Pricing Options with Mathematical Models

1. OVERVIEW

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

- What we want to accomplish:

Learn the basics of option pricing so you can:

- (i) continue learning on your own, or in more advanced courses;

- (ii) prepare for graduate studies on this topic, or for work in industry, or your own business.

- The prerequisites we need to know:

- (i) Calculus based probability and statistics, for example computing probabilities and expected values related to normal distribution.
- (ii) Basic knowledge of differential equations, for example solving a linear ordinary differential equation.
- (iii) Basic programming or intermediate knowledge of Excel

- A rough outline:

  - Basic securities: stocks, bonds
  - Derivative securities, options
  - Deterministic world: pricing fixed cash flows, spot interest rates, forward rates

- A rough outline (continued):

- Stochastic world, pricing options:

  - Pricing by no-arbitrage
  - Binomial trees
  - Stochastic Calculus, Ito's rule, Brownian motion
  - Black-Scholes formula and variations
  - Hedging
  - Fixed income derivatives

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2. Stocks, Bonds, Forwards

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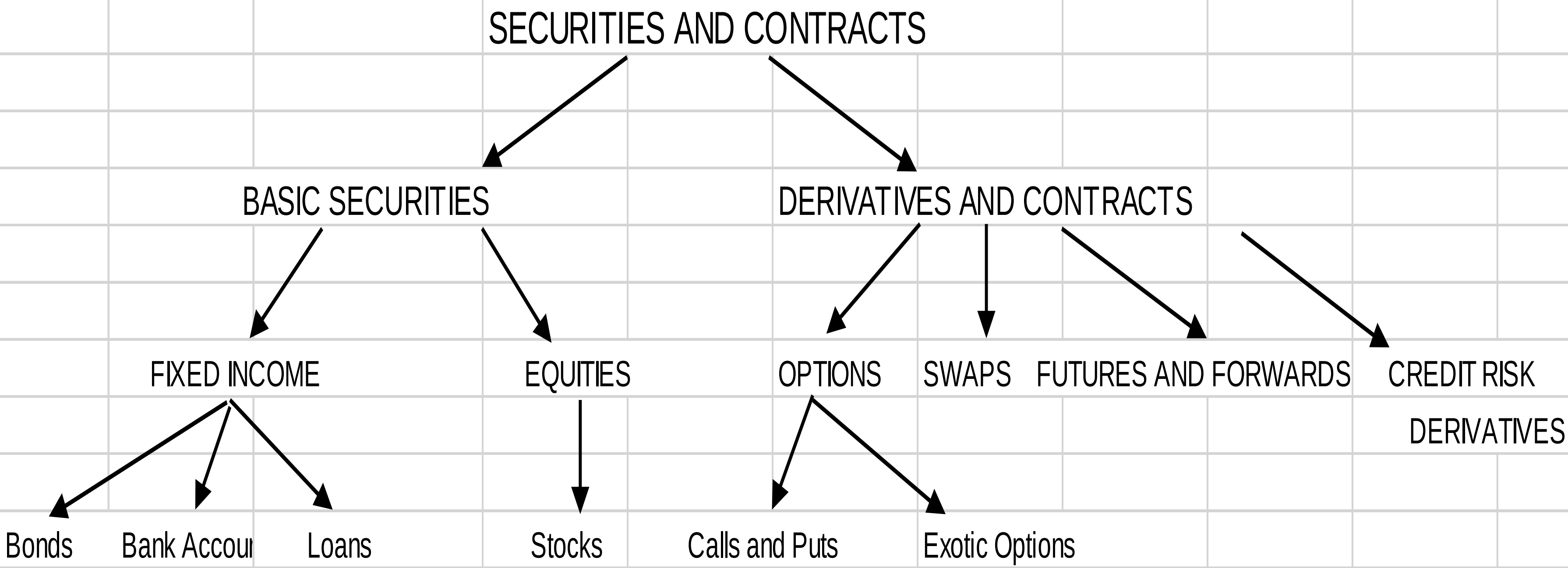
A Classification of Financial Instruments



Stocks

- Issued by firms to finance operations
- Represent ownership of the firm
- Price known today, but not in the future
- May or may not pay dividends

Bonds

- Price known today
- Future payoffs known at fixed dates
- Otherwise, the price movement is random
- Final payoff at maturity: face value/nominal value/principal
- Intermediate payoffs: coupons
- Exposed to default/credit risk

Derivatives

- Sell for a **price/value/premium** today.
- Future value **derived** from the value of the underlying securities (as a function of those).
- Traded at exchanges – standardized contracts, no credit risk;
- or, over-the-counter (OTC) – a network of dealers and institutions, can be non-standard, some credit risk.

Why derivatives?

- To hedge risk
- To speculate
- To attain “arbitrage” profit
- To exchange one type of payoff for another
- To circumvent regulations

Forward Contract

- An agreement to buy (**long**) or sell (**short**) a given **underlying** asset S:
  - At a predetermined future date T (**maturity**).
  - At a predetermined price F (**forward price**).
- F is chosen so that the contract has zero value today.
- Delivery takes place at maturity T:
  - Payoff at maturity: S(T) - F or F - S(T)
  - Price F set when the contract is established.
  - S(T) = **spot (market) price** at maturity.

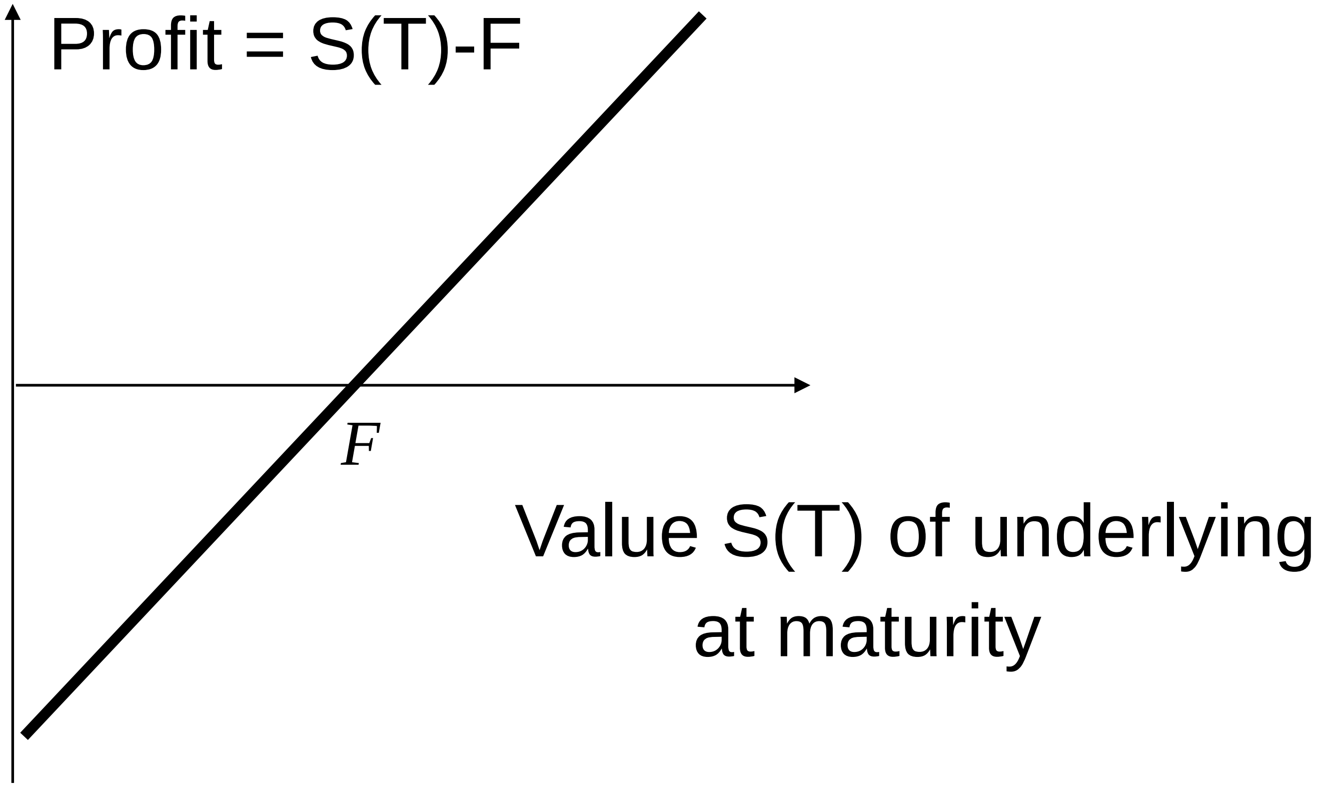
Forward Contract (continued)

- Long position: obligation to buy
- Short position: obligation to sell
- Differences with options:
  - Delivery has to take place.
  - Zero value today.

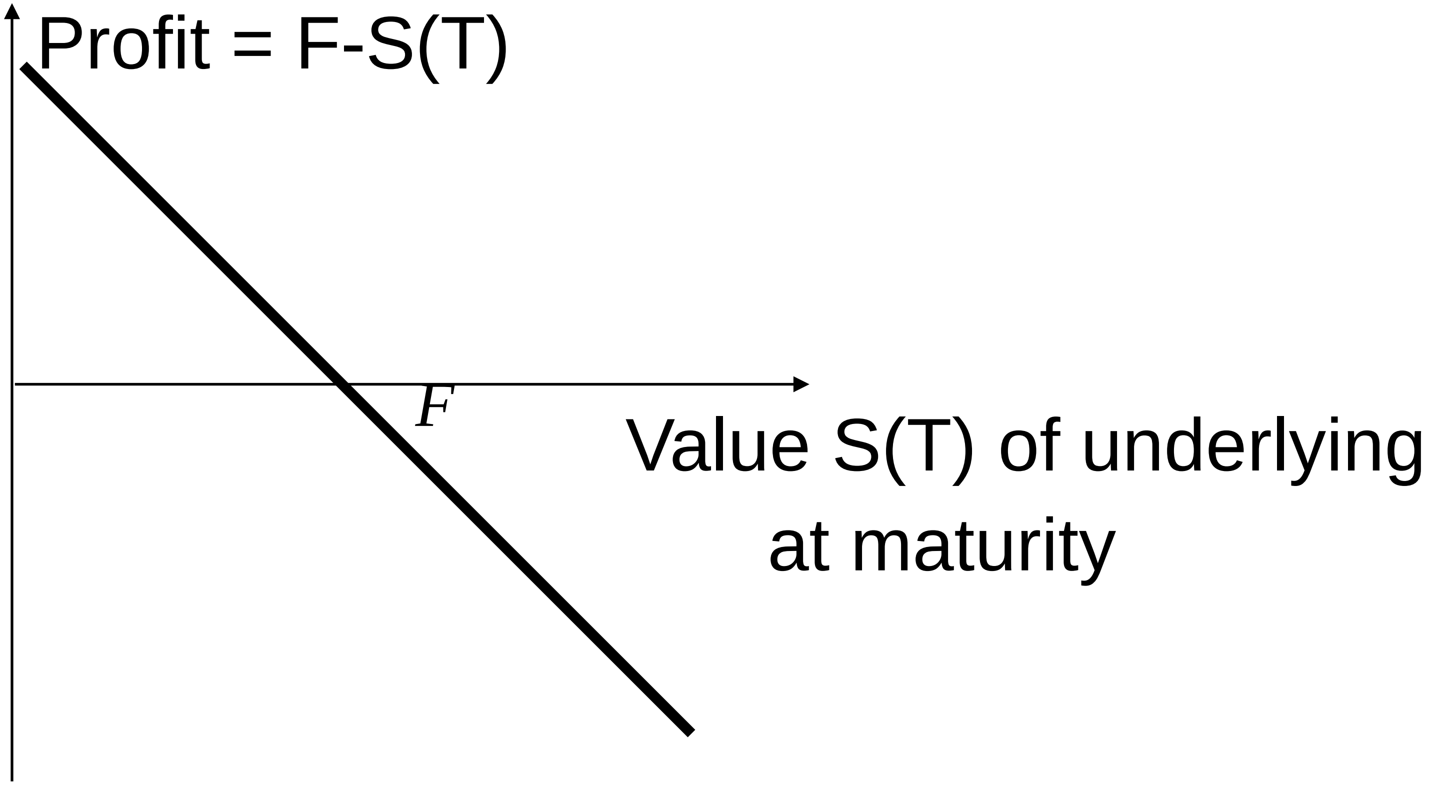
Example

- On May 13, a firm enters into a long forward contract to buy one million euros in six months at an exchange rate of 1.3
- On November 13, the firm pays F=$1,300,000 and receives S(T)= one million euros.
- How does the payoff look like at time T as a function of the dollar value of S(T) spot exchange rate?

