}V_\varepsilon \geq yI(yH_\nu)(H_\nu - H_{\lambda_y})$. We get the same result when assuming $H_\nu \leq H_{\lambda_y}$. This and assumption (11.21) imply that we can use Fatou's lemma when taking the limit as $\varepsilon \downarrow 0$ in (11.24), which gives us (11.20). $\diamond$

Now, given $y > 0$ and the optimal λ_y for the dual problem, let π_y be the portfolio of Theorem 6.4 for hedging the claim $B_{\lambda_y} = I(yH_{\lambda_y}(T))$. Lemma 11.12 implies that, in the notation of Section 6,

$$h_y(0) = V_y(0) = E[H_{\lambda_y}(T)I(yH_{\lambda_y}(T))] = \text{ initial capital for portfolio } \pi_y,$$

so (11.15) is satisfied for $x = h_y(0)$. It also implies, by (6.18), that (11.14) holds for the pair (π_y, λ_y). Therefore we have completed both steps (ii) and (iii). Step (iv) is a corollary of the following result.

11.13 Proposition: *Under the assumptions of Theorem 11.9, for any given $x > 0$, there exists $y_x > 0$ that achieves $\inf_{y>0}[\tilde{V}(y) + xy]$ and satisfies*

$$x = \mathcal{X}_{\lambda_{y_x}}(y_x). \tag{11.22}$$

For the (straightforward) proof see CK (1992, Proposition 12.2). We now put together the results of this section:

11.14 Theorem: *Under the assumptions of Theorem 11.9, for any given $x > 0$ there exists an optimal portfolio process $\hat{\pi}$ for the utility maximization problem of Definition 11.1. $\hat{\pi}$ is equal to the portfolio of Theorem 6.4 for minimaly hedging the claim $I(y_x H_{\lambda_{y_x}}(T))$, where y_x is given by Proposition 11.13 and λ_{y_x} is the optimal process for the dual problem (11.12).*

12. EXAMPLES

12.1 Example: *Logarithmic utility.* If $U(x) = \log x$, for $x \epsilon (0,\infty)$, we have $I(y) = \frac{1}{y}$, $\tilde{U}(y) = -(1 + \log y)$ and

$$\mathcal{X}_\nu(y) = \frac{1}{y}\ , \quad \mathcal{Y}_\nu(x) = \frac{1}{x} \tag{12.1}$$

and therefore, the optimal terminal wealth is

$$X_\lambda(T) = x\frac{1}{H_\lambda(T)} \tag{12.2}$$

for $\lambda \epsilon \mathcal{H}$ optimal. In particular $\mathcal{D}' = \mathcal{H}$ in this case. Therefore,

$$E\Big[\tilde{U}(\mathcal{Y}_\lambda(x)H_\nu(T))\Big] == -1 - \log\frac{1}{x} + E\Big(\log\frac{1}{H_\nu(T)}\Big)\ . \tag{12.3}$$

But

$$E\Big(\log\frac{1}{H_\nu(T)}\Big) = E\int_0^T\Big[r(s) + \delta(\nu(s)) + \frac{1}{2}||\theta(s) + \sigma^{-1}(s)\nu(s)||^2\Big]ds\ ,$$

and thus the dual problem amounts to a point-wise minimization of the convex function
$\delta(x) + \frac{1}{2}||\theta(t) + \sigma^{-1}(t)x||^2$ over $x\epsilon\tilde{K}$, for every $t \ \epsilon\ [0,T]$:

$$\lambda(t) = arg\min_{x\epsilon\tilde{K}}\ [\ 2\delta(x) + ||\theta(t) + \sigma^{-1}(t)x||^2\ ]\ . \tag{12.4}$$

Furthermore, (12.2) gives

$$H_\lambda(t)X_\lambda(t) = x; \quad 0 \le t \le T \ ,$$

and using Ito's rule to get the SDE for $H_\lambda X_\lambda$ we get, by equating the integrand in the stochastic integral term to zero, $\sigma^*(t)\hat{\pi}(t) = \theta_\lambda(t)$, $\ell \otimes P$ - a.e.

We conclude that the optimal portfolio is given by

$$\hat{\pi}(t) = (\sigma(t)\sigma^*(t))^{-1}[\lambda(t) + b(t) - r(t)\underset{\sim}{1}] \tag{12.5}$$

in terms of the market coëfficients and the process λ of (12.4).

12.2 Example: *(Constraints on borrowing)* From the point of view of applications, an interesting example is the one in which the total proportion $\sum_{i=1}^d \pi_i(t)$ of wealth invested in stocks is bounded from above by some real constant $a > 0$. For example, if we take $a = 1$, we exclude borrowing; with $a \in (1,2)$, we allow borrowing up to a fraction $1 - a$ of wealth. If we take $a = 1/2$, we have to invest at least half of the wealth in the bank.

To illustrate what happens in this situation, let again $U(x) = \log x$,, and, for the sake of simplicity, $d = 2$, $\sigma =$ unit matrix, and the constraints on the portfolio be given by

$$K = \{x \epsilon \mathbb{R}^2;\ x_1 \ge 0, x_2 \ge 0,\ x_1 + x_2 \le a\}$$

for some $a\epsilon(0,1]$ (obviously, we also exclude short-selling with this K). We have here $\delta(x) \equiv a \max\{x_1^-, x_2^-\}$, and thus $\tilde{K} = \mathbb{R}^2$. By some elementary calculus and/or by inspection, and omitting the dependence on t, we can see that the optimal dual process λ that minimizes $\frac{1}{2}||\theta_t + \nu_t||^2 + \delta(\nu_t)$, and the optimal portfolio $\pi_t = \theta_t + \lambda_t$, are given respectively by

$$\lambda = -\theta \ ; \ \pi = (0,0)^* \ \text{ if } \ \theta_1, \theta_2 \le 0$$

(do not invest in stocks if the interest rate is larger than the stocks appreciation rates),

$$\begin{aligned}
&\lambda = (0, -\theta_2)^*; \ \ \pi = (\theta_1, 0)^* \ \text{ if } \ \theta_1 \ge 0, \theta_2 \le 0,\ a \ge \theta_1 \ , \\
&\lambda = (a - \theta_1, -\theta_2)^*; \ \ \pi = (a, 0)^* \ \text{ if } \ \theta_1 \ge 0, \theta_2 \le 0,\ a < \theta_1 \ , \\
&\lambda = (-\theta_1, 0)^*; \ \ \pi = (0, \theta_2)^* \ \text{ if } \ \theta_1 \le 0, \theta_2 \ge 0,\ a \ge \theta_2 \ , \\
&\lambda = (-\theta_1, a - \theta_2)^*; \ \ \pi = (0, a)^* \ \text{ if } \ \theta_1 \le 0, \theta_2 \ge 0,\ a < \theta_2 \ ,
\end{aligned}$$

(do not invest in the stock whose rate is less than the interest rate, invest $X \min\{a, \theta_i\}$ in the i-th stock whose rate is larger than the interest rate),

$$\lambda = (0,0)^*; \ \ \pi = \theta \ \text{ if } \ \theta_1, \theta_2 \ge 0,\ \theta_1 + \theta_2 \le a$$

(invest $\theta_i X$ in the respective stocks-as in the no constraints case-whenever the optimal portfolio of the no constraints case happens to take values in K),

$$\lambda = (a - \theta_1, -\theta_2)^*;\ \pi = (a, 0)^* \ \text{ if } \ \theta_1, \theta_2 \geq 0,\ a \leq \theta_1 - \theta_2\ ,$$
$$\lambda = (-\theta_1, a - \theta_2)^*;\ \pi = (0, a)^* \ \text{ if } \ \theta_1, \theta_2 \geq 0,\ a \leq \theta_2 - \theta_1$$

(with both $\theta_1, \theta_2 \geq 0$ and $\theta_1 + \theta_2 > a$ do not invest in the stock whose rate is smaller, invest aX in the other one if the absolute value of the difference of the stocks rates is larger than a),

$$\lambda_1 = \lambda_2 = \frac{a - \theta_1 - \theta_2}{2}\ ;\ \ \pi_1 = \frac{a + \theta_1 - \theta_2}{2}\ ,\ \pi_2 = \frac{a + \theta_2 - \theta_1}{2}$$

if $\theta_1, \theta_2 \geq 0, \theta_1 + \theta_2 > a > |\theta_1 - \theta_2|$ (if none of the previous conditions is satisfied, invest the amount $\frac{a}{2}X$ in the stocks, corrected by the difference of their rates).

Let us consider now the case, where the coëfficients $r(\cdot), b(\cdot), \sigma(\cdot)$ of the market model are deterministic functions on $[0, T]$, which we shall take for simplicity to be bounded and continuous. Then there is a formal HJB (Hamilton-Jacobi-Bellman) equation associated with the dual optimization problem, namely,

$$\text{(12.6)} \qquad Q_t + \inf_{x \epsilon \tilde{K}} [\frac{1}{2} y^2 Q_{yy} ||\theta(t) + \sigma^{-1}(t)x||^2 - yQ_y \delta(x)] - yQ_y r(t) = 0\ ,$$

in $[0, T) \times (0, \infty)$;

$$\text{(12.7)} \qquad Q(T, y) = \tilde{U}(y)\ ;\ \ y\ \epsilon\ (0, \infty)\ .$$

If there exists a classical solution $Q\ \epsilon\ C^{1,2}([0, T) \times (0, \infty))$ of this equation, that satisfies appropriate growth conditions, then standard verification theorems in stochastic control (e.g. Fleming & Soner (1993)) lead to the representation

$$\text{(12.8)} \qquad \tilde{V}(y) = Q(0, y), \quad 0 < y < \infty$$

for the dual value function.

12.3 Example: *(Cone constraints)* Suppose that $\delta \equiv 0$ on $\tilde{K}$. Then

$$\text{(12.9)} \qquad \lambda(t) = arg \min_{x \epsilon \tilde{K}} ||\theta(t) + \sigma^{-1}(t)x||^2$$

is *deterministic*, the same for all $y \epsilon (0, \infty)$, and the equation (12.6) becomes

$$\text{(12.10)} \quad Q_t + \frac{1}{2} ||\theta_\lambda(t)||^2 y^2 Q_{yy} - r(t) y Q_y + \tilde{U}_1(t, y) = 0\ ;\ \ \text{in } \ [0, T) \times (0, \infty)\ .$$

Standard theory guarantees then the existence and uniqueness of a classical solution for this equation.

In the case of constant coëfficients, this solution can even be computed explicitly; see Cvitanić & Karatzas (1992).

12.4 Problem: *(Power utility)* Consider the case $U(x) = \frac{x^\alpha}{\alpha}$, $x \epsilon (0,\infty)$ for some $\alpha \epsilon (0,1)$. Then $\tilde{U}(y) = \frac{1}{\rho} y^{-\rho}$, $0 < y < \infty$ with $\rho := \frac{\alpha}{1-\alpha}$. Again, the process $\lambda(\cdot)$ is *deterministic*, namely

$$\lambda(t) = arg \min_{x \epsilon \tilde{K}} \, [\, ||\theta(t) + \sigma^{-1}(t)x||^2 + 2(1-\alpha)\delta(x) \,] \ ,$$

and is the same for all $y \epsilon (0,\infty)$. $\diamond$

Show that, in this case,

$$\pi_\lambda(t) = \frac{1}{1-\alpha}(\sigma(t)\sigma^*(t))^{-1}[b(t) - r(t)\mathbf{1} + \lambda(t)] \ .$$

12.5 Example: *(Different interest rates for borrowing and lending)* We consider the example of Section 9, with different interest rates for borrowing R, and lending r, $R(\cdot) \geq r(\cdot)$. The methodology of the previous section can still be used in the context of the model in Section 7, of which the different interest rates case is just one example. See Cvitanić & Karatzas (1992) for details. We are looking for an optimal process $\lambda_y \in \mathcal{H}$ for the corresponding dual problem, and, for any given $x \in (0,\infty)$, for an optimal portfolio $\hat{\pi}$ for the original primal control problem. In the case of logaritmic utility $U(x) = \log x$, we see as in (12.4) that $\lambda(t) = \lambda_1(t)\mathbf{1}$, where

$$\lambda_1(t) = arg \min_{r(t)-R(t) \leq x \leq 0} (-2x + ||\theta(t) + \sigma^{-1}(t)\mathbf{1}x||^2) \ .$$

With $A(t) := tr[(\sigma^{-1}(t))^*(\sigma^{-1}(t))], B(t) := \theta^*(t)\sigma^{-1}(t)\mathbf{1}$, this minimization is achieved as follows:

$$\lambda_1(t) = \left\{ \begin{array}{ccc} \frac{1-B(t)}{A(t)} & ; \ if & 0 < B(t) - 1 < A(t)(R(t) - r(t)) \\ 0 & ; \ if & B(t) \leq 1 \\ r(t) - R(t) & ; \ if & B(t) - 1 \geq A(t)(R(t) - r(t)) \end{array} \right\} \ .$$