Profit from a long forward position



Profit from a short forward position



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3. Swaps

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Swaps

- Agreement between two parties to exchange two series of payments.
- Classic interest rate swap:
  - One party pays **fixed** interest rate payments on a notional amount.
  - Counterparty pays **floating** (random) interest rate payments on the same notional amount.
- Floating rate is often linked to LIBOR (London Interbank Offer Rate), reset at every payment date.

Motivation

- The two parties may be exposed to different interest rates in different markets, or to different institutional restrictions, or to different regulations.

A Swap Example

- New pension regulations require higher investment in fixed income securities by pension funds, creating a problem: liabilities are long-term while new holdings of fixed income securities may be short-term.
- Instead of selling assets such as stocks, a pension fund can enter a swap, exchanging returns from stocks for fixed income returns.
- Or, if it wants to have an option not to exchange, it can buy **swaptions** instead.

Swap Comparative Advantage

- US firm B wants to borrow AUD, Australian firm A wants to borrow USD
- Firm B can borrow at 5% in USD, 12.6% AUD
- Firm A can borrow at 7% USD, 13% AUD
- Expected gain = (7-5) – (13-12.6) = 1.6%
- Swap:

	← USD5%		←USD6.3%	
← Firm B		BANK		Firm A→
5%	→AUD11.9%		→AUD13%	13%

- Bank gains 1.3% on USD, loses 1.1% on AUD, gain=0.2%
- Firm B gains (12.6-11.9) = 0.7%
- Firm A gains (7-6.3) = 0.7%
- Part of the reason for the gain is credit risk involved

A Swap Example: Diversifying

- Charitable foundation CF receives 50mil in stock X from a privately owned firm.
- CF does not want to sell the stock, to keep the firm owners happy
- Equity swap: pays returns on 50mil in stock X, receives return on 50mil worth of S&P500 index.
- A bad scenario: S&P goes down, X goes up; a potential cash flow problem.

Swap Example: Diversifying II

- An executive receives 500mil of stock of her company as compensation.
- She is not allowed to sell.
- Swap (if allowed): pays returns on a certain amount of the stock, receives returns on a certain amounts of a stock index.
- Potential problems: less favorable tax treatment; shareholders might not like it.

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4. Call and Put Options

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Vanilla Options

- **Call** option: a right to buy the underlying
- **Put** option: a right to sell the underlying
- **European** option: the right can be **exercised** only at **maturity**
- **American** option: can be exercised at any time before maturity

Various underlying variables

- Stock options
- Index options
- Futures options
- Foreign currency options
- Interest rate options
- Credit risk derivatives
- Energy derivatives
- Mortgage based securities
- Natural events derivatives …

Exotic options

– **Asian options**: the payoff depends on the average underlying asset price
– **Lookback options**: the payoff depends on the maximum or minimum of the underlying asset price
– **Barrier options**: the payoff depends on whether the underlying crossed a barrier or not
– **Basket options**: the payoff depends on the value of several underlying assets.

Terminology

- ***Writing*** *an option*: selling the option
- ***Premium***: price or value of an option
- Option **in/at/out of the money**:
  - *At*: strike price equal to underlying price
  - *In*: immediate exercise would be profitable
  - *Out*: immediate exercise would not be profitable

Long Call

	Outcome at maturity	
	$S(\mathrm{T}) \leq K$	$S(\mathrm{T}) > K$
Payoff:	0	$S(\mathrm{T}) - K$
Profit:	$-C(\mathrm{t}, K, \mathrm{T})$	$S(\mathrm{T}) - K - C(\mathrm{t}, K, \mathrm{T})$

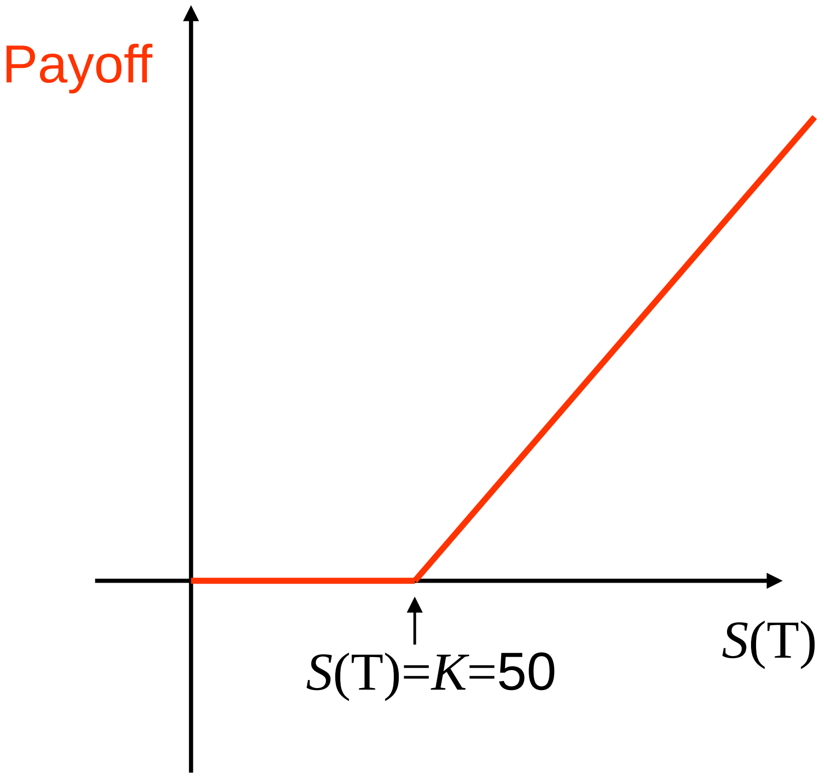
A more compact notation:

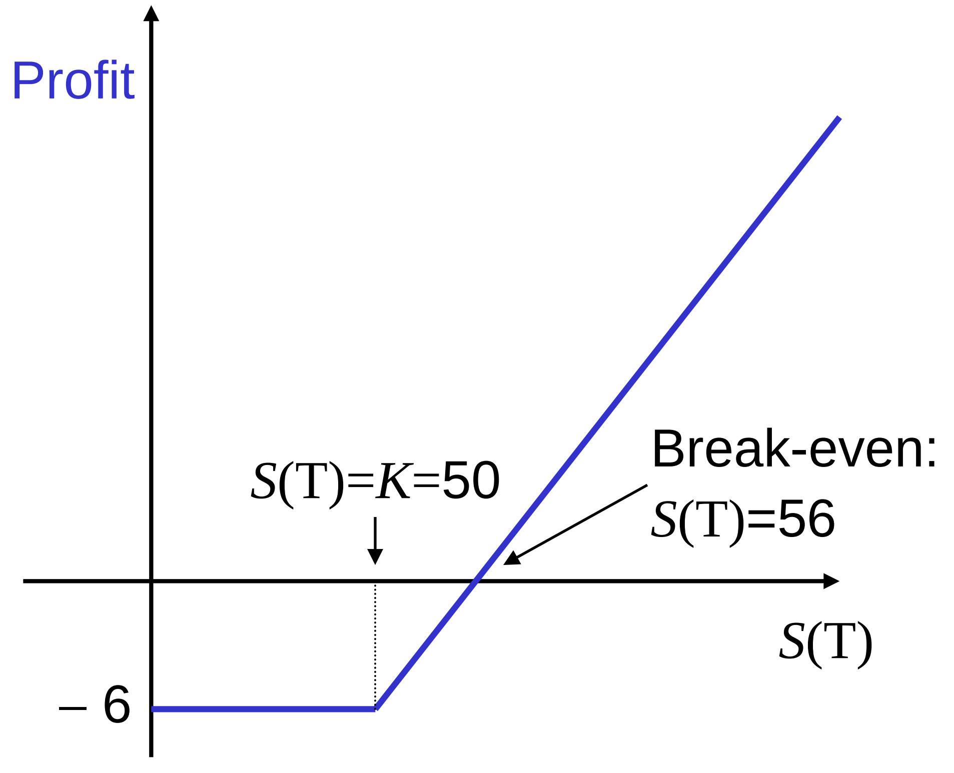
Payoff: $\max\,[\mathrm{S(T)} - K, 0] = (\mathrm{S(T)} - \mathrm{K})_+$

Profit: $\max\,[\mathrm{S(T)} - K, 0] - C(\mathrm{t}, K, \mathrm{T})$

Long Call Position

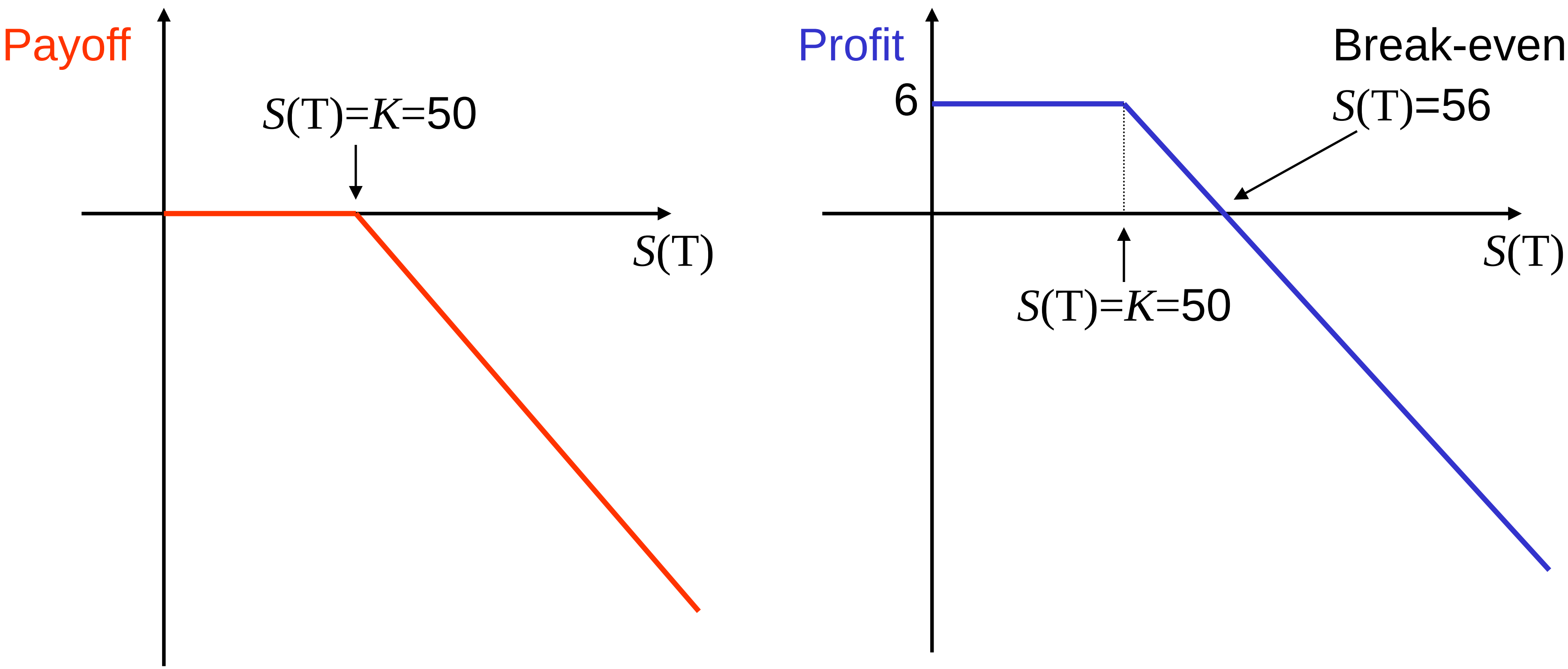
- Assume K = \$50, $C(t,K,T)$ = \$6
- Payoff: *max* [S(T) – 50, 0]
- Profit: *max* [S(T) – 50, 0] – 6