From (12.5), the optimal portfolio is then computed as

$$\hat{\pi}_t = \left\{ \begin{array}{ccc} (\sigma_t\sigma_t^*)^{-1}[b_t - (r_t + \frac{B_t - 1}{A_t})\mathbf{1}] & ; & 0 < B_t - 1 \leq A_t(R_t - r_t) \\ (\sigma_t\sigma_t^*)^{-1}[b_t - r_t\mathbf{1}] & ; & B_t \leq 1 \\ (\sigma_t\sigma_t^*)^{-1}[b_t - R_t\mathbf{1}] & ; & B_t - 1 \geq A_t(R_t - r_t) \end{array} \right.$$

In the case $U(x) = \frac{x^\alpha}{\alpha}$, for some $\alpha \in (0,1)$, we get $\lambda(t) = \lambda_1(t)\mathbf{1}$ with

$$\lambda_1(t) = arg \min_{r(t)-R(t) \leq x \leq 0} \left[-2(1-\alpha)x + ||\theta(t) + \sigma^{-1}(t)\mathbf{1}x||^2 \right]$$

$$= \left\{ \begin{array}{ccc} \frac{1-\alpha-B(t)}{A(t)} & ; \ if & 0 < B(t) - 1 + \alpha < A(t)(R(t) - r(t)) \\ 0 & ; \ if & B(t) \leq 1 - \alpha \\ r(t) - R(t) & ; \ if & B(t) - 1 + \alpha \geq A(t)(R(t) - r(t)). \end{array} \right\}$$

The optimal portfolio is given as

$$\hat{\pi}_t = \begin{cases} \frac{(\sigma_t\sigma_t^*)^{-1}}{A_t}[b_t - (r_t + \frac{B_t - 1 + \alpha}{A_t})\mathbf{1}] & ; \quad 0 < B_t - 1 + \alpha < A_t(R_t - r_t) \\ \frac{(\sigma_t\sigma_t^*)^{-1}}{1-\alpha}[b_t - r_t\mathbf{1}] & ; \quad B_t \leq 1 - \alpha \\ \frac{(\sigma_t\sigma_t^*)^{-1}}{1-\alpha}[b_t - R_t\mathbf{1}] & ; \quad B_t - 1 + \alpha \geq A_t(R_t - r_t) \end{cases}$$

13. UTILITY BASED PRICING

How to choose a price of a contingent claim B in the no-arbitrage pricing interval $[\tilde{h}(0), h(0)]$ of Theorem 6.15, in the case of incomplete markets, i.e., when the interval is non-degenerate (consists of more than just the Black-Scholes price)? There have been many attempts to provide a satisfactory answer to this question. We describe one suggested by Davis (1994), as presented in Karatzas & Kou (1996). It is based on the following "zero marginal rate of substitution" principle: Given the agent's utility function U and initial wealth x, the price $\hat{p}$ is the one that makes the agent neutral with respect to diversion of a small amount of funds into the contingent claim at time zero, while maximizing the utility from total wealth at the exercise time T. We will show that

$$\hat{p} = E[H_{\lambda_x}(T)B], \tag{13.1}$$

where λ_x is the associated optimal dual process. In particular, this price can be calculated in the context of examples of Section 12, and *does not depend on* U and x, in the case of cone constraints ($\delta \equiv 0$) and constant coefficients (Example 12.3). It can also be shown that, in this case, it gives the probability measure $\mathbf{P}^{\lambda_x}$ which minimizes the relative entropy with respect to the original measure $\mathbf{P}$.

In order to show (13.1), for a given $-x < \delta < x$ and price p of the claim, we introduce the value function

$$Q(\delta, p, x) := \sup_{\pi \in \mathcal{A}'(x-\delta)} EU(X^{x-\delta}(T) + \frac{\delta}{p}B). \tag{13.2}$$

In other words, the agent acquires δ/p units of the claim B at price p at time zero, and maximizes his/her terminal wealth at time T. Davis (1994) suggests to use price $\hat{p}$ for which

$$\left.\frac{\partial Q}{\partial \delta}(\delta, \hat{p}, x)\right|_{\delta=0} = 0, \tag{13.3}$$

so that this diversion of funds has a neutral effect on the expected utility. Since the derivative in (13.3) need not exist, we have the following

13.1 Definition: For a given $x > 0$, we call $\hat{p}$ a *weak solution* of (13.3) if, for every function $\varphi : (-x, x) \mapsto \mathbb{R}$ of class C^1 which satisfies

$$\varphi(\delta) \geq Q(\delta, \hat{p}, x), \forall \delta \in (-x, x), \quad \varphi(0) = Q(0, p, x) = V(x), \tag{13.4}$$

we have $\varphi'(0) = 0$. If it is unique, then we call it the *utility based price* of B.

We have

13.2 Theorem: *Under the conditions of Theorem 11.14, the utility based price of B is given as in (13.1).*

Proof: Denote by $\hat{X}^x(T)$ the optimal terminal wealth for the utility maximization problem of the agent, starting with $\hat{X}(0) = x$. It is not difficult to see that $X^x(T)$ is a.s. increasing in x. Also, for the concave function U, one has

$$U(z) + (y - z)U'(y) \leq U(y) \leq U(z) + (y - z)U'(z), \; 0 < z < y. \tag{13.5}$$

We get, for $0 < \delta < x$, and a given function φ as in the theorem,

$$\begin{aligned}\varphi(\delta) &\geq Q(\delta, p, x) \geq EU(\hat{X}^{x-\delta}(T) + \frac{\delta}{p}B) \\ &\geq EU(\hat{X}^{x-\delta}(T)) + \frac{\delta}{p}E[U'(\hat{X}^{x-\delta}(T) + \frac{\delta}{p}B)B].\end{aligned}$$

Since $\varphi(0) = EU(\hat{X}^0(T))$, we obtain, by monotone convergence,

$$\varphi'(0) \geq \frac{1}{p}E[U'(\hat{X}^x(T))B] - V'(x). \tag{13.6}$$

Similarly, with $-x < \delta < 0$, setting formally $U(z) = U'(z) = \infty$ for $z < 0$, we get

$$\varphi'(0) \leq \frac{1}{p}E[U'(\hat{X}^x(T))B] - V'(x). \tag{13.7}$$

Now, (13.6), (13.7) imply that $\varphi'(0) = 0$ iff

$$p = \hat{p}(x) := \frac{E[U'(\hat{X}^x(T))B]}{V'(x)}.$$

But from previous sections we know that $U'(\hat{X}^x(T)) = y_x H_{\lambda_x}(T)$, for a suitable $y_x > 0$. One can also easily show (see Cvitanić & Karatzas (1992)) that $V'(x) = y_x$. This completes the proof.

14. HEDGING AND PORTFOLIO OPTIMIZATION IN THE PRESENCE OF TRANSACTION COSTS: THE MODEL

We consider a financial market consisting of one riskless asset, *bank account* with price $B(\cdot)$ given by

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1; \tag{14.1}$$

and of one risky asset, *stock*, with price-per-share $S(\cdot)$ governed by the stochastic equation

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)]\ , \quad S(0) = p \in (0,\infty), \tag{14.2}$$

for $t \in [0,T]$. Here $W = \{W(t), 0 \le t \le T\}$ a standard, one-dimensional Brownian motion. The coefficients of the model $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot) > 0$ are assumed to be bounded and $\mathbf{F}-$progressively measurable processes; furthermore, $\sigma(\cdot)$ is also assumed to be bounded away from zero (uniformly in (t,ω)).

Now, a *trading strategy* is a pair (L, M) of $\mathbf{F}-$adapted processes on $[0,T]$, with left-continuous, nondecreasing paths and $L(0) = M(0) = 0$; $L(t)$ (respectively, $M(t)$) represents the total amount of funds transferred from bank-account to stock (respectively, from stock to bank-account) by time t. Given *proportional transaction costs* $0 < \lambda, \mu < 1$ for such transfers, and initial holdings x, y in bank and stock, respectively, the portfolio holdings $X(\cdot) = X^{x,L,M}(\cdot), Y(\cdot) = Y^{y,L,M}(\cdot)$ corresponding to a given trading strategy (L, M), evolve according to the equations:

$$X(t) = x - (1+\lambda)L(t) + (1-\mu)M(t) + \int_0^t X(u)r(u)du,\ 0 \le t \le T \tag{14.3}$$

$$Y(t) = y + L(t) - M(t) + \int_0^t Y(u)[b(u)du + \sigma(u)dW(u)],\ 0 \le t \le T. \tag{14.4}$$

14.1 Definition: A *contingent claim* is a pair (C_0, C_1) of $\mathcal{F}(T)-$measurable random variables. We say that a trading strategy (L, M) *hedges the claim* (C_0, C_1) *starting with* (x, y) as initial holdings, if $X(\cdot), Y(\cdot)$ of (14.3), (14.4) satisfy

$$X(T) + (1-\mu)Y(T) \ge C_0 + (1-\mu)C_1 \tag{14.5}$$

$$X(T) + (1+\lambda)Y(T) \ge C_0 + (1+\lambda)C_1. \tag{14.6}$$

Interpretation: Here C_0 (respectively, C_1) is understood as a target-position in the bank-account (resp., the stock) at the terminal time $t = T$: for example

$$C_0 = -q\mathbf{1}_{\{S(T)>q\}},\ C_1 = S(T)\mathbf{1}_{\{S(T)>q\}} \tag{14.7}$$

in the case of a European call-option; and

$$C_0 = q\mathbf{1}_{\{S(T)<q\}},\ C_1 = -S(T)\mathbf{1}_{\{S(T)<q\}} \tag{14.8}$$

for a European put-option (both with exercise price $q \ge 0$).

"Hedging", in the sense of (14.5) and (14.6), simply means that "one is able to cover these positions at $t = T$". Indeed, assume that we have both $Y(T) \ge C_1$ and (14.5), in the form

$$X(T) + (1-\mu)[Y(T) - C_1] \ge C_0\ ; \tag*{(14.5)$'$}$$

then (14.6) holds too, and $(14.5)'$ shows that we can cover the position in the bank-account as well, by transferring the amount $Y(T) - C_1 \geq 0$ to it. Similarly for the case $Y(T) < C_1$.

14.3 Remark: The equations (14.3), (14.4) can be written in the equivalent form

$$(14.9) \qquad d\left(\frac{X(t)}{B(t)}\right) = \left(\frac{1}{B(t)}\right)[(1-\mu)dM(t) - (1+\lambda)dL(t)] \, , \quad X(0) = x$$

$$(14.10) \qquad d\left(\frac{Y(t)}{S(t)}\right) = \left(\frac{1}{S(t)}\right)[dL(t) - dM(t)] \, , \quad Y(0) = y$$

in terms of "number-of-shares" (rather than amounts) held.

15. AUXILIARY MARTINGALES.

Consider the class $\mathcal{D}$ of pairs of strictly positive $\mathbf{F}-$martingales $(Z_0(\cdot), Z_1(\cdot))$ with

$$(15.1) \qquad Z_0(0) = 1 \, , \quad z := Z_1(0) \in [p(1-\mu), p(1+\lambda)]$$

and

$$(15.2) \qquad 1-\mu \leq R(t) := \frac{Z_1(t)}{Z_0(t)P(t)} \leq 1+\lambda, \quad \forall \; 0 \leq t \leq T,$$

where

$$(15.3) \quad P(t) := \frac{S(t)}{B(t)} = p + \int_0^t P(u)[(b(u)-r(u))du + \sigma(u)dW(u)] \, , \quad 0 \leq t \leq T$$

is the *discounted stock price.*

The martingales $Z_0(\cdot), Z_1(\cdot)$ are the feasible *state-price densities* for holdings in bank and stock, respectively, in this market with transaction costs; as such, they reflect the "constraints" or "frictions" inherent in this market, in the form of condition (15.2). From the martingale representation theorem there exist $\mathbf{F}-$progressively measurable processes $\theta_0(\cdot), \theta_1(\cdot)$ with $\int_0^T (\theta_0^2(t) + \theta_1^2(t))dt < \infty$ a.s. and

$$(15.4) \qquad Z_i(t) = Z_i(0) \exp\left\{\int_0^t \theta_i(s)dW(s) - \frac{1}{2}\int_0^t \theta_i^2(s)ds\right\} \, , \quad i = 0, 1;$$

thus, the process $R(\cdot)$ of (15.2) has the dynamics

$$(15.5) \qquad \begin{aligned} dR(t) =& R(t)[\sigma^2(t) + r(t) - b(t) - (\theta_1(t) - \theta_0(t))(\sigma(t) + \theta_0(t))]dt \\ &+ R(t)(\theta_1(t) - \sigma(t) - \theta_0(t))dW(t), \quad R(0) = z/p. \end{aligned}$$

15.1 Remark: A rather "special" pair $(Z_0^*(\cdot), Z_1^*(\cdot)) \in \mathcal{D}$ is obtained, if we take in (15.4) the processes $(\theta_0(\cdot), \theta_1(\cdot))$ to be given as

$$\theta_0^*(t) := \frac{r(t) - b(t)}{\sigma(t)} \ , \quad \theta_1^*(t) := \sigma(t) + \theta_0^*(t) \ , \quad 0 \le t \le T, \tag{15.6}$$

and let $Z_0^*(0) = 1,\ p(1-\mu) \le Z_1^*(0) = z \le p(1+\lambda)$. Because then, from (15.5), $R^*(\cdot) := \frac{Z_1^*(\cdot)}{Z_0^*(\cdot)P(\cdot)} \equiv \frac{z}{p}$; in fact, the pair of (15.6) and $z = p$ provide the *only* member $(Z_0^*(\cdot), Z_1^*(\cdot))$ of $\mathcal{D}$, if $\lambda = \mu = 0$. Notice that the processes $\theta_0^*(\cdot)$, $\theta_1^*(\cdot)$ of (15.6) are *bounded.*