Short Call Position

- K = \$50, C(t,K,T) = \$6
- Payoff: – *max* [S(T) – 50, 0]
- Profit: 6 – *max* [S(T) – 50, 0]



Long Put

	Outcome at maturity	
	$S(\mathrm{T}) \le K$	$S(\mathrm{T}) > K$
Payoff:	$K - S(\mathrm{T})$	0
Profit:	$K - S(\mathrm{T}) - P(\mathrm{t}, K, \mathrm{T})$	$-P(\mathrm{t}, K, \mathrm{T})$

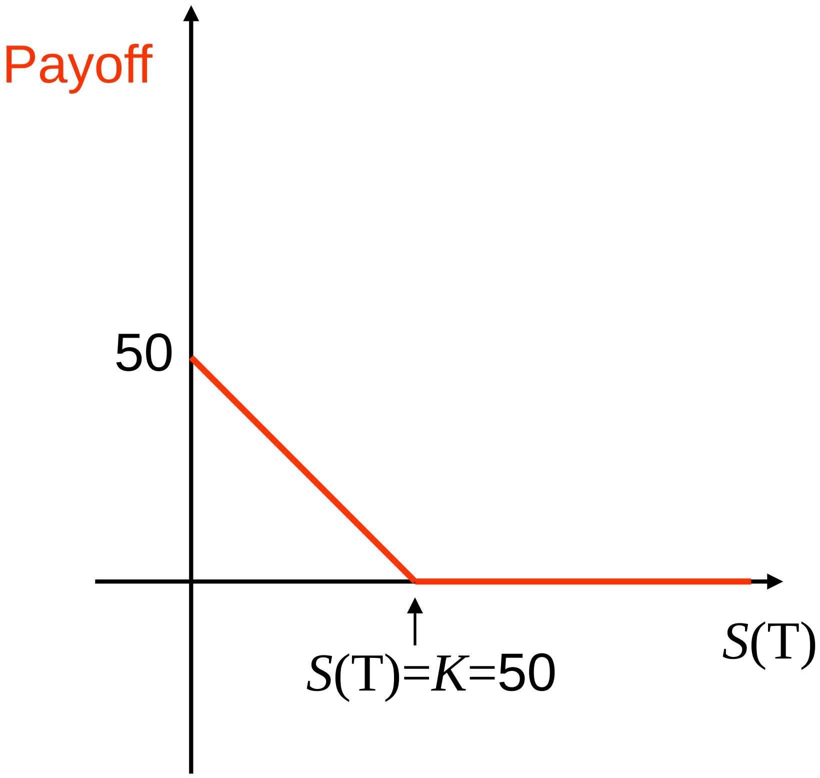
A more compact notation:

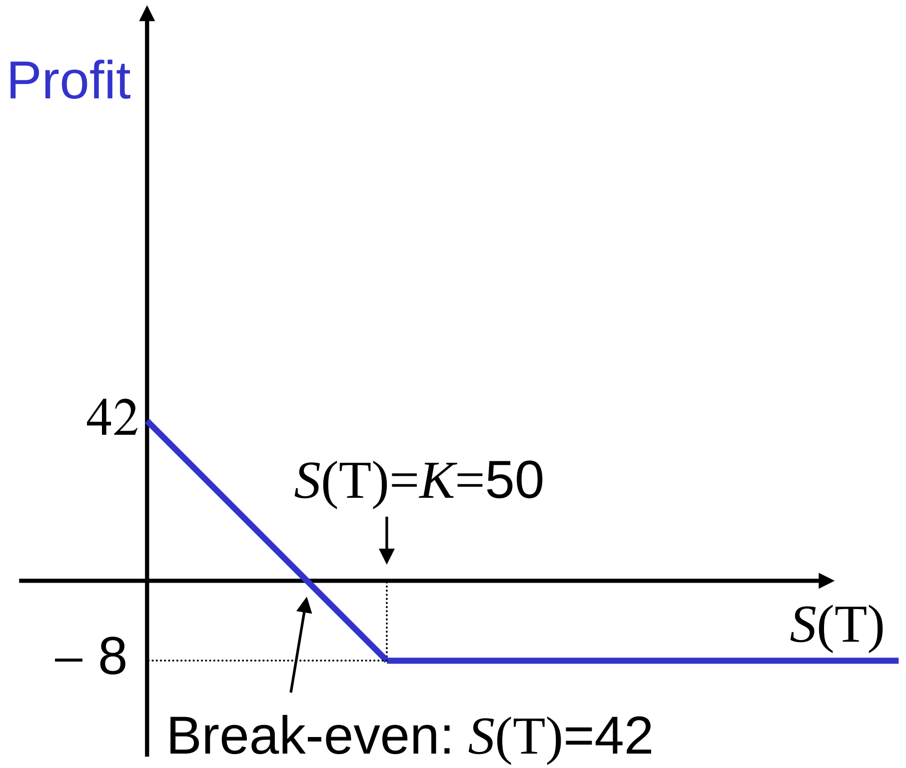
Payoff: $max\,[K - \mathrm{S(T)}, 0] = (\mathrm{K-S(T)})_{+}$

Profit: $max\,[K - \mathrm{S(T)}, 0] - P(\mathrm{t},K,\mathrm{T})$

Long Put Position

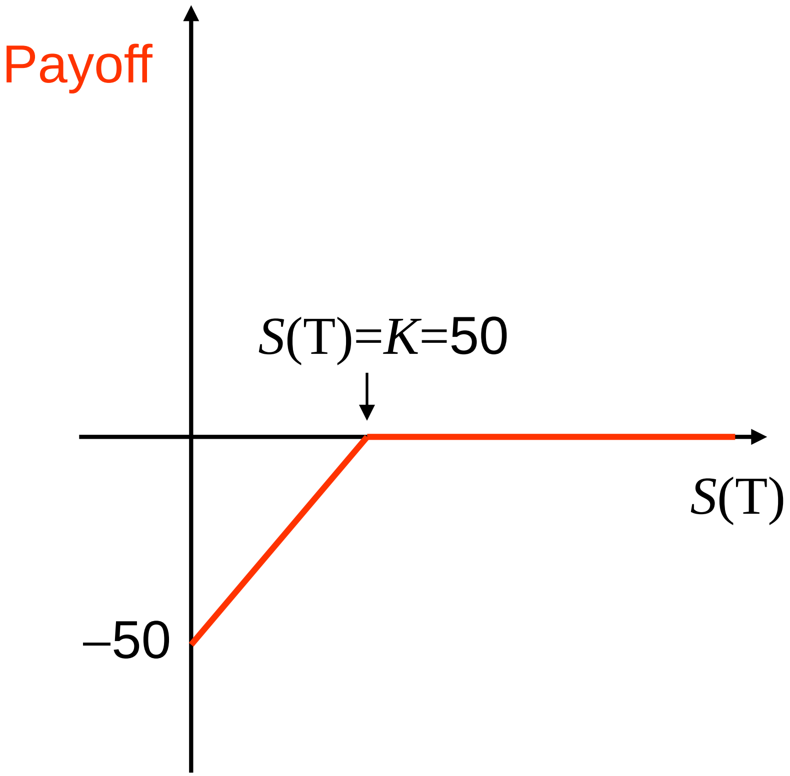
- Assume K = \$50, P(t,K,T) = \$8
- Payoff: *max* [50 – S(T), 0]
- Profit: *max* [50 – S(T), 0] – 8

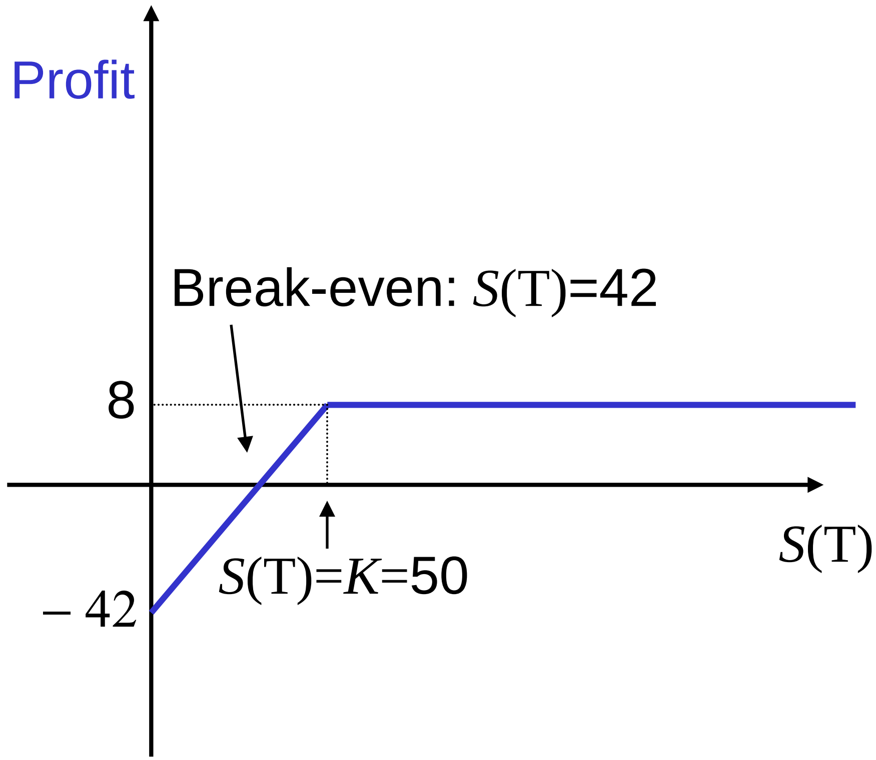




Short Put Position

- K = \$50, P(t,K,T) = \$8
- Payoff: $-\ max\ [50 - S(T), 0]$
- Profit: $8 - max\ [50 - S(T), 0]$





Implicit Leverage: Example

- Consider two securities
  - Stock with price S(0) = $100
  - Call option with price C(0) = $2.5 (K = $100)
- Consider three possible outcomes at t=T:
  - Good: S(T) = $105
  - Intermediate: S(T) = $101
  - Bad: S(T) = $98

Implicit Leverage: Example (continued)

Suppose we plan to invest $100

Invest in:	Stocks	Options
Units	1	40
Return in:		
Good State	5%	100%
Mid State	1%	-60%
Bad State	-2%	-100%

EQUITY LINKED BANK DEPOSIT

- Investment =10,000
- Return = 10,000 if an index below the current value of 1,300 after 5.5 years
- Return = 10,000 × (1+ 70% of the percentage return on index)
- Example: Index=1,500. Return = $=10{,}000 \cdot (1+(1{,}500/1{,}300-1) \cdot 70\%)=11{,}077$
- Payoff = Bond + call option on index

HEDGING EXAMPLE

Your bonus compensation: 100 shares of the company, each worth $150.