15.2 Remark: Let us observe also that the martingales $Z_0(\cdot), Z_1(\cdot)$ play the role of *adjoint processes* to the "number-of-share holdings" processes $X(\cdot)/B(\cdot)$, $Y(\cdot)/S(\cdot)$, respectively, in the sense that

(15.7)

$$\begin{aligned} Z_0(t)\frac{X(t)}{B(t)} + Z_1(t)\frac{Y(t)}{S(t)} &+ \int_0^t \frac{Z_0(s)}{B(s)}[(1+\lambda) - R(s)]dL(s) \\ &+ \int_0^t \frac{Z_0(s)}{B(s)}[R(s) - (1-\mu)]dM(s) \\ &= x + \frac{yz}{p} + \int_0^t \frac{Z_0(s)}{B(s)}[X(s)\theta_0(s) + R(s)Y(s)\theta_1(s)]dW(s),\ t \in [0,T] \end{aligned}$$

is a $\mathbf{P}-$local martingale, for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and any trading strategy (L, M); this follows directly from (14.9), (14.10), (15.4) and the product rule. Equivalently, (15.7) can be re-written as

(15.8)

$$\begin{aligned} &\frac{X(t) + R(t)Y(t)}{B(t)} + \int_0^t \frac{(1+\lambda) - R(s)}{B(s)}dL(s) + \int_0^t \frac{R(s) - (1-\mu)}{B(s)}dM(s) \\ &= x + \frac{yz}{p} + \int_0^t \frac{R(s)Y(s)}{B(s)}(\theta_1(s) - \theta_0(s))dW_0(s) = \mathbf{P_0} - \text{local martingale}, \end{aligned}$$

where

$$W_0(t) := W(t) - \int_0^t \theta_0(s)ds, \quad 0 \le t \le T \tag{15.9}$$

is (by Girsanov's theorem), a Brownian motion under the equivalent probability measure

$$\mathbf{P_0}(A) := E[Z_0(T)\mathbf{1}_A],\ A \in \mathcal{F}(T). \tag{15.10}$$

15.3 Remark: We shall denote by $Z_0^*(\cdot), W_0^*(\cdot)$ and $\mathbf{P_0^*}$ the processes and probability measure, respectively, corresponding to the process $\theta_0^*(\cdot)$ of (15.6), via the equations (15.4) (with $Z_0^*(0) = 1$), (15.9) and (15.10). With this notation, (15.3) becomes $dP(t) = P(t)\sigma(t)dW_0^*(t)$, $P(0) = p$.

15.4 Definition: Let $\mathcal{D}_\infty$ be the class of positive martingales $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, for which the random variable

$$\frac{Z_0(T)}{Z_0^*(T)}, \qquad \text{and thus also} \quad \frac{Z_1(T)}{Z_0^*(T)P(T)}\ , \tag{15.11}$$

is essentially bounded.

15.5 Definition: We shall say that a given trading strategy (L, M) is *admissible* for (x, y), and write $(L, M) \in \mathcal{A}(x, y)$, if

$$\frac{X(\cdot) + R(\cdot)Y(\cdot)}{B(\cdot)} \quad \text{is a } \mathbf{P_0} - \text{supermartingale}, \ \ \forall \ (Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty. \tag{15.12}$$

Consider, for example, a trading strategy (L, M) that satisfies the no-bankruptcy conditions

$$X(t) + (1+\lambda)Y(t) \geq 0 \ \text{ and } \ X(t) + (1-\mu)Y(t) \geq 0, \ \ \forall\ 0 \leq t \leq T.$$

Then $X(\cdot) + R(\cdot)Y(\cdot) \geq 0$ for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ (recall (15.2), and note Remark 15.6 below); this means that the $\mathbf{P_0}-$local martingale of (15.8) is non-negative, hence a $\mathbf{P_0}-$supermartingale. But the second and the third terms

$$\int_0^\cdot \frac{1+\lambda - R(s)}{B(s)} dL(s), \quad \int_0^\cdot \frac{R(s) - (1-\mu)}{B(s)} dM(s)$$

in (15.8) are increasing processes, thus the first term $\frac{X(\cdot)+R(\cdot)Y(\cdot)}{B(\cdot)}$ is also a $\mathbf{P_0}-$supermartingale, for every pair $(Z_0(\cdot), Z_1(\cdot))$ in $\mathcal{D}$. The condition (15.12) is actually weaker, in that it requires this property only for pairs in $\mathcal{D}_\infty$. This provides a motivation for Definition 15.4, namely, *to allow for as wide a class of trading strategies as possible, and still exclude arbitrage opportunities.* This is usually done by imposing a lower bound on the wealth process; however, that excludes simple strategies of the form "trade only once, by buying a fixed number of shares of the stock at a specified time t", which may require (unbounded) borrowing. We shall have occasion, to use such strategies in the sequel; see, for example, (16.20).

15.6 Remark: Here is a trivial (but useful) observation: if $x + (1-\mu)y \geq a + (1-\mu)b$ and $x + (1+\lambda)y \geq a + (1+\lambda)b$, then $x + ry \geq a + rb, \forall\, 1-\mu \leq r \leq 1+\lambda$.

16. HEDGING PRICE.

Suppose that we are given an initial holding $y \in \mathbb{R}$ in the stock, and want to hedge a given contingent claim (C_0, C_1) with strategies which are admissible (in the sense of Definitions 14.1, 15.4). What is the smallest amount of holdings in the bank

(16.1)

$$h(C_0, C_1; y) := \inf\{x \in \mathbb{R} / \ \exists (L, M) \in \mathcal{A}(x, y) \text{ and } \ (L, M) \text{ hedges } (C_0, C_1)\}$$

that allows to do this? We call $h(C_0, C_1; y)$ the *hedging price* of the contingent claim (C_0, C_1) for initial holding y in the stock, and with the convention that $h(C_0, C_1; y) = \infty$ if the set in (16.1) is empty.

Suppose this is not the case, and let $x \in \mathbb{R}$ belong to the set of (16.1); then for any $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$ we have from (15.12), the Definition 14.1 of hedging, and Remark 15.6:

$$x + \frac{y}{p} E Z_1(T) = x + \frac{y}{p} z \geq E_0 \left[\frac{X(T) + R(T)Y(T)}{B(T)} \right]$$
$$\geq E_0 \left[\frac{C_0 + R(T)C_1}{B(T)} \right] = E \left[\frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) \right],$$

so that $x \geq E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p} Z_1(T)\right]$. Therefore

$$h(C_0, C_1; y) \geq \sup_{\mathcal{D}_\infty} E \left[\frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) - \frac{y}{p} Z_1(T) \right], \tag{16.2}$$

and this inequality is clearly also valid if $h(C_0, C_1; y) = \infty$.

16.1 Lemma: *If the contingent claim (C_0, C_1) is bounded from below, in the sense*

(4.3) $C_0 + (1+\lambda)C_1 \geq -K$ *and* $C_0 + (1-\mu)C_1 \geq -K$, *for some* $0 \leq K < \infty$

then

(16.4)

$$\sup_{\mathcal{D}_\infty} E\left[\frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) - \frac{y}{p} Z_1(T) \right]$$
$$= \sup_{\mathcal{D}} E \left[\frac{Z_0(T)}{B(T)} (C_0 + R(T)C_1) - \frac{y}{p} Z_1(T) \right].$$

Proof: Start with arbitrary $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$ and define the sequence of stopping times $\{\tau_n\} \uparrow T$ by

$$\tau_n := \inf\{t \in [0, T] \;/\; \frac{Z_0(t)}{Z_0^*(t)} \geq n\} \wedge T, \; n \in \mathbb{N}.$$

Consider also, for $i = 0, 1$ and in the notation of (15.6):

$$\theta_i^{(n)}(t) := \left\{ \begin{array}{ll} \theta_i(t), & 0 \leq t < \tau_n \\ \theta_i^*(t), & \tau_n \leq t \leq T \end{array} \right\}$$

and

$$Z_i^{(n)}(t) = z_i \exp\{\int_0^t \theta_i^{(n)}(s) dW(s) - \frac{1}{2} \int_0^t (\theta_i^{(n)}(s))^2 ds\}$$

with $z_0 = 1$, $z_1 = Z_1(0) = E Z_1(T)$. Then, for every $n \in \mathbb{N}$, both $Z_0^{(n)}(\cdot)$ and $Z_1^{(n)}(\cdot)$ are positive martingales, $R^{(n)}(\cdot) = \frac{Z_1^{(n)}(\cdot)}{Z_0^{(n)}(\cdot)P(\cdot)} = R(\cdot \wedge \tau_n)$ takes values in $[1-\mu, 1+\lambda]$ (by (15.2) and Remark 15.1), and $Z_0^{(n)}(\cdot)/Z_0^*(\cdot)$ is bounded by

n (in fact, constant on $[\tau_n, T]$). Therefore, $(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot)) \in \mathcal{D}_\infty$. Now let κ denote an upper bound on $K/B(T)$, and observe, from Remark 15.6, (16.3) and Fatou's lemma:

$$
\begin{aligned}
& E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p}Z_1(T)\right] + \frac{y}{p}Z_1(0) + \kappa \\
& = E\left[Z_0(T)\left\{\frac{C_0 + R(T)C_1}{B(T)} + \kappa\right\}\right] \\
& = E\left[\lim_n Z_0^{(n)}(T)\left\{\frac{C_0 + R^{(n)}(T)C_1}{B(T)} + \kappa\right\}\right] \qquad (16.5) \\
& \leq \underline{\lim}_n E\left[Z_0^{(n)}(T)\left\{\frac{C_0 + R^{(n)}(T)C_1}{B(T)} + \kappa\right\}\right] \\
& = \underline{\lim}_n E\left[\frac{Z_0^{(n)}(T)}{B(T)}(C_0 + R^{(n)}(T)C_1) - \frac{y}{p}Z_1^{(n)}(T)\right] + \frac{y}{p}Z_1(0) + \kappa.
\end{aligned}
$$

This shows that the left-hand-side dominates the right-hand-side in (16.4); the reverse inequality is obvious. $\diamond$

Remark: Formally taking $y = 0$ in (16.5), we deduce

$$E_0\left(\frac{C_0 + R(T)C_1}{B(T)}\right) \leq \underline{\lim}_{n\to\infty} E_0^{(n)}\left(\frac{C_0 + R^{(n)}(T)C_1}{B(T)}\right), \qquad (16.6)$$

where $E_0, E_0^{(n)}$ denote expectations with respect to the probability measures $\mathbf{P_0}$ of (15.10) and $\mathbf{P_0^{(n)}}(\cdot) = E[Z_0^{(n)}(T)\mathbf{1}_\cdot]$, respectively.

Here is the main result of this section.

16.2 Theorem: (Cvitanić & Karatzas (1996)) *Under the conditions (16.3) and*

$$E_0^*(C_0^2 + C_1^2) < \infty\ , \qquad (16.7)$$

we have

$$h(C_0, C_1; y) = \sup_{\mathcal{D}} E\left[\frac{Z_0(T)}{B(T)}(C_0 + R(T)C_1) - \frac{y}{p}Z_1(T)\right]. \qquad (16.8)$$

In (16.7), E_0^* denotes expectation with respect to the probability measure $\mathbf{P_0^*}$. The conditions (16.3), (16.7) are both easily verified for a European call or put. In fact, one can show that if a pair of admissible terminal holdings $(X(T), Y(T))$ hedges a pair $(\tilde{C}_0, \tilde{C}_1)$ satisfying (16.7) (for example, $(\tilde{C}_0, \tilde{C}_1) \equiv (0,0)$), then necessarily the pair $(X(T), Y(T))$ also satisfies (16.7) – and so does any other pair of random variables (C_0, C_1) which are bounded from below and are hedged by $(X(T), Y(T))$. In other words, any strategy which satisfies the "no-bankruptcy" condition of hedging $(0,0)$, necessarily results in a square-integrable final wealth. In this sense, the condition (16.7) is consistent with the standard "no-bankruptcy" condition, hence not very restrictive.

It would be of significant interest to be able to describe the least expensive hedging strategy associated with a general hedgeable contingent claim; Our *functional-analytic proof*, which takes up the remainder of this section and was inspired by similar arguments in Kusuoka (1995), does not provide the construction of such a strategy.

Proof: In view of Lemma 16.1 and the inequality (16.2), it suffices to show

$$h(C_0, C_1; y) \le \sup_{\mathcal{D}} E\left[Z_0(T)\frac{C_0}{B(T)} + Z_1(T)\left(\frac{C_1}{S(T)} - \frac{y}{p}\right)\right] =: R. \tag{16.9}$$

And in order to alleviate somewhat the (already rather heavy) notation, we shall take $p = 1$, $r(\cdot) \equiv 0$, thus $B(\cdot) \equiv 1$, for the remainder of the section; the reader will verify easily that this entails no loss of generality.

We start by taking an arbitrary $b < h(C_0, C_1; y)$ and considering the sets

(16.10)
$$A_0 := \{(U,V) \in (\mathbf{L}_2^*)^2 : \exists (L,M) \in \mathcal{A}(0,0) \text{ that hedges } (U,V) \text{ starting with } x = 0, y = 0\}$$

$$A_1 := \{(C_0 - b, C_1 - yS(T))\}, \tag{16.11}$$

where $\mathbf{L}_2^* = \mathbf{L}_2(\Omega, \mathcal{F}(T), \mathbf{P}_\mathbf{0}^*)$. It is not hard to prove (see below) that

$$A_0 \textit{ is a convex cone, and contains the origin } (0,0), \textit{ in } (\mathbf{L}_2^*)^2, \tag{16.12}$$

$$A_0 \cap A_1 = \emptyset. \tag{16.13}$$

It is, however, considerably harder to establish that

$$A_0 \textit{ is closed in } (\mathbf{L}_2^*)^2. \tag{16.14}$$

The proof can be found in the appendix of Cvitanić & Karatzas(1996). From (16.12)-(16.14) and the Hahn-Banach theorem there exists a pair of random variables $(\rho_0^*, \rho_1^*) \in (\mathbf{L}_2^*)^2$, not equal to $(0,0)$, such that

$$E_0^*[\rho_0^* V_0 + \rho_1^* V_1] = E[\rho_0 V_0 + \rho_1 V_1] \le 0, \quad \forall\ (V_0, V_1) \in A_0 \tag{16.15}$$

$$E_0^*[\rho_0^*(C_0 - b) + \rho_1^*(C_1 - yS(T))] = E[\rho_0(C_0 - b) + \rho_1(C_1 - yS(T))] \ge 0, \tag{16.16}$$

where $\rho_i := \rho_i^* Z_0^*(T),\ i = 0, 1$. It is also not hard to check (see below) that

$$(1-\mu)E[\rho_0|\mathcal{F}(t)] \le \frac{E[\rho_1 S(T)|\mathcal{F}(t)]}{S(t)} \le (1+\lambda)E[\rho_0|\mathcal{F}(t)],\ \forall\ 0 \le t \le T \tag{16.17}$$

$$\rho_1 \ge 0,\ \rho_0 \ge 0 \text{ and } E\rho_0 > 0,\ E(\rho_1 S(T)) > 0. \tag{4.18}$$

In view of (16.18), we may take $E\rho_0 = 1$, and then (16.16) gives

$$b \le E[\rho_0 C_0 + \rho_1(C_1 - yS(T))]. \tag{16.19}$$

Consider now *arbitrary* $0 < \varepsilon < 1$, $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, and define

$$\tilde{Z}_0(t) := \varepsilon Z_0(t) + (1-\varepsilon)E[\rho_0|\mathcal{F}(t)],\ \tilde{Z}_1(t) := \varepsilon Z_1(t) + (1-\varepsilon)E[\rho_1 S(T)|\mathcal{F}(t)],$$

for $0 \le t \le T$. Clearly these are positive martingales, and $\tilde{Z}_0(0) = 1$; on the other hand, multiplying in (16.17) by $1-\varepsilon$, and in $(1-\mu)Z_0(t) \le Z_1(t)/S(t) \le (1+\lambda)Z_0(t),\ 0 \le t \le T$ by ε, and adding up, we obtain $(\tilde{Z}_0(\cdot), \tilde{Z}_1(\cdot)) \in \mathcal{D}$. Thus, in the notation of (16.9),

$$\begin{aligned} R &\ge E\left[\tilde{Z}_0(T)C_0 + \tilde{Z}_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\ &= (1-\varepsilon)E[\rho_0 C_0 + \rho_1(C_1 - yS(T))] + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \\ &\ge b(1-\varepsilon) + \varepsilon E\left[Z_0(T)C_0 + Z_1(T)\left(\frac{C_1}{S(T)} - y\right)\right] \end{aligned}$$

from (16.19); letting $\varepsilon \downarrow 0$ and then $b \uparrow h(C_0, C_1; y)$, we obtain (16.9), as required to complete the proof of Theorem 16.2.

Proof of (16.13): Suppose that $A_0 \cap A_1$ is not empty, i.e., that there exists $(L, M) \in \mathcal{A}(0,0)$ such that, with $X(\cdot) = X^{0,L,M}(\cdot)$ and $Y(\cdot) = Y^{0,L,M}(\cdot)$, the process $X(\cdot) + R(\cdot)Y(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$, and we have:

$$\begin{aligned} X(T) + (1-\mu)Y(T) &\ge (C_0 - b) + (1-\mu)(C_1 - yS(T)), \\ X(T) + (1+\lambda)Y(T) &\ge (C_0 - b) + (1+\lambda)(C_1 - yS(T)). \end{aligned}$$

But then, with

$$\tilde{X}(\cdot) := X^{b,L,M}(\cdot) = b + X(\cdot),\ \tilde{Y}(\cdot) := Y^{y,L,M}(\cdot) = Y(\cdot) + yS(\cdot)$$

we have, from above, that $\tilde{X}(\cdot) + R(\cdot)\tilde{Y}(\cdot) = X(\cdot) + R(\cdot)Y(\cdot) + b + yZ_1(\cdot)/Z_0(\cdot)$ is a $\mathbf{P_0}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}_\infty$, and that

$$\begin{aligned} \tilde{X}(T) + (1-\mu)\tilde{Y}(T) &\ge C_0 + (1-\mu)C_1, \\ \tilde{X}(T) + (1+\lambda)\tilde{Y}(T) &\ge C_0 + (1+\lambda)C_1. \end{aligned}$$

In other words, (L, M) belongs to $\mathcal{A}(b, y)$ and hedges (C_0, C_1) starting with (b, y) – a contradiction to the definition (16.1), and to the fact $h(C_0, C_1; y) > b$.

Proof of (16.17), (16.18): Fix $t \in [0, T)$ and let ξ be an arbitrary bounded, nonnegative, $\mathcal{F}(t)-$measurable random variable. Consider the strategy of starting with $(x, y) = (0, 0)$ and buying ξ shares of stock at time $s = t$, otherwise doing nothing ("buy-and-hold strategy"); more explicitly, $M^\xi(\cdot) \equiv 0$, $L^\xi(s) = \xi S(t)\mathbf{1}_{(t,T]}(s)$ and thus

$$\begin{aligned} X^\xi(s) &:= X^{0,L^\xi,M^\xi}(\cdot) = -\xi(1+\lambda)S(t)\mathbf{1}_{(t,T]}(s), \\ Y^\xi(s) &:= Y^{0,L^\xi,M^\xi}(s) = \xi S(s)\mathbf{1}_{(t,T]}(s), \end{aligned} \tag{16.20}$$

for $0 \le s \le T$. Consequently, $Z_0(s)[X^\xi(s) + R(s)Y^\xi(s)] = \xi[Z_1(s) - (1+\lambda)S(t)Z_0(s)]\mathbf{1}_{(t,T]}(s)$ is a $\mathbf{P}-$supermartingale for every $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$, since, for instance with $t < s \leq T$:

$$\begin{aligned} E[Z_0(s)(X_s^\xi + R_s Y_s^\xi)|\mathcal{F}_t] &= \xi\,(E[Z_1(s)|\mathcal{F}_t] - (1+\lambda)S_t E[Z_0(s)|\mathcal{F}_t]) \\ &= \xi[Z_1(t) - (1+\lambda)S(t)Z_0(t)] = \xi S(t)Z_0(t)[R(t) - (1+\lambda)] \\ &\leq 0 = Z_0(t)[X^\xi(t) + R(t)Y^\xi(t)]. \end{aligned}$$

Therefore, $(L^\xi, M^\xi) \in \mathcal{A}(0,0)$, thus $(X^\xi(T), Y^\xi(T))$ belongs to the set A_0 of (16.10), and, from (16.15):

$$\begin{aligned} 0 &\geq E[\rho_0 X^\xi(T) + \rho_1 Y^\xi(T)] = E[\xi(\rho_1 S(T) - (1+\lambda)\rho_0 S(t))] \\ &= E\big[\xi\big(E[\rho_1 S(T)|\mathcal{F}(t)] - (1+\lambda)S(t)E[\rho_0|\mathcal{F}(t)]\big)\big]. \end{aligned}$$

From the arbitrariness of $\xi \geq 0$, we deduce the inequality of the right-hand side in (16.17), and a dual argument gives the inequality of the left-hand side, for given $t \in [0,T)$. Now all three processes in (16.17) have continuous paths; consequently, (16.17) is valid for *all* $t \in [0,T]$.

Next, we notice that (16.17) with $t = T$ implies $(1-\mu)\rho_0 \leq \rho_1 \leq (1+\lambda)\rho_0$, so that ρ_0, hence also ρ_1, is nonnegative. Similarly, (16.17) with $t = 0$ implies $(1-\mu)E\rho_0 \leq E[\rho_1 S(T)] \leq (1+\lambda)E\rho_0$, and therefore, since (ρ_0, ρ_1) is not equal to $(0,0)$, $E\rho_0 > 0$, hence also $E[\rho_1 S(T)] > 0$. This proves (16.18). $\diamond$

16.3 Example: Consider the *European call option* of (14.7). From (16.8) with $y = 0$, we have

(16.21)
$$h(C_0, C_1) \equiv h(C_0, C_1; 0) = \sup_{\mathcal{D}} E\left[Z_1(T)\mathbf{1}_{\{S(T)>q\}} - q\frac{Z_0(T)}{B(T)}\mathbf{1}_{\{S(T)>q\}}\right],$$

and therefore, $h(C_0, C_1) \leq \sup_{\mathcal{D}} EZ_1(T) = \sup_{\mathcal{D}} Z_1(0) \leq (1+\lambda)p$. The number $p(1+\lambda)$ corresponds to the cost of the "buy-and-hold strategy", of acquiring one share of the stock at $t = 0$ (at a price $p(1+\lambda)$, due to the transaction cost), and holding on to it until $t = T$. Davis & Clark (1993) conjectured that this hedging strategy is actually the cheapest:

$$h(C_0, C_1) = (1+\lambda)p. \tag{16.22}$$

The conjecture (16.22) was proved by Soner, Shreve & Cvitanić (1995), as well as by Levental & Skorohod (1995). It is an open question *to derive (16.22) directly from the representation (16.21)*; in other words, to find a sequence $\{(Z_0^{(n)}(\cdot), Z_1^{(n)}(\cdot))\}_{n\in\mathbf{N}}$ with

$$\mathbf{P_0}^{(n)}[S(T) > q] \to 0, \quad E[Z_1^{(n)}(T)\mathbf{1}_{\{S(T)>q\}}] \to 1, \quad Z_1^{(n)}(0) \to 1+\lambda,$$

as $n \uparrow \infty$.

17. MAXIMIZING EXPECTED UTILITY FROM TERMINAL WEALTH.

Consider now a small investor, who can make decisions in the context of the market model of (13.1), (13.2) as described in section 13, and who derives utility $U(X(T+))$ from his *terminal wealth*

$$X(T+) := X(T) + f(Y(T)), \text{ where } f(u) := \begin{Bmatrix} (1+\lambda)u \ ; \ u \le 0 \\ (1-\mu)u \ ; \ u > 0 \end{Bmatrix}. \tag{17.1}$$

In other words, this agent liquidates at the end of the day his position in the stock, incurs the appropriate transaction cost, and collects all the money in the bank-account. For a given initial holding $y \ge 0$ in the stock, his optimization problem is to find an admissible pair $(\hat{L}, \hat{M}) \in \mathcal{A}^+(x, y)$ that *maximizes expected utility from terminal wealth*, i.e., attains the supremum

$$V(x; y) := \sup_{(L,M) \in \mathcal{A}^+(x,y)} EU\big(X^{x,L,M}(T) + f(Y^{y,L,M}(T))\big), \ 0 < x < \infty, \tag{17.2}$$

where $\mathcal{A}^+(x, y)$ is the class of processes $(L, M) \in \mathcal{A}(x, y)$ for which $X^{x,L,M}(T) + f(Y^{y,L,M}(T)) \ge 0$. It can be shown, using standard convex/functional analysis arguments, that the supremum of (17.2) is attained, i.e., that there exists an optimal pair $(\hat{L}, \hat{M})$ for this problem, and that $V(x, y) < \infty$. Our purpose in this section is to describe the nature of this optimal pair, by using results of section 16 in the context of the *dual problem*

$$\tilde{V}(\zeta; y) := \inf_{(Z_0, Z_1) \in \mathcal{D}} E\left[\tilde{U}\left(\zeta \frac{Z_0(T)}{B(T)}\right) + \frac{y}{p}\zeta Z_1(T)\right], \ 0 < \zeta < \infty, \tag{17.3}$$

under the following assumption.