Your hedging strategy: buy 50 put options with strike K = 150

If share value falls to $100: you lose $5,000 in stock, win $2,500 minus premium in options

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5. Options Combinations

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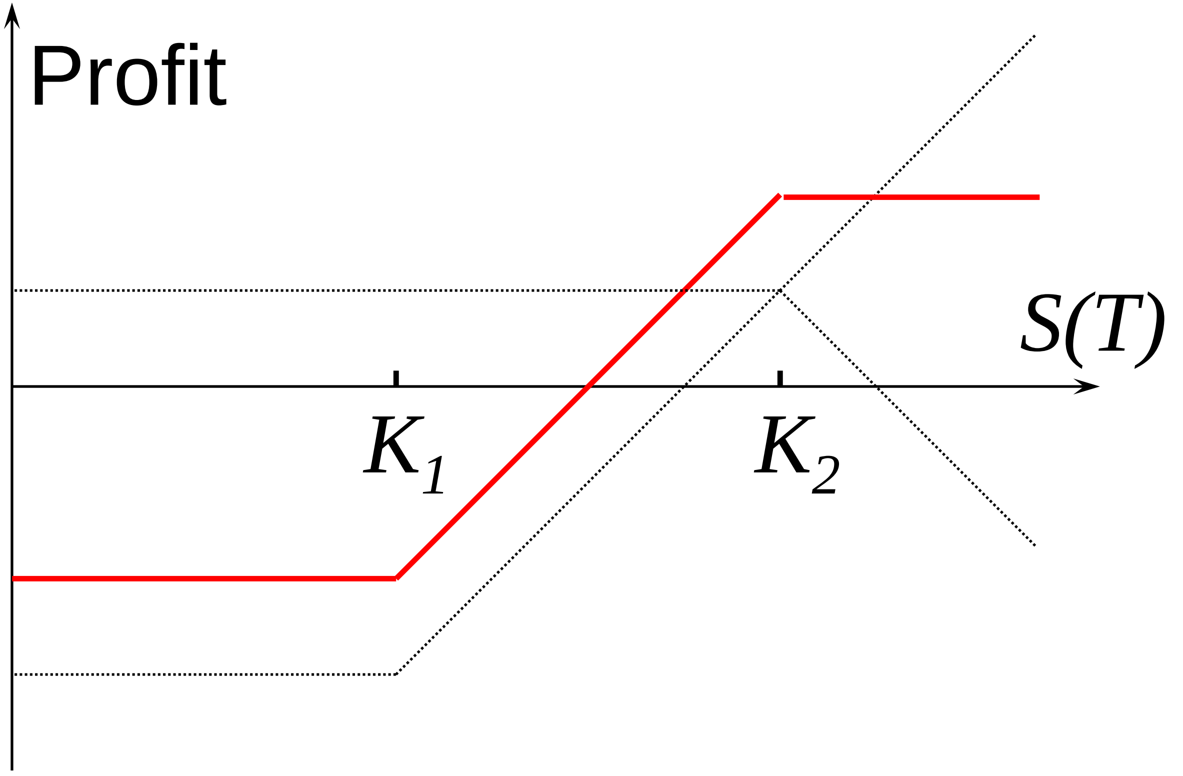
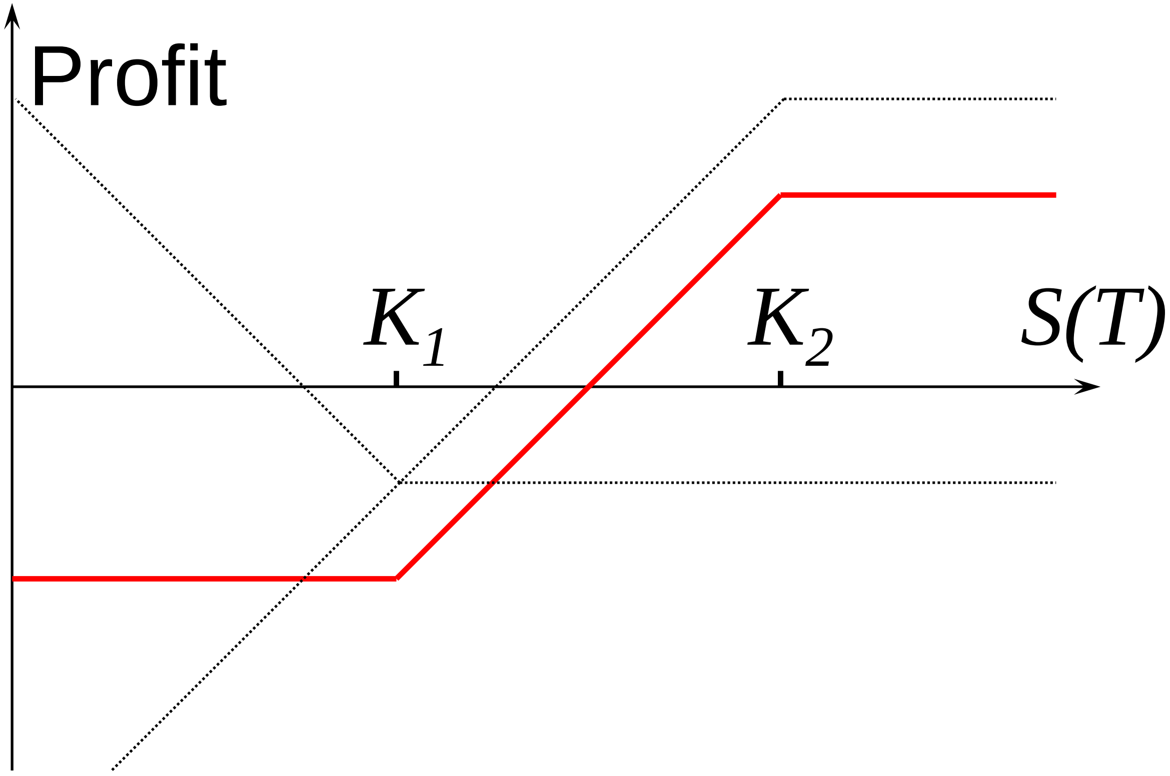
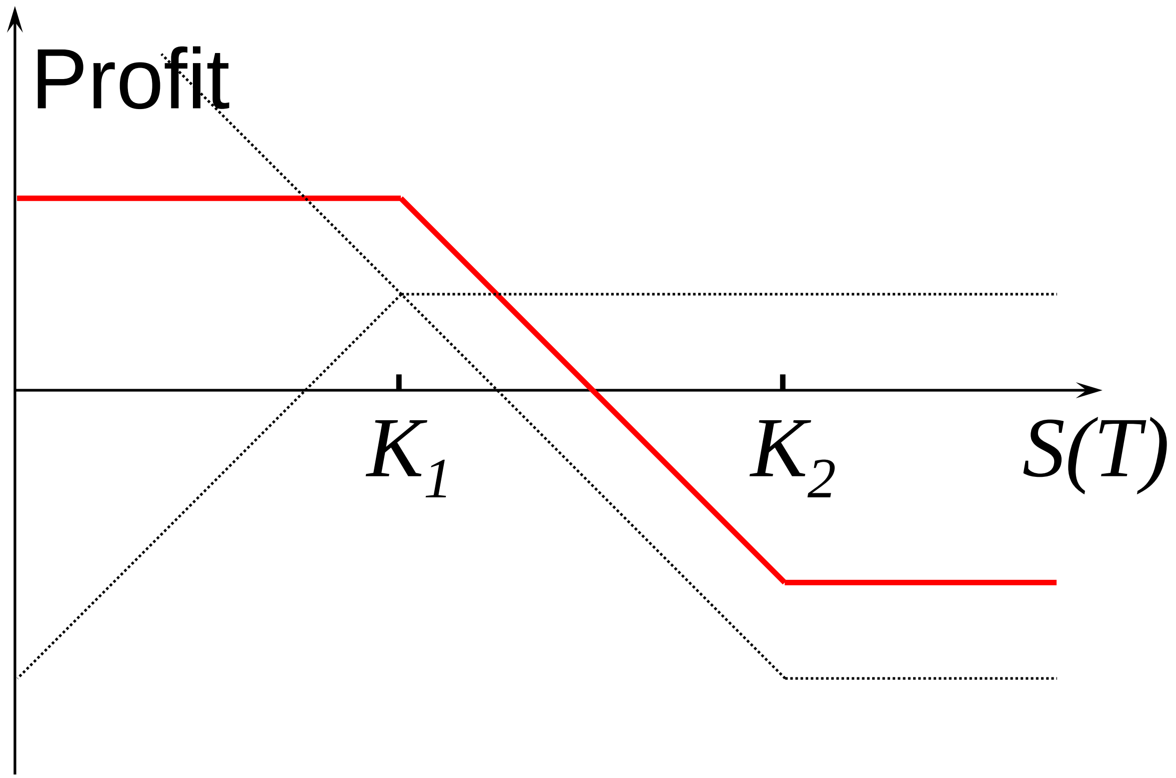
Bull Spread Using Calls



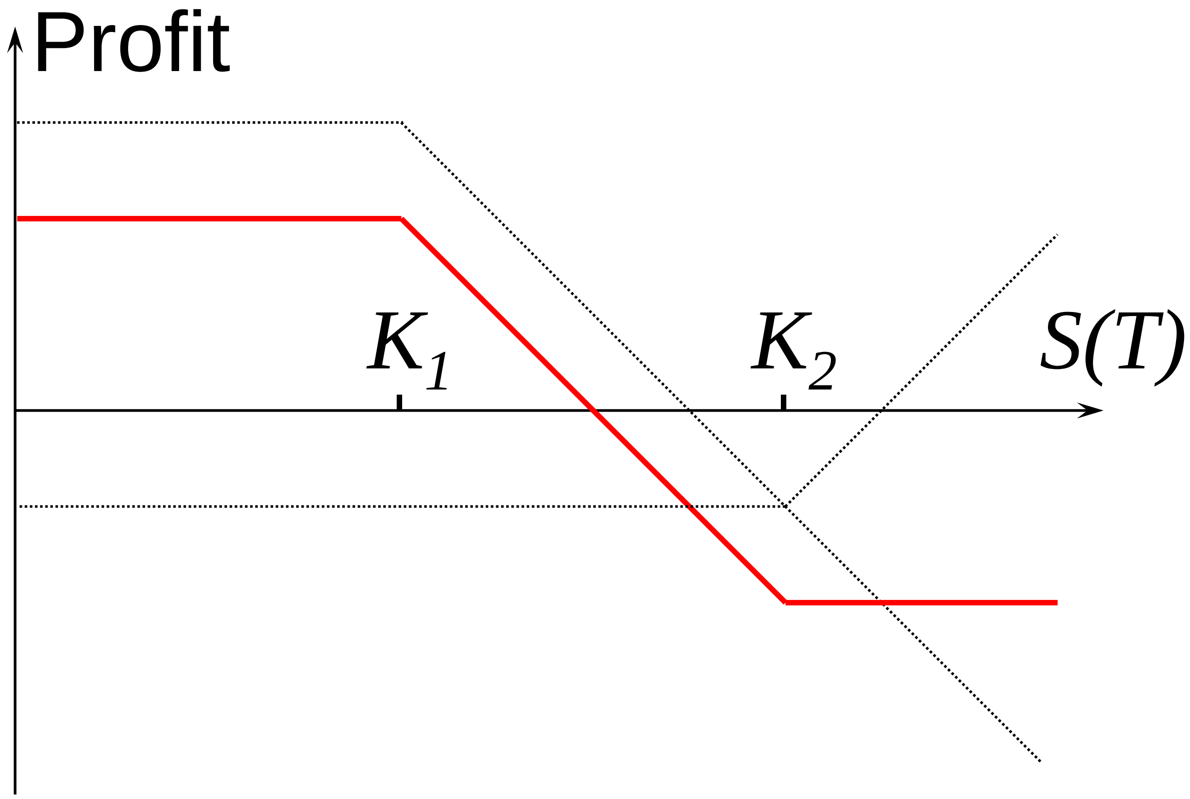
Bull Spread Using Puts



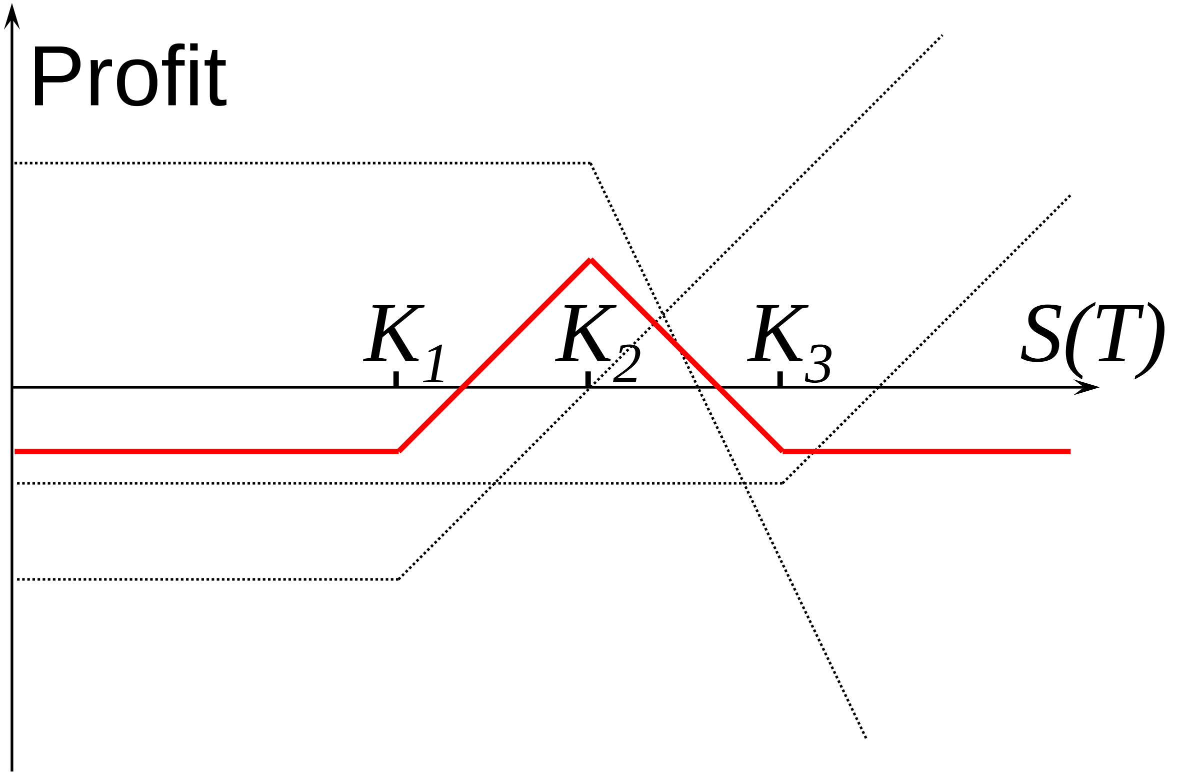
Bear Spread Using Puts



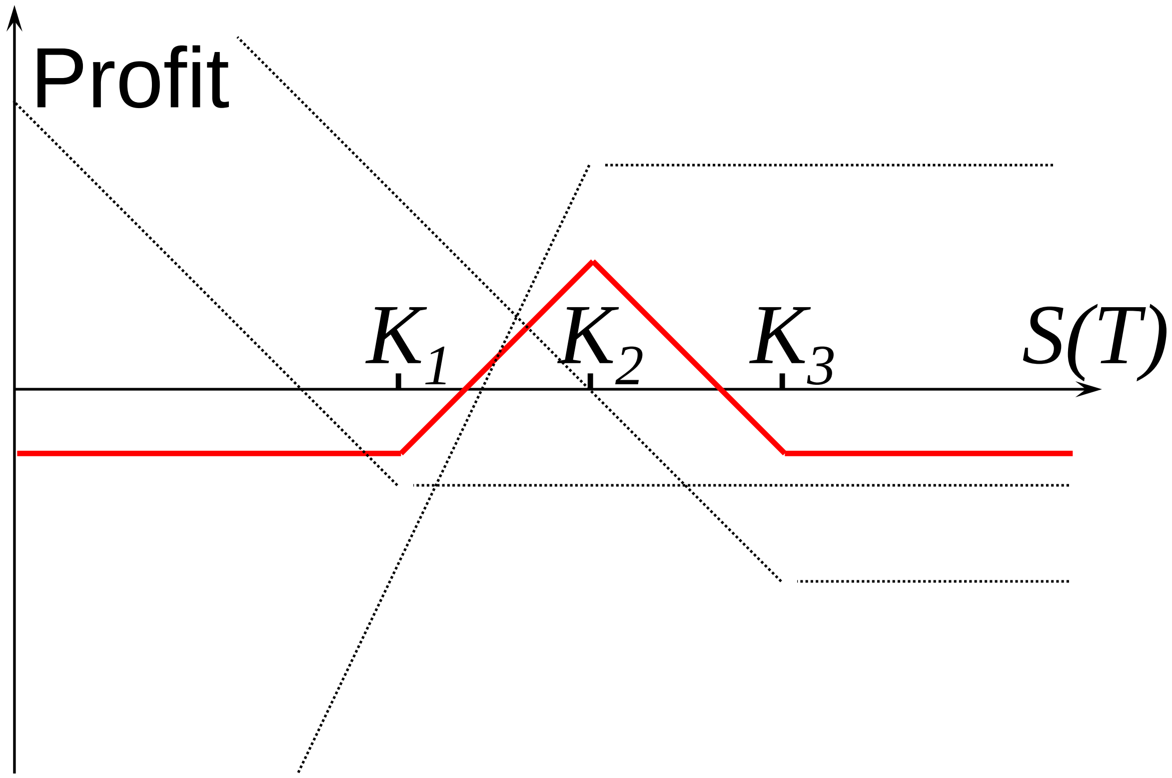
Bear Spread Using Calls



Butterfly Spread Using Calls



Butterfly Spread Using Puts



Bull Spread (Calls)

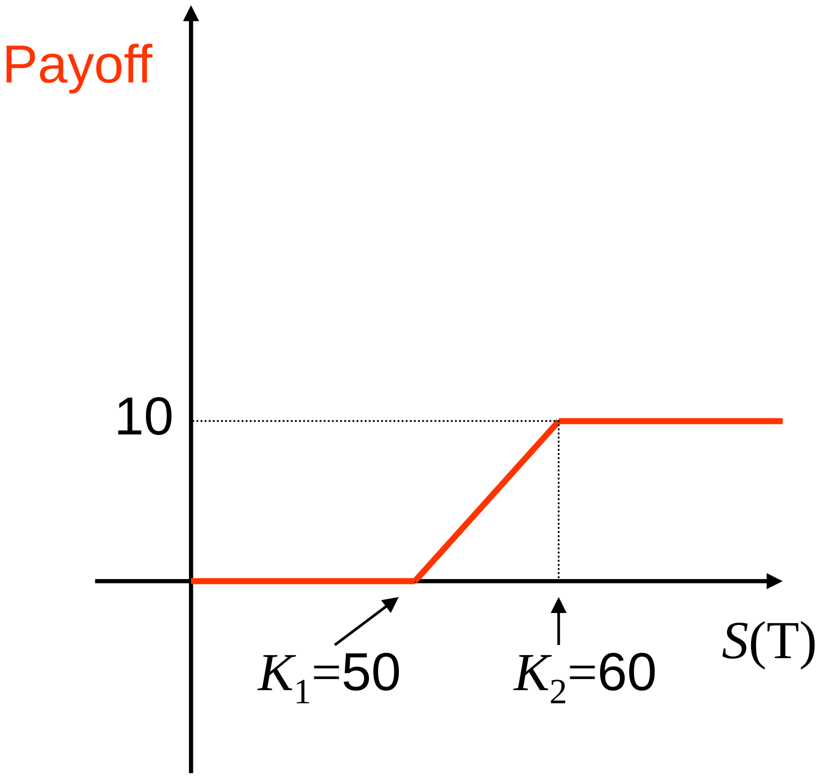
- Two strike prices: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $C(K_1)$, $C(K_2)$

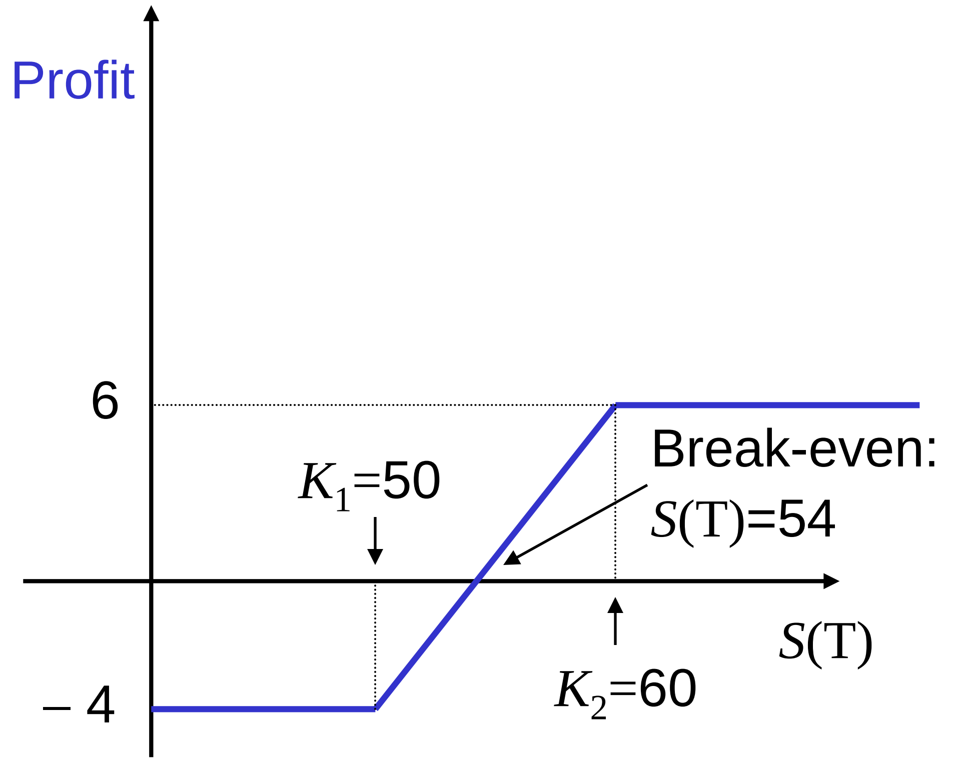
Outcome at Expiration

	$S(\mathrm{T}) \le K_1$	$K_1 < S(\mathrm{T}) \le K_2$	$S(\mathrm{T}) > K_2$
Payoff:	0	$S(\mathrm{T}) - K_1$	$S(\mathrm{T}) - K_1 - (S(\mathrm{T}) - K_2) =$ $= K_2 - K_1$
Profit:	$C(K_2) - C(K_1)$	$C(K_2) - C(K_1)$ $+ S(\mathrm{T}) - K_1$	$C(K_2) - C(K_1) + K_2 - K_1$

Bull Spread (Calls)

- Assume K_1 = \$50, K_2 = \$60, $C(K_1)$ = \$10, $C(K_2)$ = \$6
- Payoff: *max* [S(T) – 50, 0] – *max* [S(T) – 60, 0]
- Profit: (6–10) + *max* [S(T)–50,0] – *max* [S(T)–60,0]