17.1 Assumption: There exists a pair $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot)) \in \mathcal{D}$, that attains the infimum in (17.3), and does so for all $0 < \zeta < \infty$. Moreover, for all $0 < \zeta < \infty$, we have

$$\tilde{V}(\zeta; y) < \infty \quad \text{and} \quad E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\frac{\zeta Z_0^*(T)}{B(T)}\right)\right] < \infty.$$

17.2 Remark: The assumption that the infimum of (17.3) is attained is a big one; we have not yet been able to obtain a general existence result to this effect, only very simple examples that can be solved explicitly. The assumption that the minimization in (17.3) can be carried out for all $0 < \zeta < \infty$ simultaneously, is made only for simplicity; it can be dispensed with using methods analogous to those in previous sections. Note, however, that this latter assumption is satisfied if $y = 0$ and either $U(x) = \log x$ or $U(x) = \frac{1}{\delta}x^\delta$ for $0 < \delta < 1$. It should also be mentioned that the optimal pair $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot))$ of Assumption 17.1 need not be unique; thus, in the remainder of this section, $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot))$ will denote *any* pair that attains the infimum in (17.3), as in Assumption 17.1.

For any such pair, we have then the following property, proved similarly as the corresponding result in the utility maximization under constraints.

17.3 Lemma: *Under the Assumption 17.1 and the condition*

$$xU'(x) \le a + (1-b)U(x), \quad 0 < x < \infty,$$

for some $a \ge 0$, $0 < b \le 1$, we have
(17.4)
$$E\left[\frac{Z_0(T)}{B(T)} I\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - \frac{y}{p} Z_1(T)\right] \le E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\zeta \frac{\hat{Z}_0(T)}{B(T)}\right) - \frac{y}{p} \hat{Z}_1(T)\right] < \infty,$$

for all $0 < \zeta < \infty$, for every $(Z_0(\cdot), Z_1(\cdot))$ in $\mathcal{D}$.

Now, because the function $\zeta \mapsto E\left[\frac{\hat{Z}_0(T)}{B(T)} I(\zeta \frac{\hat{Z}_0(T)}{B(T)})\right] : (0,\infty) \to (0,\infty)$ is continuous and strictly decreasing, there exists a unique $\hat{\zeta} = \hat{\zeta}(x; y, U) \in (0,\infty)$ that satisfies

$$E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\hat{\zeta} \frac{\hat{Z}_0(T)}{B(T)}\right)\right] = x + \frac{y}{p} E\hat{Z}_1(T). \tag{17.5}$$

And with

$$\hat{C}_0 := I\left(\hat{\zeta} \frac{\hat{Z}_0(T)}{B(T)}\right), \quad \hat{C}_1 := 0, \tag{17.6}$$

it follows from (17.4) that

$$\begin{aligned} \sup_{(Z_0,Z_1)\in\mathcal{D}} & E\Big[Z_0(T)\frac{\hat{C}_0}{B(T)} + Z_1(T)\Big(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\Big)\Big] \\ &= E\Big[\hat{Z}_0(T)\frac{\hat{C}_0}{B(T)} + \hat{Z}_1(T)\Big(\frac{\hat{C}_1}{S(T)} - \frac{y}{p}\Big)\Big] \\ &= x. \end{aligned} \tag{17.7}$$

Consequently, if in addition we have $\hat{C}_0 \in \mathbf{L}_2^*$, then Theorem 16.2 gives $h(\hat{C}_0, \hat{C}_1; y) = x$. Now it can be shown (see Cvitanić & Karatzas (1996)) that *the infimum in (16.1) is actually attained*; in other words, there exists a pair $(\hat{L}, \hat{M}) \in \mathcal{A}(x,y)$ such that, with $\hat{X}(\cdot) \equiv X^{x,\hat{L},\hat{M}}(\cdot)$, $\hat{Y}(\cdot) \equiv Y^{y,\hat{L},\hat{M}}(\cdot)$, we have

$$\hat{X}(T) + (1-\mu)\hat{Y}(T) \ge \hat{C}_0, \quad \hat{X}(T) + (1+\lambda)\hat{Y}(T) \ge \hat{C}_0. \tag{17.8}$$

17.4 Theorem: (Cvitanić & Karatzas (1996)) *Under assumptions of Lemma 17.3 and the condition*

$$E_0^*[\hat{C}_0^2] = E_0^*\left[I^2(\hat{\zeta}\hat{Z}_0(T)/B(T))\right] < \infty, \tag{17.9}$$

the above pair $(\hat{L}, \hat{M}) \in \mathcal{A}(x,y)$ is optimal for the problem of (17.2), and satisfies

$$\hat{X}(T+) := \hat{X}(T) + f(\hat{Y}(T)) = I(\hat{\zeta}\hat{Z}_0(T)/B(T)) = \hat{C}_0 \tag{17.10}$$

(17.11) $\hat{L}(\cdot)$ *is flat off the set* $\{0 \le t \le T / \hat{R}(t) = 1 + \lambda\}$

(17.12) $\hat{M}(\cdot)$ *is flat off the set* $\{0 \le t \le T / \hat{R}(t) = 1-\mu\}$

$$\frac{\hat{X}(t)+\hat{R}(t)\hat{Y}(t)}{B(t)} = \hat{E}_0\left[\frac{I(\hat{\zeta}\hat{Z}_0(T)/B(T))}{B(T)}\middle|\mathcal{F}(t)\right],\ 0 \le t \le T, \tag{17.13}$$

where $\hat{R}(\cdot) := \frac{\hat{Z}_1(\cdot)}{\hat{Z}_0(\cdot)P(\cdot)}$. *Furthermore, we have* $\tilde{V}(\hat{\zeta}; y) = V(x;y) - x\hat{\zeta} < \infty$.

Proof: As we just argued, (17.9) and Theorem 16.2 imply the existence of a pair $(\hat{L}, \hat{M}) \in \mathcal{A}(x,y)$, so that (17.8) is satisfied; and from (17.8), we know that both

$$\hat{X}(T)+\hat{R}(T)\hat{Y}(T) \ge \hat{C}_0, \quad \hat{X}(T) + f(\hat{Y}(T)) \ge \hat{C}_0 \tag{17.14}$$

hold. On the other hand, (15.12) implies that the process

$$\frac{\hat{X}(\cdot)+\hat{R}(\cdot)\hat{Y}(\cdot)}{B(\cdot)} \text{ is a } \hat{\mathbf{P}}_{\mathbf{0}} - \text{supermartingale.} \tag{17.15}$$

Therefore, from (17.5), (17.14) and (17.15) we have

$$\begin{aligned} x + \frac{y}{p}E\hat{Z}_1(T) &= E\left[\frac{\hat{Z}_0(T)}{B(T)} I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] = \hat{E}_0\left(\frac{\hat{C}_0}{B(T)}\right) \\ &\le \hat{E}_0\left(\frac{\hat{X}(T)+\hat{R}(T)\hat{Y}(T)}{B(T)}\right) \le x + \frac{y}{p}E\hat{Z}_1(T), \end{aligned} \tag{17.16}$$

whence

$$\hat{X}(T)+\hat{R}(T)\hat{Y}(T) = \hat{C}_0. \tag{17.17}$$

But now from (17.8), (17.14) we deduce $\hat{R}(T) = 1-\mu$ on $\{\hat{Y}(T) > 0\}$, and $\hat{R}(T) = 1+\lambda$ on $\{\hat{Y}(T) < 0\}$; thus

$$\begin{aligned} \hat{C}_0 &= \hat{X}(T)+\hat{R}(T)\hat{Y}(T) \\ &= \hat{X}(T) + \hat{Y}(T)[(1+\lambda)\mathbf{1}_{\{\hat{Y}(T)\le 0\}} + (1-\mu)\mathbf{1}_{\{\hat{Y}(T)>0\}}] = \hat{X}(T) + f(\hat{Y}(T)), \end{aligned}$$

and (17.10) follows.

It develops from (17.15), (17.16) that the process $\frac{\hat{X}(\cdot)+\hat{R}(\cdot)\hat{Y}(\cdot)}{B(\cdot)}$ is a $\hat{\mathbf{P}}_{\mathbf{0}}-$super -martingale with constant expectation, thus a $\hat{\mathbf{P}}_{\mathbf{0}}-$martingale; from this and (17.17), we obtain (17.13), as well as the fact that this process is nonnegative, hence that the $\hat{\mathbf{P}}_{\mathbf{0}}-$local martingale

$$\frac{\hat{X}(t)+\hat{R}(t)\hat{Y}(t)}{B(t)} + \int_0^t \frac{1+\lambda-\hat{R}(s)}{B(s)}d\hat{L}(s) + \int_0^t \frac{\hat{R}(s)-(1-\mu)}{B(s)}d\hat{M}(s),\ 0 \le t \le T \tag{17.18}$$

is also nonnegative. Hence, the process of (17.18) is a $\hat{\mathbf{P}}_{\mathbf{0}}-$supermartingale with $\hat{\mathbf{P}}_{\mathbf{0}}-$expectation at most $x + \frac{y}{p}E\hat{Z}_1(T)$ at $t = T$; but this is equal to the

$\hat{\mathbf{P}}_{\mathbf{0}}-$expectation of $\frac{\hat{X}(T)+\hat{R}(T)\hat{Y}(T)}{B(T)}$ by (17.16), whence the nonnegative terms

$$\int_0^T \frac{1+\lambda-\hat{R}(s)}{B(s)}d\hat{L}(s)\ ,\quad \int_0^T \frac{\hat{R}(s)-(1-\mu)}{B(s)}d\hat{M}(s)$$

must have $\hat{\mathbf{P}}_{\mathbf{0}}-$expectation equal to zero. The claims (17.11), (17.12) follow.

Now for the optimality of the pair $(\hat{L},\hat{M})$: we have from (17.10) and (17.5)

$$\begin{aligned} EU(\hat{X}(T)+f(\hat{Y}(T)))) &= EU(\hat{C}_0) = EU\left(I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right) \\ &= E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}E\left[\frac{\hat{Z}_0(T)}{B(T)}I\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)\right] \\ &= E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}x+\hat{\zeta}\frac{y}{p}E\hat{Z}_1(T)=\tilde{V}(\hat{\zeta};y)+x\hat{\zeta}. \end{aligned} \tag{17.19}$$

Consider also the holdings processes $X(\cdot)\equiv X^{x,L,M}(\cdot)$, $Y(\cdot)\equiv Y^{y,L,M}(\cdot)$ corresponding to an *arbitrary* strategy $(L,M)\in\mathcal{A}(x,y)$. We have

$$U(X(T)+(1-\mu)Y(T))\le \tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}[X(T)+(1-\mu)Y(T)]$$

$$U(X(T)+(1+\lambda)Y(T))\le \tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}[X(T)+(1+\lambda)Y(T)]$$

and thus, in conjunction with Remark 15.6, (16.6) and (15.12),

(17.20)

$$\begin{aligned} EU(X(T)+f(Y(T))) &\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\hat{E}_0\left(\frac{X(T)+\hat{R}(T)Y(T)}{B(T)}\right) \\ &\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}\,\underline{\lim}_{n\to\infty}\,\hat{E}_0^{(n)}\left(\frac{X(T)+\hat{R}^{(n)}(T)Y(T)}{B(T)}\right) \\ &\le E\tilde{U}\left(\hat{\zeta}\frac{\hat{Z}_0(T)}{B(T)}\right)+\hat{\zeta}(x+\frac{y}{p}E\hat{Z}_1(T)) \\ &= \tilde{V}(\hat{\zeta};y)+x\hat{\zeta}. \end{aligned}$$

The optimality of $(\hat{L},\hat{M})\in\mathcal{A}(x,y)$ for the problem of (17.2), as well as the equality $V(x;y)=\tilde{V}(\hat{\zeta};y)+x\hat{\zeta}$, follow now directly from (17.19) and (17.20). $\diamond$

Notice that, if $r(\cdot)$ is deterministic, then Jensen's inequality gives

$$\text{(17.21)} \qquad \begin{aligned} E\left[\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right)+\frac{y}{p}\zeta Z_1(T)\right] &\geq \tilde{U}\left(\frac{\zeta}{B(T)}EZ_0(T)\right)+\frac{y}{p}\zeta Z_1(0) \\ &\geq \tilde{U}\left(\frac{\zeta}{B(T)}\right)+y\zeta(1-\mu), \end{aligned}$$

for all $(Z_0(\cdot), Z_1(\cdot)) \in \mathcal{D}$. We shall use this observation to find examples, in which the optimal strategy $(\hat{L}, \hat{M})$ of Theorem 17.4 never trades.