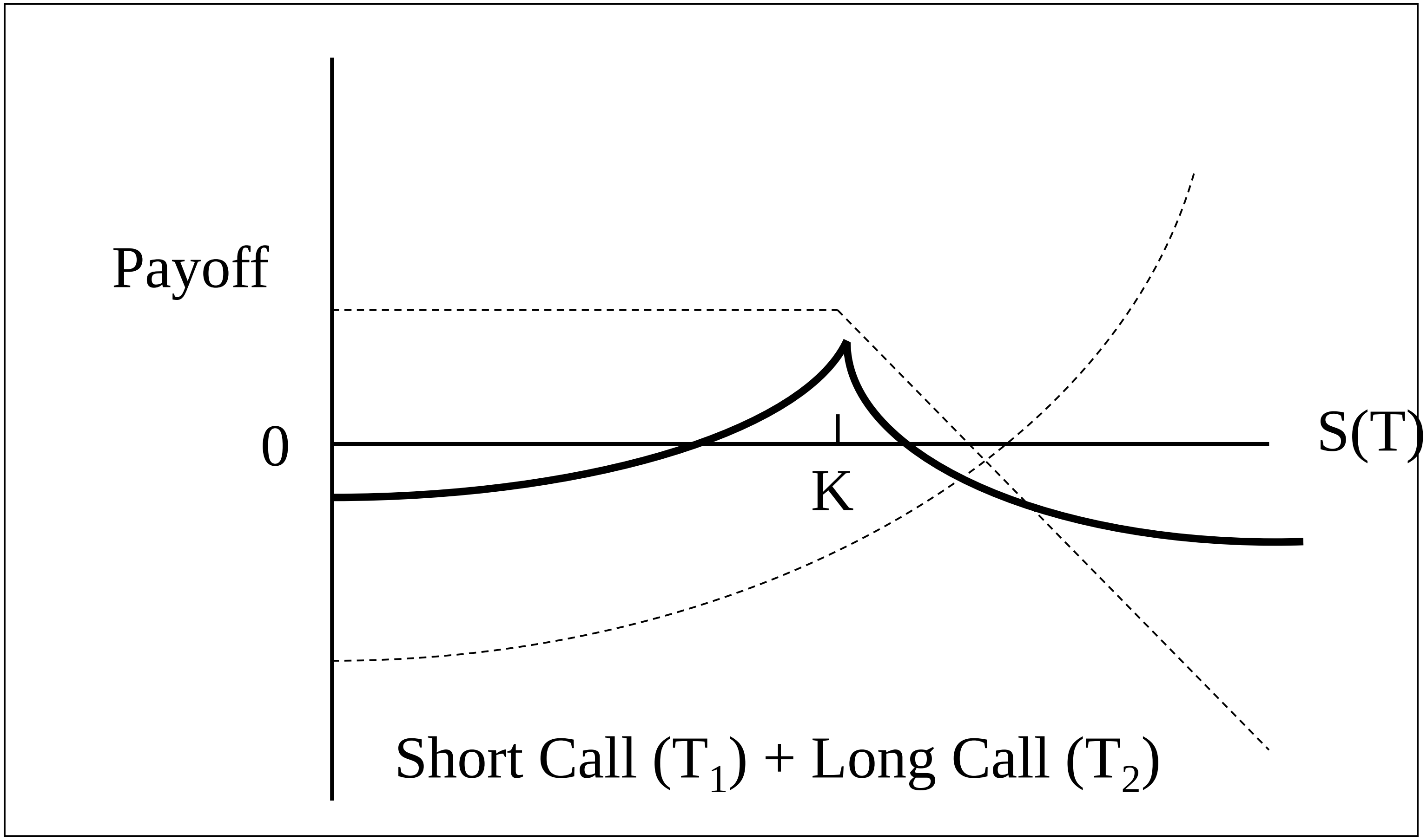
Bear Spread (Puts)

- Again two strikes: K_1, K_2 with $K_1 < K_2$
- Short-hand notation: $P(K_1)$, $P(K_2)$

Outcome at Expiration

	$S(\mathrm{T}) \leq K_1$	$K_1 < S(\mathrm{T}) \leq K_2$	$S(\mathrm{T}) > K_2$
Payoff:	$K_2 - S(\mathrm{T}) - (K_1 - S(\mathrm{T})) =$ $= K_2 - K_1$	$K_2 - S(\mathrm{T})$	0
Profit:	$P(K_1) - P(K_2) + K_2 - K_1$	$P(K_1) - P(K_2) +$ $+ K_2 - S(\mathrm{T})$	$P(K_1) - P(K_2)$

Calendar Spread



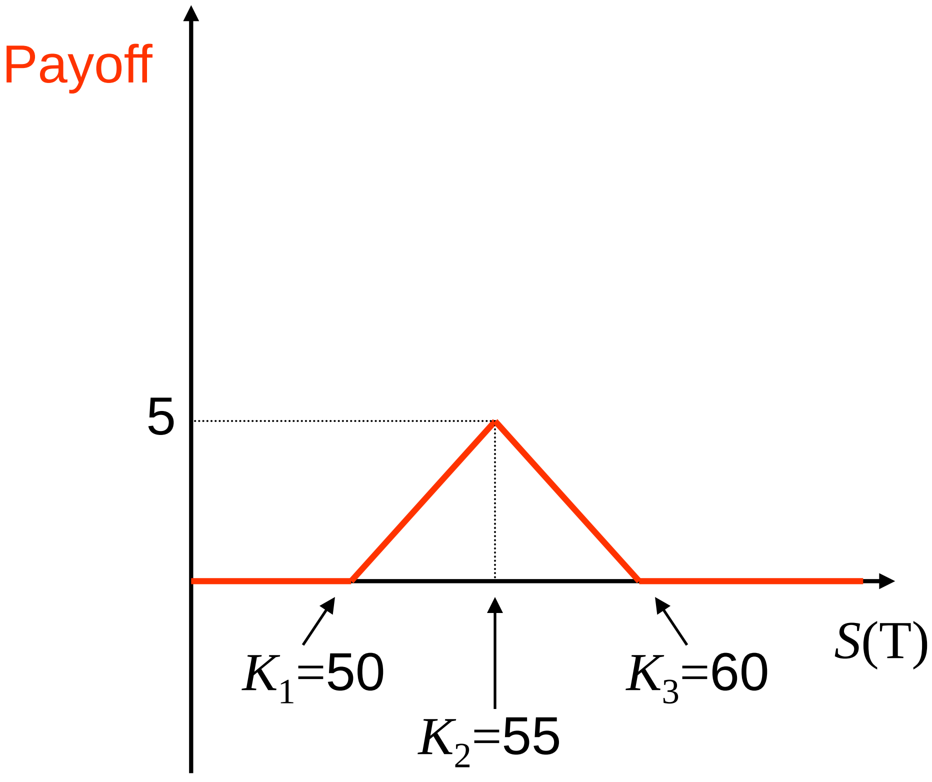
Butterfly Spread

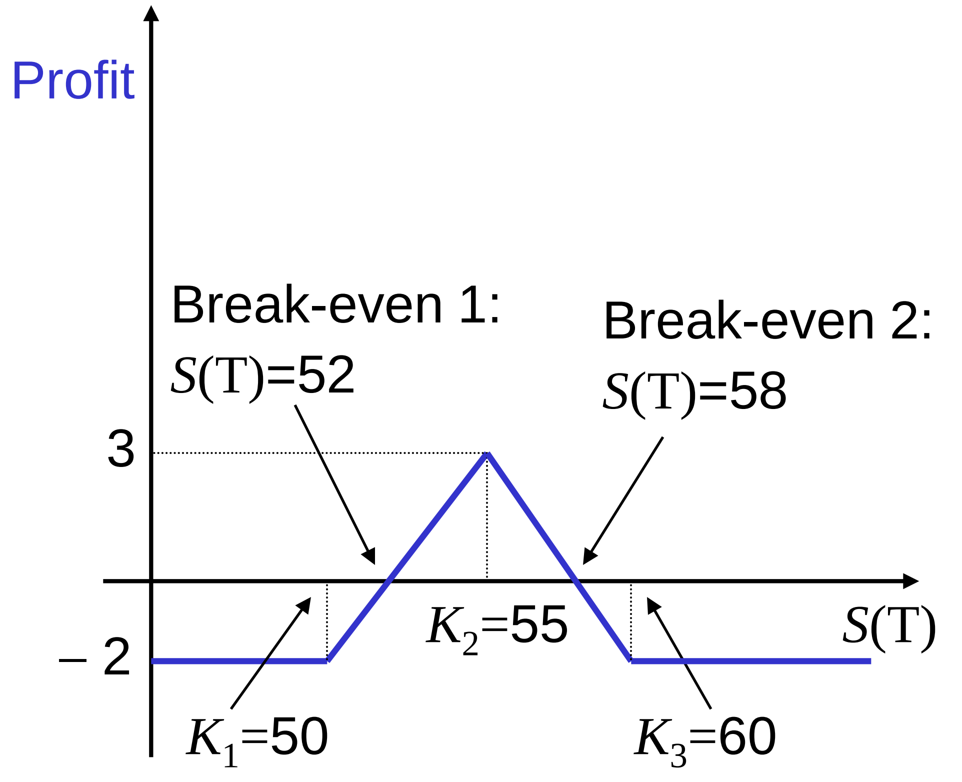
- Positions in **three** options of the same class, with same maturities but different strikes K_1, K_2, K_3
  - Long butterfly spreads: buy one option each with strikes K_1, K_3, sell two with strike K_2
- $K_2 = (K_1 + K_3)/2$

$\Leftrightarrow$

Long Butterfly Spread (Puts)

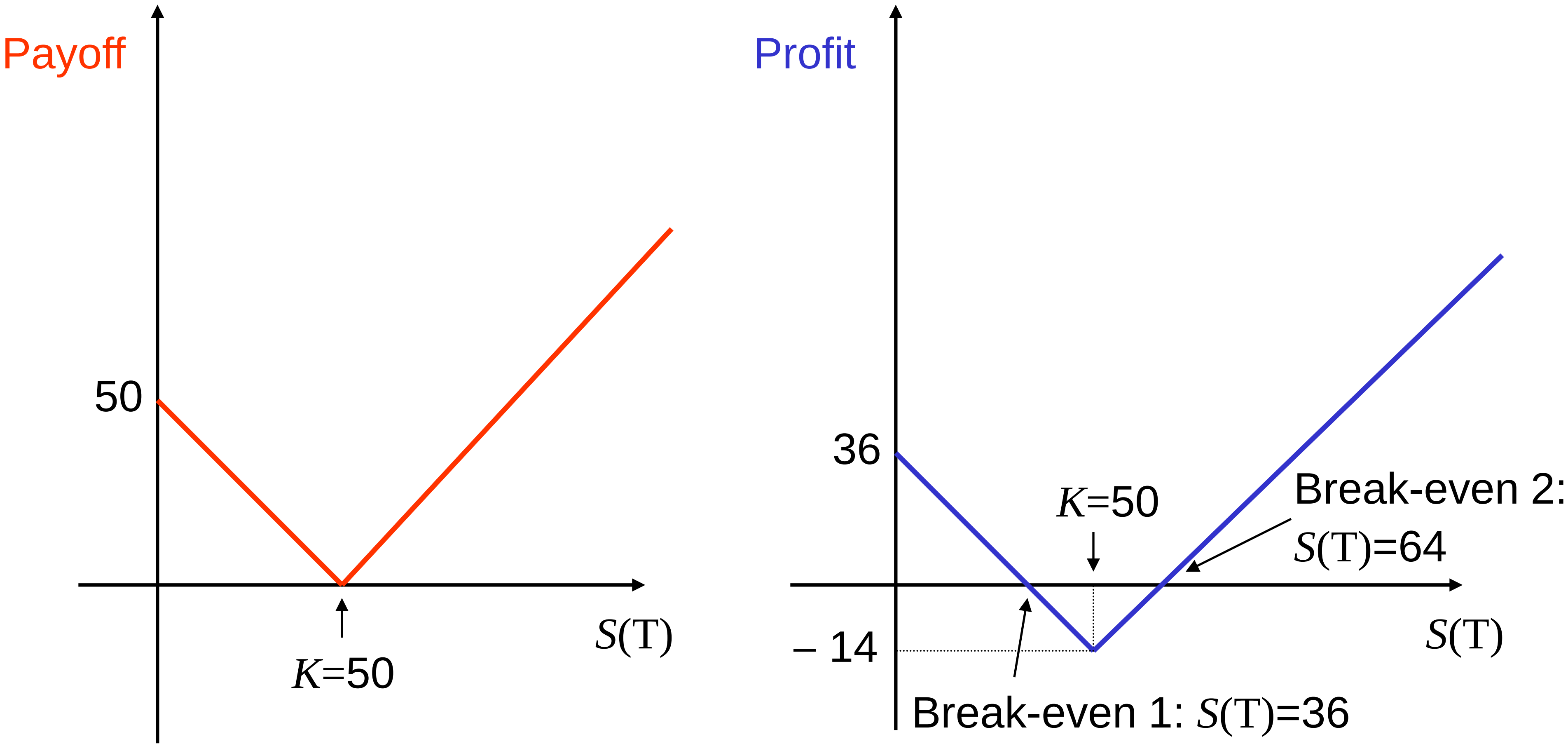
- K_1 = \$50, K_2 = \$55, K_3 = \$60
- $P(K_1)$ = \$4, $P(K_2)$ = \$6, $P(K_3)$ = \$10