17.5 Example: $r(\cdot)$ *deterministic,* $y = 0$. In this case we see from (17.21) that

$$\tilde{V}(\zeta;0) = \inf_{(Z_0,Z_1)\in\mathcal{D}} E\tilde{U}\left(\zeta\frac{Z_0(T)}{B(T)}\right) \geq \tilde{U}(\zeta/B(T)),$$

and the infimum is achieved by taking $\hat{Z}_0(\cdot) \equiv 1$, i.e., by any pair $(1, \hat{Z}_1(\cdot)) \in \mathcal{D}$ that satisfies $1-\mu \leq \hat{R}(\cdot) = \hat{Z}_1(\cdot)/P(\cdot) \leq 1+\lambda$, if such exists. In particular, one can take $\hat{Z}_1(0) = (1+\lambda)p$ and $\hat{\theta}_1(\cdot) \equiv \sigma(\cdot)$, in which case $(1, \hat{Z}_1(\cdot)) \in \mathcal{D}$ if and only if

$$\text{(17.22)} \qquad 0 \leq \int_0^t (b(s)-r(s))ds \leq \log\frac{1+\lambda}{1-\mu} \ , \forall\ 0 \leq t \leq T.$$

Furthermore, from (17.10) and (17.5),(17.6) we have

$$\hat{X}(T) + f(\hat{Y}(T)) = I(\hat{\zeta}/B(T)) = \hat{C}_0 = xB(T).$$

All the conditions (17.4), (17.9) and the Assumption 17.1 are satisfied rather trivially; and the *no-trading-strategy* $\hat{L} \equiv 0$, $\hat{M} \equiv 0$ is optimal, from Theorem 17.4 (and gives $\hat{X}(T) = xB(T)$, $\hat{Y}(T) = 0$). The condition (17.22) is satisfied, for instance, if

$$\text{(17.23)} \qquad r(\cdot) \leq b(\cdot) \leq r(\cdot) + \rho \ , \ \text{for some } 0 \leq \rho \leq \frac{1}{T}\log\frac{1+\lambda}{1-\mu}.$$

If $b(\cdot) = r(\cdot)$ the result is not surprising – even without transaction costs, it is then optimal not to trade. However, for $b(\cdot) > r(\cdot)$ the optimal portfolio always invests a positive amount in the stock, if there are no transaction costs; the same is true even in the presence of transaction costs, if one is maximizing expected discounted utility from consumption over an infinite time-horizon, and if the market coefficients are constant – see Shreve & Soner (1994), Theorem 11.6.

The situation here, on the finite time-horizon $[0, T]$, is quite different: if the excess rate of return $b(\cdot) - r(\cdot)$ is positive but small relative to the transaction costs, and/or if the time-horizon is small, in the sense of (17.23), then it is optimal not to trade.

Remark: In the infinite time-horizon case with constant market coefficients, as in Davis & Norman (1990), Shreve & Soner (1994), the ratio $\hat{X}/\hat{Y}$ of optimal holdings is a reflected diffusion process in a fixed interval; more precisely, one trades only when this ratio hits the endpoints of the interval, and in such a way as to keep the ratio inside the interval. In our case, under the assumptions of Example 17.5, and with $U(x) = \log x$, one obtains from (17.13) that

$$(\hat{X}(t) + \hat{R}(t)\hat{Y}(t))/B(t) = (\hat{\zeta}\hat{Z}_0(t))^{-1}, \quad 0 \le t \le T.$$

Comparing the stochastic integral representation of $(\hat{\zeta}\hat{Z}_0(\cdot))^{-1}$ with the equation (15.8), one obtains

$$\frac{\hat{R}(t)\hat{Y}(t)}{B(t)}(\hat{\theta}_1(t) - \hat{\theta}_0(t)) = -\frac{\hat{\theta}_0(t)}{\hat{\zeta}\hat{Z}_0(t)}, \quad 0 \le t \le T.$$

The last two equations imply

$$\hat{X}(t)/\hat{Y}(t) = -\hat{R}(t)\left(\frac{\hat{\theta}_1(t)}{\hat{\theta}_0(t)}\right), \quad 0 \le t \le T,$$

provided $\hat{Y}(t)\hat{\theta}_0(t) \neq 0$, $\forall\, t \in [0,T]$. While $\hat{R}(\cdot)$ is a reflected process in a fixed interval, it is not clear what happens to the second factor, either for fixed T or as $T \to \infty$.

17.6 Remark: Assuming $y = 0$, it can be argued, using the arguments as in the case of constraints, that the utility based price $V(0)$ of a claim $C = (C_0, C_1)$ in our setting should be the expected value of the discounted claim evaluated under the optimal shadow state-price densities of the dual problem, i.e., by

$$V(0) = E\left[\hat{Z}_0(T)\frac{C_0}{B(T)} + \hat{Z}_1(T)\frac{C_1}{S(T)}\right],$$

provided that the dual optimization problem of (17.3) has a *unique* solution $(\hat{Z}_0(\cdot), \hat{Z}_1(\cdot))$. Notice that this price does not depend on the initial holdings x in stock, if the solution to the dual problem does not depend on ζ, as in Assumption 17.1. However, $V(0)$ *does* depend in general on the return rate of the stock $b(\cdot)$.

18. ON PORTFOLIO OPTIMIZATION UNDER "DRAWDOWN" CONSTRAINTS.

Let us consider again the standard model of a financial market $\mathcal{M}$, as in Section 2. We give a different representation for the wealth process of a financial agent with initial capital $x > 0$:

(18.1)

$$dX^\pi(t) = r(t)X^\pi(t)dt + \left(X^\pi(t) - \frac{\alpha M^\pi(t)}{\gamma_0(t)}\right)\pi^*(t)\left[(b(t) - r(t)\mathbf{1})dt + \sigma(t)dW(t)\right]$$

$$X^\pi(0) = x,$$

and we require that it satisfies the "drawdown constraint"

$$P\left[\gamma_0(t)X^\pi(t) > \alpha M^\pi(t), \quad \forall 0 \le t < \infty\right] = 1. \tag{18.2}$$

Here $\alpha \in (0,1)$ is a given constant, and

$$M^\pi(t) := \max_{0\le s\le t}(\gamma_0(s)X^\pi(s)). \tag{18.3}$$

The interpretation is this: the agent does not tolerate the "drawdown $1 - \frac{\gamma_0(t)X^\pi(t)}{M^\pi(t)}$ of his discounted wealth, from its maximum-to-date", to be greater than or equal to the constant $1-\alpha$, *at any time* $t \ge 0$; thus, he imposes the (almost sure) constraint (18.2).

More precisely, we say that a portfolio $\pi = (\pi_1, \dots, \pi_d)$ is admissible if $\pi_i(t)$ is the proportion of the difference $X^\pi(t) - \alpha\frac{M^\pi(t)}{\gamma_0(t)} > 0$ invested in the ith stock, $i = 1, \dots, d$, and the remainder is invested in the bank account; we also require that $\pi(\cdot)$ is measurable and adapted process, and that, for any $T > 0$, $\int_0^T \|\pi(t)\|^2 dt < \infty$. We denote by $\mathcal{A}_\alpha(x)$ the class of admissible portfolios.

18.1 Lemma: *If $\pi(\cdot) \in \mathcal{A}_\alpha(x)$, then (18.1) has a unique solution and (18.2) is satisfied.*

Proof: Without loss of generality we set $r(\cdot) \equiv 0$ and $\sigma(\cdot)$ equal to identity matrix. We therefore have to show that there is a unique solution to

$$dX(t) = (X(t) - \alpha M(t))\pi^*(t)dW_0(t), \quad M(t) = \max_{0\le s\le t} X(s); \quad X(0) = x, \tag{18.4}$$

that satisfies a.s.

$$X(t) > \alpha M(t), \quad \forall 0 \le t < \infty. \tag{18.5}$$

Suppose that $X(\cdot)$ is an adapted process that satisfies (18.4), (18.5). Observe that

$$d\left(\frac{X(t)}{M(t)} - \alpha\right) = \left(\frac{X(t)}{M(t)} - \alpha\right)\pi^*(t)dW_0(t) - \frac{dM(t)}{M(t)},$$

whence

$$d\left(\log\left(\frac{X(t)}{M(t)} - \alpha\right)\right) = d\xi(t) - \frac{1}{1-\alpha}\frac{dM(t)}{M(t)},$$

where

$$\xi(t) := \int_0^t \pi^*(s)dW_0(s) - \frac{1}{2}\int_0^t \|\pi(s)\|^2 ds.$$

Therefore,

$$0 \le R(t) := \log(1-\alpha) - \log\left(\frac{X(t)}{M(t)} - \alpha\right) = -\xi(t) + \log\left(\frac{M(t)}{x}\right)^{\frac{1}{1-\alpha}}.$$

Clearly, the continuous increasing process $K(t) := \log\left(\frac{M(t)}{x}\right)^{\frac{1}{1-\alpha}}$ is flat away from the set $\{t \ge 0 / X(t) = M(t)\}$, i.e., away from the zero-set of the continuous nonnegative process $R(\cdot)$. From the theory of the Skorohod equation (e.g. Karatzas & Shreve (1991), §3.6) we have then $K(t) = \max_{0\le s\le t}\xi(s)$, and from this:

$$M(t) \equiv \tilde{M}(t) := x\exp\left\{(1-\alpha)\max_{0\le s\le t}\xi(s)\right\},$$

$$X_t \equiv \tilde{X}_t := x \exp\left\{(1-\alpha) \max_{0\le s\le t} \xi_s\right\} \left[\alpha + (1-\alpha)\exp\left\{\xi_t - \max_{0\le s\le t} \xi_s\right\}\right].$$

It is straightforward to check that $\tilde{X}(\cdot)$ satisfies (18.4), (18.5).

18.2 Problem *(Grossman & Zhou (1993))*: For some given $0<\delta<1$, maximize the long-term rate of growth

$$\mathcal{R}(\pi) := \overline{\lim_{T\to\infty}} \frac{1}{T} \log E(X^\pi(T))^\delta \tag{18.6}$$

of expected power-utility, over $\pi \in \mathcal{A}_\alpha(x)$. In particular, compute

$$v(\alpha) := \sup_{\pi\in\mathcal{A}_\alpha(x)} \mathcal{R}(\pi) \tag{18.7}$$

and find $\pi \in \mathcal{A}_\alpha(x)$, for which the limit $\lim_{T\to\infty} \frac{1}{T}\log E(X^{\hat\pi}(T))^\delta = \mathcal{R}(\hat\pi)$ exists and achieves the supremum in (18.7).

In order to solve Problem 18.2, we introduce an auxiliary process and problem, as follows: For any portfolio process $\pi \in \mathcal{A}_\alpha(x)$, consider the auxiliary process

$$N_\alpha^\pi(t) := \left(X^\pi(t) - \alpha\frac{M^\pi(t)}{\gamma_0(t)}\right)(M^\pi(t))^{\frac{\alpha}{1-\alpha}}, \quad 0 \le t < \infty. \tag{18.8}$$

Because the increasing process $M^\pi(\cdot)$ is flat off the set $\{t\ge 0/\gamma_0(t)X^\pi(t) = M^\pi(t)\}$, we have:

$$d(\gamma_0(t)N_\alpha^\pi(t)) = (\gamma_0(t)N_\alpha^\pi(t))\pi^*(t)\sigma(t)dW_0(t), \tag{18.9}$$

Notice also, in the notation of the proof of Lemma 18.1, that

$$\gamma_0(t)N_\alpha^{\hat\pi}(t) = (1-\alpha)x^{\frac{1}{1-\alpha}}e^{\xi(t)}. \tag{18.10}$$

Recall also the notation

$$Z_0(t) := \exp\left\{-\int_0^t \theta^*(s)dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\}, \quad H_0(t) := \gamma_0(t)Z_0(t).$$

From the product rule we obtain: $d(H_0(t)N_\alpha^\pi(t)) = H_0(t)N_\alpha^\pi(t)(\pi^*(t)\sigma(t) - \theta^*(t))dW(t)$. In other words, for any $\pi\in\mathcal{A}_\alpha(x)$ the process

$$\begin{aligned} &H_0(t)N_\alpha^\pi(t) \\ &\quad = (1-\alpha)x^{\frac{1}{1-\alpha}}\exp\left\{\int_0^t (\pi^*\sigma-\theta^*)(s)dW(s) - \frac{1}{2}\int_0^t \|\pi^*\sigma-\theta^*\|^2(s)ds\right\} \end{aligned}$$

is a positive local martingale, hence supermartingale, which thus satisfies

$$E\left[H_0(T)N_\alpha^\pi(T)\right] \le (1-\alpha)x^{\frac{1}{1-\alpha}}, \quad \forall T\in(0,\infty). \tag{18.11}$$

We now pose an auxiliary stochastic control problem, involving the process $N_\alpha^\pi(\cdot)$.