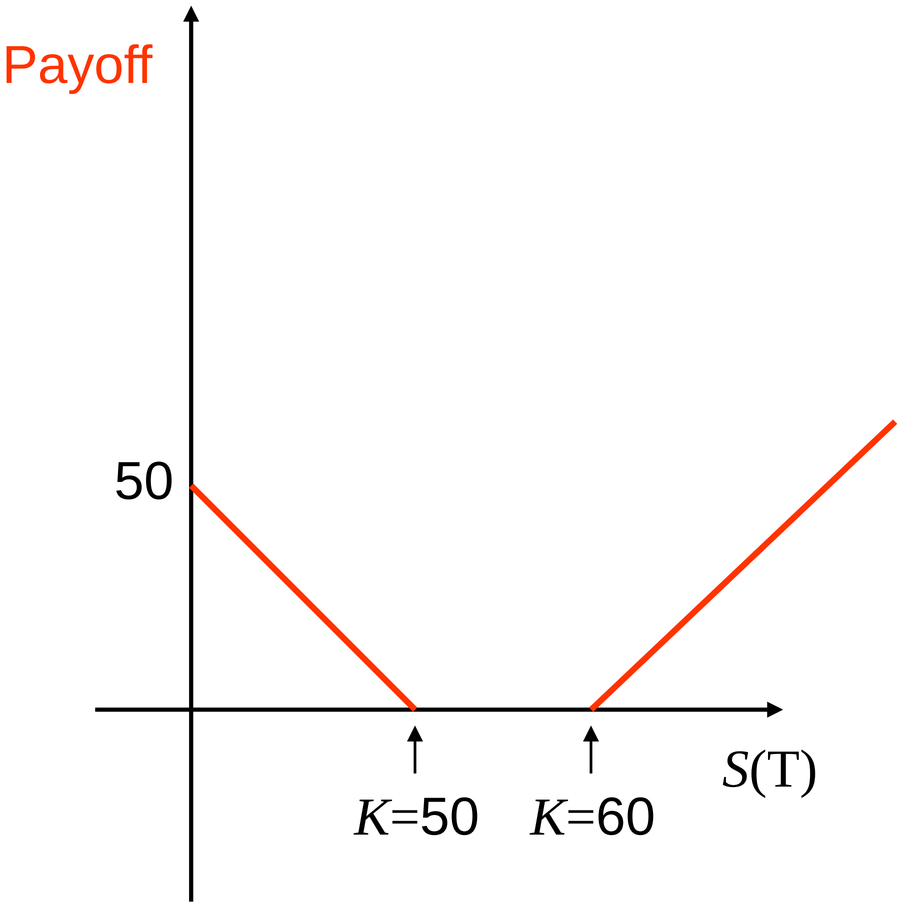
Bottom Straddle

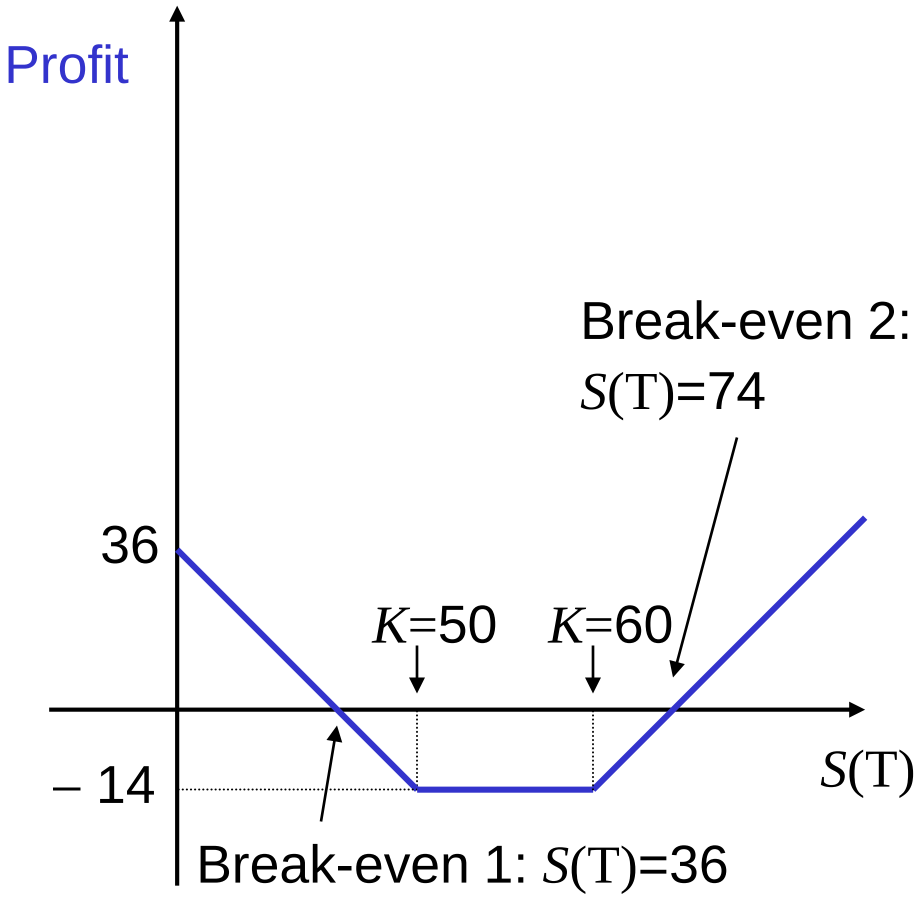
Assume K = \$50, $P(K)$ = \$8, $C(K)$ = \$6



Bottom Strangle

Assume K_1 = \$50, K_2 = \$60, $P(K_1)$ = \$8, $C(K_2)$ = \$6





Arbitrary payoff shape

- Suppose we want to have a payoff of the form

f (S(T)) for some function f ( ). Assume that call options written on S(T) are traded for all possible strike values K.

- CLAIM: If f ( ) is smooth and $f'(\infty) \cdot 0 = 0$, then

- $f(s) = f(0) + f'(0)s + \int_0^\infty f''(K) \max(S - K, 0)\, dK$

Proof sketch

$$\int_0^\infty f''(K)\max(s-K,0)\,dK$$

= (integration by parts) =

$$= f'(\infty) \bullet 0 - f'(0) \bullet \mathrm{s} - \int_0^\infty f'(K)\,\mathrm{d}[\max(s-K,0)]$$

$$= - f'(0) \bullet \mathrm{s} + \int_0^s f'(K)dK$$

$$= - f'(0) \bullet \mathrm{s} + f(s) - f(0)\ .$$

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6. Pricing deterministic payoffs

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Future value with constant interest rate

- Suppose you can lend money at annual interest rate r, so that

Present Value (PV)=$1.00 → Future Value (FV) after 1 year=$$(1+r)$

- Different conventions:

- Simple interest: after T years, FV $= 1 + T \times r$

- Interest compounded once a year: after T years, FV $= (1+r)^T$

- Interest compounded n times a year: after m compounding periods,

$$\text{FV} = (1 + r/n)^m$$

- **Effective annual interest rate** r':

$$(1 + r/n)^n = 1 + r'$$

EXAMPLE: Quarterly compounding at **nominal** annual rate r = 8%,

$$(1 + 0.08/4)^4 = 1.0824 = 1 + 0.0824$$

Thus, the effective annual interest rate is 8.24%.

- **Continuous compounding**:

  - after one year, FV $= \lim_{n \to \infty} (1 + r/n)^n = e^r$, e = 2.718 …
  - after T years, FV $= \lim_{n \to \infty} (1 + r/n)^{nT} = e^{rT}$

EXAMPLE: r = 8%, e^r = 1.0833, r' = 8.33%

Price as Present Value

- **LAW OF ONE PRICE**: If two cash flows deliver the same payments in the future, they have the same price (value) today.
- **PRICE DEFINITION**: If one can guarantee having \$X(T) at time T by investing \$X(0) today, then, today's price of X(T) is X(0).
- For deterministic X(T), X(0) is called the **present value**, PV(X(T)).
- Thus, if one can invest at compounded rate of r/n, and T=m periods,

$$X(0) = PV\big(X(T)\big) = X(T)/(1 + r/n)^m$$

because this is equivalent to $X(0)(1 + r/n)^m = X(T)$.

- **Discount factor**: $1/(1 + r/n)^m$, e^{-rT}

PV of cash flows

- Cash flow X(0), X(1), X(2), …, X(m), paid at compounding intervals:

$$PV = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

- When X(0)=0 and X(i) = X, we have a geometric series:
- $PV = X \cdot \left(\frac{1}{\left(1+\frac{r}{n}\right)} + \frac{1}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{1}{\left(1+\frac{r}{n}\right)^m}\right)$

$$= X \cdot \frac{1}{r/n} \cdot \left(1 - \frac{1}{\left(1+\frac{r}{n}\right)^m} \right)$$

Example

- We want to estimate the value of leasing a gold mine for 10 years. It is estimated that the mine will produce 10,000 ounces of gold per year, at a cost of $200 per ounce, and that the gold will sell for $400 per ounce. We also estimate that, if not invested in the mine, we could invest elsewhere at r=10% return per year.

Annual profit = 10,000 (400-200) = 2 mil

$$PV = \sum_{k=1}^{10} \frac{2\ mil}{(1+0.1)^k} = 2 \cdot 10 \cdot \left(1 - \frac{1}{(1+0.1)^{10}} \right) = 12.29 \text{ mil}$$

Loan payments

- Suppose you take a loan with value PV = $V, and the loan is supposed to be paid off (amortized) in equal amounts X over m periods at interest rate $\frac{r}{n}$.
- Inverting the PV formula, we get

$$X = V \cdot \frac{\frac{r}{n}\left(1+\frac{r}{n}\right)^{m}}{\left(1+\frac{r}{n}\right)^{m} - 1}$$

EXAMPLE 1. You take a 30-year loan on $400,000, at annual rate of 8%, compounded monthly. What is the amount X of your monthly payments?

With 12 months in a year, the number of periods is m=30 · 12=360. The rate per period is 0.08/12=0.0067. The value of the loan is V=400,000. We compute X= $2,946 in monthly payments, approximately.