18.3 An Auxiliary, Finite-Horizon, Control Problem: *For a given $T\in(0,\infty)$ and utility function $U:(0,\infty)\to\mathbb{R}$, denote by $\mathcal{A}_\alpha(x,T)$ the class of* *admissible portfolios* $\pi(\cdot)$ *on the finite horizon* $[0,T]$, *and find* $\hat{\pi}(\cdot) \in \mathcal{A}_\alpha(x,T)$ *which achieves*

$$V(\alpha;T,x) := \sup_{\pi \in \mathcal{A}_\alpha(x,T)} EU(N_\alpha^\pi(T)). \tag{18.12}$$

As we know from earlier sections, the solution is found as follows: Choose $\hat{y}$ such that

$$E[H_0(T)I(\hat{y}H_0(T))] = (1-\alpha)x^{1/1-\alpha}.$$

Choose portfolio $\hat{\pi} \in \mathcal{A}_\alpha(x)$ by introducing the positive martingale

$$\mathcal{Q}(t) := E[H_0(T)I(\hat{y}H_0(T))\|\mathcal{F}(t)] = (1-\alpha)x^{\frac{1}{1-\alpha}} + \int_0^t \mathcal{Q}(s)\varphi^*(s)dW(s),$$

for $0 \le t \le T$, and setting

$$\hat{\pi}(\cdot) = \left((\theta^* + \varphi^*)\sigma^{-1}\right)^*(\cdot) \in \mathcal{A}_\alpha(x,T).$$

18.4 Remark: Let us consider now Problem 18.2 with utility function

$$U(x) = \frac{1}{\gamma}x^\gamma \quad \text{for } \gamma := \delta(1-\alpha), \quad 0 < \delta < 1.$$

Then, with $\mu := \frac{\gamma}{1-\gamma}$, we get

(18.13)

$$\hat{y}^{-\frac{1}{1-\gamma}} = \frac{(1-\alpha)x^{\frac{1}{1-\alpha}}}{E[(H_0(T))^{-\mu}]}, \quad V(\alpha;T,x) = \frac{1}{\gamma}\left((1-\alpha)x^{\frac{1}{1-\alpha}}(E(H_0(T))^{-\mu})^{1/\mu}\right)^\gamma.$$

If, in addition, the coefficients $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ are deterministic, then

$$\begin{aligned}(H_0(t))^{-\mu} = &\exp\left[\mu\int_0^t \theta^*(s)dW(s) - \frac{\mu^2}{2}\int_0^t \|\theta(s)\|^2 ds\right] \\ &\times \exp\left\{\mu\int_0^t \left(r(s) + \frac{1+\mu}{2}\|\theta(s)\|^2\right)ds\right\}\end{aligned}$$

and we obtain

(18.14)

$$\mathcal{Q}(t) = (1-\alpha)x^{\frac{1}{1-\alpha}}\exp\left\{\mu\int_0^t \theta^*(s)dW(s) - \frac{\mu^2}{2}\int_0^t \|\theta(s)\|^2 ds\right\}, \quad \varphi(t) = \mu\theta(t)$$

$$\hat{\pi}^*(t)\sigma(t) = (1+\mu)\theta^*(t) = \frac{1}{1-\delta(1-\alpha)}\theta^*(t), \quad \text{independent of } T, \tag{18.15}$$

(18.16)

$$V(\alpha;T,x) = \frac{1}{\gamma}\left((1-\alpha)x^{\frac{1}{1-\alpha}}\exp\left\{\int_0^T \left(r(t) + \frac{1+\mu}{2}\|\theta(t)\|^2\right)dt\right\}\right)^\gamma.$$

In order to solve the original, Grossman-Zhou problem, we shall assume that the coefficients $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ are deterministic, and $r_* := \lim_{T\to\infty} \frac{1}{T}\int_0^T r(s)ds$, $\|\theta_*\|^2 := \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\theta(s)\|^2 ds$ exist and are finite.

18.5 Theorem: (Grossman & Zhou (1993), Cvitanić & Karatzas (1995)) *Under the above assumptions, the portfolio $\hat{\pi}(\cdot)$ of (18.15) is optimal for the Problem 18.2. In fact, we have*

$$\lim_{T\to\infty} \frac{1}{T} \log E(X^{\hat{\pi}}(T))^{\delta} = \mathcal{R}(\hat{\pi}) = v(\alpha) = V(\alpha) + \alpha\delta r_*, \tag{18.17}$$

where

$$\begin{aligned} V(\alpha) &:= \lim_{T\to\infty} \frac{1}{T} \log V(\alpha; T, x) = \gamma r_* + \frac{\gamma}{2}(1+\mu)\|\theta_*\|^2 \\ &= \delta(1-\alpha)\left[r_* + \frac{\|\theta_*\|^2}{2}\frac{1}{1-\delta(1-\alpha)}\right]. \end{aligned} \tag{18.18}$$

In order to establish this result, it will be helpful to consider the auxiliary problem

$$\bar{v}(\alpha) := \sup_{\pi\in\mathcal{A}_\alpha(x)} \bar{\mathcal{R}}_\alpha(\pi), \quad \bar{\mathcal{R}}_\alpha(\pi) := \varlimsup_{T\to\infty} \frac{1}{T} \log E(N^{\pi}(T))^{\delta(1-\alpha)}. \tag{18.19}$$

From the fact that the portfolio $\hat{\pi}(\cdot)$ of (18.15) does not depend on the horizon $T \in (0,\infty)$, it is clear that

$$\lim_{T\to\infty} \frac{1}{T} \log E(N^{\hat{\pi}}_\alpha(T))^{\delta(1-\alpha)} = \bar{\mathcal{R}}_\alpha(\hat{\pi}) = \bar{v}(\alpha) = V(\alpha). \tag{18.20}$$

It will also be helpful to note that

$$(N^{\pi}_\alpha(t))^{\delta(1-\alpha)} = (\gamma_0(t))^{\alpha\delta}(X^{\pi}(t))^{\delta}\left(f_\alpha\left(\frac{\alpha M^{\pi}(t)}{\gamma_0(t)X^{\pi}(t)}\right)\right)^{\delta}, \tag{18.21}$$

where the function $f_\alpha(x) := \left(\frac{x}{\alpha}\right)^{\alpha}(1-x)^{1-\alpha}$, $0 \le x \le 1$ is strictly increasing on $(0,\alpha)$ and strictly decreasing on $(\alpha, 1)$.

Proof of Theorem 18.5: From (18.21) we obtain

$$E(N^{\pi}_\alpha(T))^{\delta(1-\alpha)} \le (\gamma_0(T))^{\alpha\delta}(1-\alpha)^{\delta(1-\alpha)}E(X^{\pi}(T))^{\delta},$$

whence

$$\bar{\mathcal{R}}_\alpha(\pi) \le \mathcal{R}(\pi) - \alpha\delta r_* \le v(\alpha) - \alpha\delta r_*, \quad \forall \pi \in \mathcal{A}_\alpha(x)$$

and therefore $V(\alpha) \le v(\alpha) - \alpha\delta r_*$. In order to establish the reverse inequality, take $\eta \in (0,\alpha)$ close enough to α so that $f_\eta(\eta) \ge f_\eta(\eta/\alpha)$, and observe from (18.21) that for an arbitrary $\pi \in \mathcal{A}_\alpha(x)$ $(\subseteq \mathcal{A}_\eta(x))$ we have

$$\begin{aligned} E(N^{\pi}_\eta(T))^{\delta(1-\eta)} &\ge (\gamma_0(T))^{\eta\delta}\left(f_\eta(\eta/\alpha)\right)^{\delta} E(X^{\pi}(T))^{\delta} \\ &= (\gamma_0(T))^{\eta\delta}\left(\alpha^{-\eta}(1-\eta/\alpha)^{1-\eta}\right)^{\delta} E(X^{\pi}(T))^{\delta}. \end{aligned}$$

Consequently

$$V(\eta) \geq \bar{\mathcal{R}}_\eta(\pi) \geq \mathcal{R}(\pi) - \eta\delta r_*, \quad \forall \pi \in \mathcal{A}_\alpha(x),$$

whence $V(\eta) \geq v(\alpha) - \eta\delta r_*$; letting $\eta \uparrow \alpha$ and invoking the continuity of the function $V(\cdot)$, we obtain $V(\alpha) \geq v(\alpha) - \alpha\delta r_*$ and thus the third equality of (18.17):

$$v(\alpha) = V(\alpha) + \alpha\delta r_* = \delta r_* + \frac{\|\theta_*\|^2}{2} \frac{\delta(1-\alpha)}{1-\delta(1-\alpha)}.$$

To obtain the second equality, it suffices to observe that

$$v(\alpha) \geq \mathcal{R}(\hat{\pi}) \geq \bar{\mathcal{R}}_\alpha(\hat{\pi}) + \alpha\delta r_* = V(\alpha) + \alpha\delta r_* = v(\alpha).$$

Finally, the first equality, i.e., the existence of the indicated limit, follows from the double inequality

$$\begin{aligned}
&-\frac{\delta(1-\alpha)}{T}\log(1-\alpha) + \frac{\alpha\delta}{T}\int_0^T r(s)ds + \frac{1}{T}\log E(N^{\hat{\pi}}(T))^{\delta(1-\alpha)} \\
&\leq \frac{1}{T}\log E(X^{\hat{\pi}}(T))^\delta \\
&\quad \leq -\frac{\delta}{T}\log\left(\alpha^{-\eta}(1-\frac{\eta}{\alpha})^{1-\eta}\right) + \frac{\eta\delta}{T}\int_0^T r(s)ds + \frac{1}{T}\log E(N^{\hat{\pi}}(T))^{\delta(1-\eta)}
\end{aligned}$$

by passing to the limit as $T \to \infty$ and then letting $\eta \uparrow \alpha$. ◇

The methods above can also be used to show that the portfolio

$$\pi_*(t) = (\theta^*(t)\sigma^{-1}(t))^*, \quad 0 \leq t < \infty$$

is optimal for the problem of *maximizing the long-term rate of expected logarithmic utility* under the drawdown constraint:

(18.22)

$$\overline{\lim_{T\to\infty}} \frac{1}{T} E(\log X^\pi(T)) \leq \lim_{T\to\infty} \frac{1}{T} E(\log X^{\pi_*}(T)) = (1-\alpha)\left(\bar{r} + \frac{\|\bar{\theta}\|^2}{2}\right) + \alpha\bar{r},$$

for all $\pi \in \mathcal{A}_\alpha(x)$ (with $\bar{r}, \|\bar{\theta}\|^2$ defined below). It turns out that this holds for *general random, adapted coefficients* $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$, for which the conditions of the model are satisfied and the limits

$$\bar{r} := \lim_{T\to\infty} \frac{1}{T}\int_0^T Er(t)dt, \quad \|\bar{\theta}\|^2 := \lim_{T\to\infty}\frac{1}{T}\int_0^T E\|\theta(t)\|^2 dt$$

exist and are finite.

More important than the optimality property (18.22), however, is the fact that the portfolio $\pi_*(\cdot)$ *maximizes the long-term growth rate from investment*

(18.23)

$$\mathcal{S}(\pi) := \overline{\lim_{T\to\infty}} \frac{1}{T}\log X^\pi(T) \leq \lim_{T\to\infty}\frac{1}{T}\log X^{\pi_*}(T) = (1-\alpha)\left(r^* + \frac{\|\theta^*\|^2}{2}\right) + \alpha r^*,$$

over all $\pi \in \mathcal{A}_\alpha(x)$. Again, this comparison is valid for *general random, adapted coefficients* in the model, under the proviso that the limits

$$r^* := \lim_{T\to\infty} \frac{1}{T}\int_0^T r(t)dt, \quad \|\theta^*\|^2 := \lim_{T\to\infty} \frac{1}{T}\int_0^T \|\theta(t)\|^2 dt$$

exist and are finite, almost surely.

In order to prove this, let us start by noticing that $\Lambda(t) := N_\alpha^\pi(t)/N_\alpha^{\pi_*}(t)$, $0 \le t < \infty$ satisfies the stochastic equation

$$d\Lambda(t) = \Lambda(t)(\pi^*(t)\sigma(t) - \theta^*(t))dW(t), \quad \Lambda(0) = 1$$

and is thus a positive supermartingale, for any $\pi \in \mathcal{A}_\alpha(x)$. It follows readily from this (see Karatzas (1989), p.1243)) that $\overline{\lim}_{t\to\infty} \frac{1}{t}\log\Lambda(t) \le 0$, or equivalently (18.24)

$$\begin{aligned}\bar{\mathcal{S}}_\alpha(\pi) := \overline{\lim_{T\to\infty}} \frac{1}{T}\log(N_\alpha^\pi(T))^{1-\alpha} &\le \lim_{T\to\infty}\frac{1}{T}\log(N^{\pi_*}(T))^{1-\alpha} = \\ &= (1-\alpha)\left(r^* + \frac{\|\theta^*\|^2}{2}\right) =: \bar{s}(\alpha), \text{ a.s.}\end{aligned}$$

The existence of this last limit, and its value follows from (18.10). On the other hand, similarly as above, one gets

$$\begin{aligned}&\frac{1}{T}\log(N_\alpha^\pi(T))^{1-\alpha} + \frac{\alpha}{T}\int_0^T r(s)ds - \frac{1-\alpha}{T}\log(1-\alpha) \le \frac{1}{T}\log X^\pi(T) \\ &\le \frac{1}{T}\log(N_\eta^\pi(T))^{1-\eta} + \frac{\eta}{T}\int_0^T r(s)ds - \frac{1}{T}\log\left(\alpha^{-\eta}(1-\eta/\alpha)^{1-\eta}\right)\end{aligned} \tag{18.25}$$

almost surely, for any $\pi \in \mathcal{A}_\alpha(x) \subseteq \mathcal{A}_\eta(x)$ and any $\eta \in (0,\alpha)$ sufficiently close to α. In particular,

$$\bar{\mathcal{S}}_\alpha(\pi) + \alpha r^* \le s(\alpha) := \operatorname{esssup}_{\pi\in\mathcal{A}_\alpha(x)} \lim \mathcal{S}(\pi), \text{ a.s.}$$

whence $\bar{s}(\alpha) + \alpha r^* \le s(\alpha)$, *a.s.*; similarly,

$$\mathcal{S}(\pi) - \eta r^* \le \bar{\mathcal{S}}_\eta(\pi) \le \bar{s}(\eta), \text{ whence } s(\alpha) - \eta r^* \le \bar{s}(\eta)$$

and in the limit as $\eta \uparrow \alpha : s(\alpha) - \alpha r^* \le \bar{s}(\alpha)$, a.s. It develops that $s(\alpha) = \bar{s}(\alpha) + \alpha r^* = (1-\alpha)\left(r^* + \frac{1}{2}\|\theta^*\|^2\right) + \alpha r^*$, and it remains to show the existence of the limit and the equality in (18.23). But both of these follow by writing the double inequality (18.25) with $\pi \equiv \pi_*$, letting $T \to \infty$ to obtain in conjunction with (18.24)

$$s(\alpha) = \bar{s}(\alpha) + \alpha r^* \le \underline{\lim}_{T\to\infty}\frac{1}{T}\log X^{\pi_*}(T) \le \overline{\lim_{T\to\infty}}\frac{1}{T}\log X^{\pi_*}(T) \le s(\eta),$$

and then letting $\eta \uparrow \alpha$ to conclude $\lim_{T\to\infty}\frac{1}{T}\log X^{\pi_*}(T) = s(\alpha)$, almost surely.

A. APPENDIX.

Most of the results in this section are taken from Karatzas & Shreve (1991) and Protter (1990).

A.1 Definition: A real valued continuous process $W^{(1)}(\cdot)$ is called a *standard Brownian motion* if $W^{(1)}(0) = 0$, if it has independent increments, and, for all $u, t \geq 0$, and the law of the increment $W^{(1)}(t) - W^{(1)}(u)$ is normal with mean zero and variance $t - u$. A vector process W of d independent Brownian motions, $W = (W^{(1)}, \ldots, W^{(d)})^*$, where a^* is vector a transposed, is a *standard Brownian motion in* $\mathbb{R}^d$. These are defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$, and we shall denote by $\{\mathcal{F}_t\}$ the $\mathbf{P}$-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s);\ 0 \leq s \leq t)$ generated by W. All our processes $X(\cdot)$ (unless otherwise mentioned) will be *adapted* to the filtration $\{\mathcal{F}_t\}$, i.e., for all $t \geq 0$, $X(t)$ is $\mathcal{F}_t-$measurable random variable. They will also always be *right-continuous* (sometimes *left-continuous*).

Brownian motion in $\mathbb{R}$ is a *martingale*, namely

$$E[W^{(1)}(t)|\mathcal{F}_s] = W(s), \quad s \leq t. \tag{A.1}$$

Processes X for which we have $\leq$ (resp., $\geq$) instead of equality in (A.1) are called *supermartingales* (resp., *submartingales*) with respect to the filtration $\{\mathcal{F}_t\}_{0 \leq t \leq \infty}$. If the property holds only for processes $X^{(n)}(t) = X(t \wedge \tau_n)$, for each n, where τ_n is a nondecreasing sequence of *stopping times* converging to infinity, then we say that $X(\cdot)$ is a *local martingale* (local super/sub-martingale). A stopping time τ is a nonnegative random variable for which $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \geq 0$.

A.2 Lemma: *A local martingale bounded from below is a supermartingale.*

Proof: Fatou's lemma for conditional expectations.

A.3 Theorem (Doob-Meyer decomposition): *A supermartingale X can be uniquely decomposed as*

$$X = X(0) + M - A \tag{A.2}$$

where M is a local martingale, A is nondecreasing (locally natural) process, $EA(T) < \infty$, $M(0) = A(0) = 0$. If X is also positive and the family $\{X(\tau) : \tau$ a stopping time bounded by $\alpha\}$ *is uniformly integrable for all $\alpha > 0$ (we say that $X(\cdot)$ is of class DL), then M is a martingale.*

From now on (unless otherwise mentioned) we restrict ourselves to the finite time-horizon $[0, T]$, $T < \infty$. For a given $\mathbb{R}^d-$valued adapted process $\pi(\cdot)$ with $\int_0^T |\pi(s)|^2 ds < \infty$, it is possible to define *Ito stochastic integral* $\int_0^t \pi(s) dW(s)$, which is a continuous local martingale process. If, moreover, $E\int_0^T |\pi(s)|^2 ds < \infty$, then $E[\int_0^t \pi(s) dW(s)]^2 = E\int_0^t \pi^2(s) ds$. Integrals with respect to finite variation processes are defined ω by ω in the usual way.

A.4 Theorem (martingale representation theorem): *Let $M(\cdot)$ be an $\mathcal{F}_t-$ adapted local martingale with RCLL (right-continuous with left limits) paths and $M(0) = 0$. Then there exist an $\mathbb{R}^d$-valued process $\varphi(\cdot)$ such that*

$$M(t) = \int_0^t \varphi(s) dW(s), \quad \int_0^T |\varphi(s)|^2 ds < \infty.$$

In particular, M has to be continuous. Moreover, if $EM^2(T) < \infty$, then

$$E\int_0^T |\varphi(s)|^2 ds < \infty,$$

and $M(\cdot)$ is a martingale. If $\tilde{\varphi}(\cdot)$ is another such a process, then

$$\int_0^T |\varphi(s) - \tilde{\varphi}(s)|^2 ds = 0.$$

1.5 Definition: A *semimartingale* $X(\cdot)$ is a process of the form

$$X = X(0) + M + A,$$

where $M(t) = M(0) + \int_0^t \varphi(s)dW(s)$ is a local martingale and A is a process of finite total variation. Its *quadratic variation process* is given by

$$< X, X > (t) =< M, M > (t) := \int_0^t \varphi^2(s) ds.$$

A *cross-variation* of semimartingales X and $\tilde{X}$ (with the corresponding representation) is given by

$$< X, \tilde{X} > (t) := \int_0^t \varphi^*(s)\tilde{\varphi}(s) ds.$$

In particular, cross-variation of process A of finite variation and any other semimartingale is zero.

A.6 Theorem (Ito's rule): Let $X = (X_1, \dots, X_k)$ be a vector of continuous semimartingales of the form

$$X_i(t) = X_i(0) + \int_0^t \varphi_i(s) dW(s) + A_i(t)$$

and let $g : \mathbb{R}^k \mapsto \mathbb{R}$ be twice continuously differentiable function. Then

$$\begin{aligned} g(X(t)) =& g(X(0)) + \sum_{i=1}^k \int_0^t \frac{\partial g}{\partial x_i}(X(u)) dX_i(u) \\ &+ \frac{1}{2} \sum_{i,j=1}^k \int_0^t \frac{\partial^2 g}{\partial x_i \partial x_j}(X(u)) \varphi_i^*(u) \varphi_j(u) du. \end{aligned}$$

In particular,

$$\begin{aligned} X_1(t)X_2(t) =& X_1(0)X_2(0) + \int_0^t X_1(u)dX_2(u) + \int_0^t X_2(u)dX_1(u) \\ &+ \int_0^t \varphi_1^*(u)\varphi_2(u)du. \end{aligned}$$

A.7 Proposition: *Consider the following one-dimensional linear Stochastic Differential Equation with possibly random, locally bounded coefficients* A, a, S_j, σ_j*:*

$$dX(t) = [A(t)X(t) + a(t)]dt + \sum_{j=1}^{d}[S_j(t)X(t) + \sigma_j(t)]dW^{(j)}(t),$$

which is understood as $X(t) = X(0) + \int_0^t \dots ds + \int_0^t \dots dW^{(j)}(s)$. *Its unique solution is given by*

$$X_t = Y_t \left[X_0 + \int_0^t \frac{1}{Y_u}\{a(u) - \sum_{j=1}^{d} S_j(u)\sigma_j(u)\}du + \sum_{j=1}^{d}\int_0^t \frac{\sigma_j(u)}{Y_u}dW_u^{(j)} \right]$$

where

$$Y(t) = \exp\{\int_0^t A(u)du\}Z(t),$$

and

$$Z(t) = \exp\{\sum_{j=1}^{d}\int_0^t S_j(u)dW^{(j)}(u) - \frac{1}{2}\sum_{j=1}^{d}\int_0^t S_j^2(u)du\}. \tag{A.3}$$

Notice then, denoting $S = (S_1, \dots, S_d)$, that Z itself is a solution to

$$dZ(t) = S(t)Z(t)dW(t), \quad Z(0) = 1. \tag{A.4}$$

Therefore, $Z(\cdot)$ is a positive local martingale, hence a supermartingale. If it is also a martingale, then one can define a new probability measure $\tilde{\mathbf{P}}$ by $\frac{d\tilde{\mathbf{P}}}{d\mathbf{P}} = Z(T)$. We have the following *Bayes rule*:

A.8 Lemma: *If* $0 \le s \le t \le T$ *and* Y *is an* $\mathcal{F}_t$*-measurable random variable satisfying* $\tilde{E}|Y| < \infty$ *(where* $\tilde{E}$ *is the expectation under the* $\tilde{\mathbf{P}}$ *measure), then*

$$\tilde{E}[Y|\mathcal{F}_s] = \frac{1}{Z(s)}E[YZ(t)|\mathcal{F}_s].$$

A.9 Theorem (Girsanov-Cameron-Martin): *Define a process* $\tilde{W}$ *by*

$$\tilde{W}^{(i)}(t) := W^{(i)}(t) - \int_0^t S_i(s)ds.$$

Then the process $\{\tilde{W}(t), \mathcal{F}_t; 0 \le t \le T\}$ *is a* $d-$*dimensional Brownian motion on* $(\Omega, \mathcal{F}_T, \tilde{\mathbf{P}})$.

A.10 Remark: In general, the filtration $\{\tilde{\mathcal{F}}_t\}$ corresponding to $\tilde{W}$ will not be equal to the filtration $\{\mathcal{F}_t\}$. Nevertheless, one can show, using the Bayes rule, that the martingale representation theorem still holds, namely that the $\tilde{\mathbf{P}}$-(local) martingales can be represented as stochastic integrals with respect to $\tilde{W}$, with $\{\mathcal{F}_t\}$-adapted integrands.

We finish this section by some results on connections with PDE's (Partial Differential Equations). Consider the SDE in $\mathbb{R}^k$

$$(A.5) \qquad dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t),$$

where the coefficients $b_i(t,x), \sigma_{ij}(t,x) : [0,T] \times \mathbb{R}^k \mapsto \mathbb{R}$ are Lipshitz in x (uniformly in t) and of linear growth in x. Define the "infinitesimal generator" of $X(\cdot)$ as the second-order differential operator

$$(A.6)\ \ \mathcal{A}_t V(x) := \frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}(t,x)\frac{\partial^2 V(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{k} b_i(t,x)\frac{\partial V(x)}{\partial x_i}; \quad V \in C^2(\mathbb{R}^k),$$

where $a(t,x)$ is a $k \times k$ matrix $a(t,x) := \sigma(t,x)\sigma^*(t,x)$.

A.11 Theorem (Feynman-Kac): *Let $f : \mathbb{R}^k \mapsto [0,\infty)$, $g, h : [0,T] \times \mathbb{R}^k \mapsto [0,\infty)$ be given continuous functions and let $V(t,x) : [0,T] \times \mathbb{R}^k \mapsto \mathbb{R}$ be a function which is continuous, twice continuously differentiable on $[0,T) \times \mathbb{R}^k$ and of polynomial growth in x, uniformly in t, satisfying the PDE*

$$\frac{\partial V}{\partial t} + \mathcal{A}_t V - hV + g = 0, \ \text{in } [0,T) \times \mathbb{R}^k,$$

with the terminal condition

$$V(T,x) = f(x); \ x \in \mathbb{R}^k.$$

Then, $V(t,x)$ admits the stochastic representation

$$\begin{aligned} V(t,x) = &E^{t,x}\Big[f(X(T))\exp\Big\{-\int_t^T h(s,X_s)ds\Big\} \\ &+ \int_t^T g(u,X_u)\exp\Big\{-\int_t^u h(s,X_s)ds\Big\}du\Big]. \end{aligned}$$

In particular, such a solution is unique.

Here, $E^{t,x}$ means that we are taking expectations of the functional of the process $X(\cdot)$ which is assumed to start at x at time t, $X(t) = x$.

Suppose now that we can control our SDE (A.5) by choosing $\{\mathcal{F}_t\}-$adapted processes $\pi(\cdot)$, which are now arguments of the coefficients, i.e., (A.5) becomes

$$(A.5)' \qquad dX(t) = b(t, X(t), \pi(t))dt + \sigma(t, X(t), \pi(t))dW(t),$$

Suppose also that we are considering the following *stochastic control problem*:

$$V(t,x) = \sup_{\pi(\cdot)} E^{t,x}\left[\int_t^T g(s, X_s, \pi_s)ds + f(X(T))\right],$$

for appropriate functions f, g. Then, under suitable conditions, the *value function* $V(t,x)$ solves the *Hamilton-Jacobi-Bellman* equation

$$(A.7) \qquad \frac{\partial V}{\partial t} + \sup_{\pi}[\mathcal{A}_t V + g] = 0, \; in \; [0,T) \times \mathbb{R}^k,$$

with the terminal condition

$$V(T,x) = f(x); \; x \in \mathbb{R}^k.$$

Moreover, if the supremum in (A.7) is attained for some $\pi = \pi(t,x)$, then, again under suitable conditions, the *feedback control* process $\pi(t, X(t))$ is the optimal control.

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