The loan balance is computed as follows:

- Before the end of the first month, balance =400,000+0.0067 · 400,000=402,680
- After the first installment of $2,946 is paid, balance = 402,680-2,946=399,734.
- Before the end of the second month, balance = 399,734(1+0.0067), and so on.

The future value corresponding to these payments thirty years from now is

$400{,}000 \cdot (1 + 0.0067)^{360}$ = 4,426,747 .

EXAMPLE 2 (Loan fees). For mortgage products, there are usually two rates listed: the mortgage interest rate and the APR, or annual percentage rate. The latter rate includes the fees added to the loan amount and also paid through the monthly installments. Consider the previous example with the rate of 7.8% compounded monthly, and the APR of 8.00%. As computed above, the monthly payment at this APR is 2,946. Now, we use this monthly payment of 2,946 and the rate of 7.8/12=0.65% in the formula for V, to find that the total balance actually being paid is 407,851.10. This means that the total fees equal

$$407{,}851.10 - 400{,}000 = 7{,}851.10 \; .$$

Perpetual annuity

- Pays amount X at the end of each period, for ever. If the interest rate per period is r,

$$\text{PV} = \sum_{k=1}^{\infty} \frac{X}{(1+r)^k} = \frac{X}{r}$$

Internal rate of return

- The rate r for which

$$0 = X(0) + \frac{X(1)}{\left(1+\frac{r}{n}\right)} + \frac{X(2)}{\left(1+\frac{r}{n}\right)^2} + \ldots + \frac{X(m)}{\left(1+\frac{r}{n}\right)^m}$$

Pricing Options with Mathematical Models

7. Bonds

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Bond yield

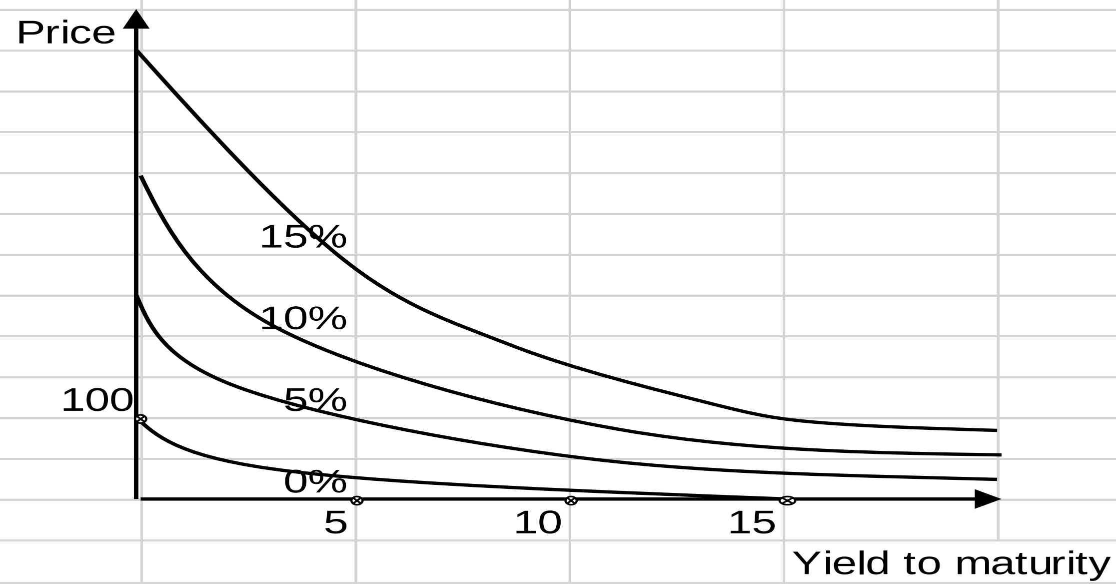
- **Yield to maturity** (YTM) of a bond is the internal rate of return of the bond, or the rate that makes the bond price equal to the present value of its future payments.
- Suppose the bond pays a **face value** V at maturity $T = m$ periods and n identical coupons a year in the amount of C/n, and its price today is P. Then, the bond's annualized yield corresponding to compounding n times a year is the value y that satisfies

$$P = \frac{V}{\left(1+\frac{y}{n}\right)^m} + \sum_{k=1}^{m} \frac{C/n}{\left(1+\frac{y}{n}\right)^k} = \frac{V}{\left(1+\frac{y}{n}\right)^m} + \frac{C}{y} \cdot \left(1 - \frac{1}{\left(1+\frac{y}{n}\right)^m} \right)$$

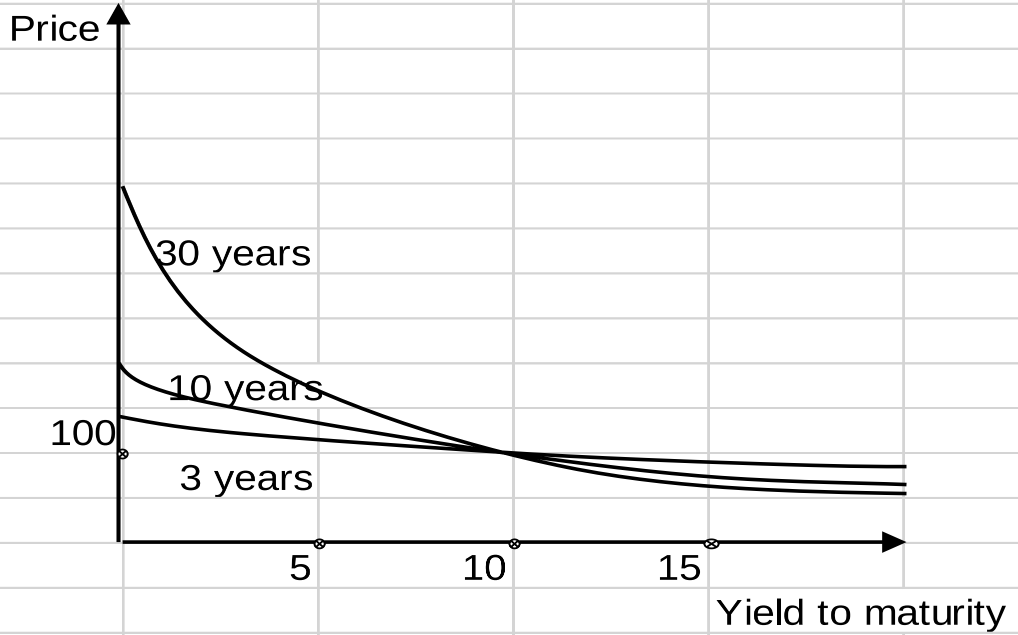
- Higher price corresponds to lower yield.

Price-yield curve

- Terminology: ‘a 10% five-year bond’ is a bond that pays 10% of its face value per year, for five years, plus the face value at maturity.



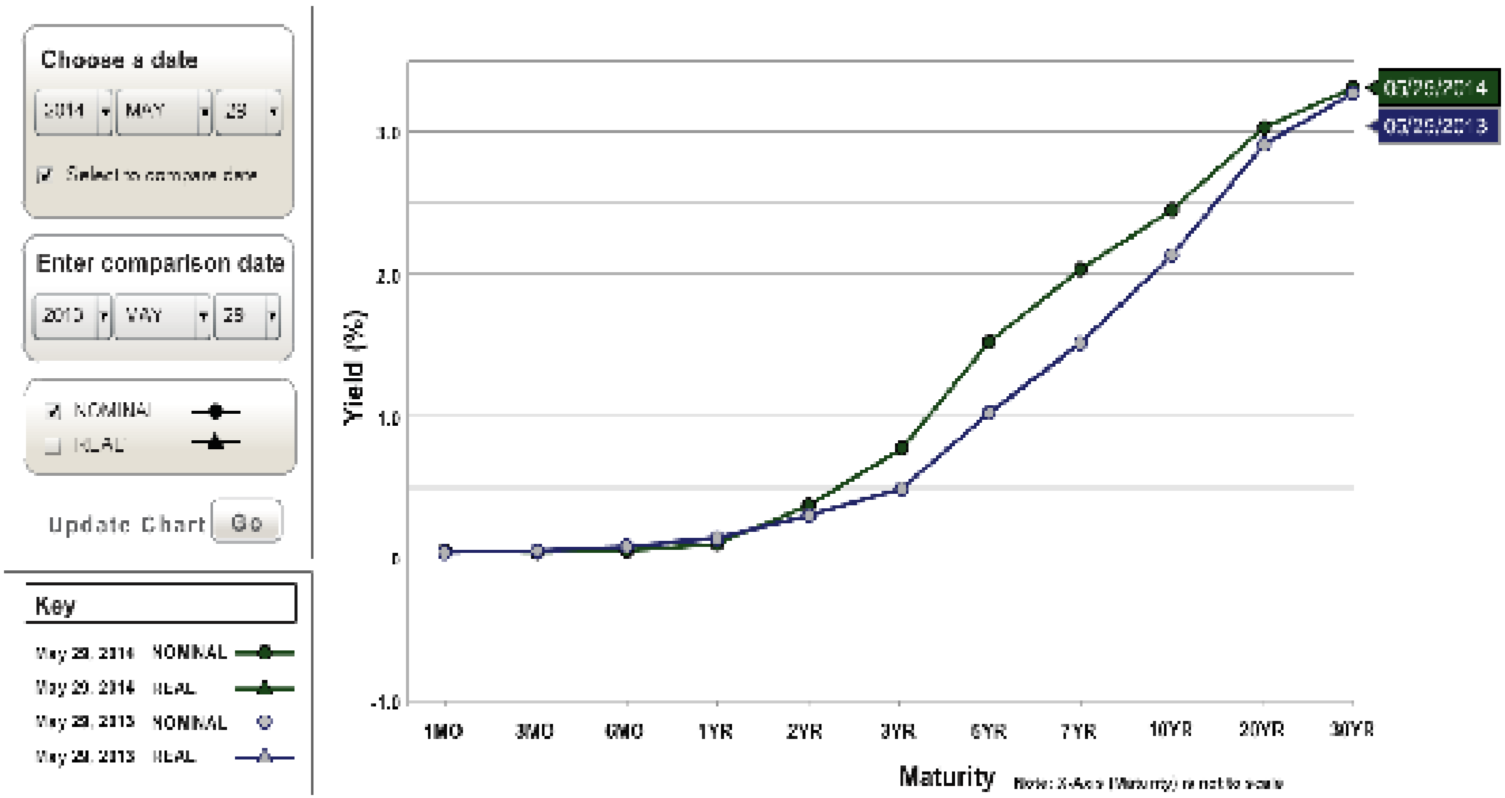
Price yield curves for 30-year bonds with various coupon rates



Price yield curves for 10% coupon bonds with various maturities

- Why do they all intersect at the point (10, 100) ?
- Hint: Set $C = y \cdot V$ in the formula for P. We say 'the bond trades at par'.

Yield curve (term structure of interest rates)



Spot rates and arbitrage

- Spot rate = yield of a **zero-coupon bond** (pure discount bond)
- **Arbitrage** (of strong kind) = making positive sure profit with zero investment
- EXAMPLE: A 6-month zero-coupon bond with face value 100 trades at 98.00. A coupon bond that pays 3.00 in 6 months and 103 in 12 months trades at 101.505. What should be the yield of the 1-year zero coupon bond with face value 100?
- REPLICATION: Find a combination of the traded bonds to replicate exactly the payoff of the 1-year bond.
- BUY: one coupon bond; SELL (short): 0.03 units of the 6-month bond
- COST = 101.505-0.03·98 = 98.565
- In 6 months: pay 3.00 for the short bond, receive 3.00 as a coupon
- In 12 months: receive 103.00
- $98.565 = \frac{103}{1+r}$, $r = 4.4996\ \%$; Otherwise arbitrage!

Alternative computation

- First, compute the 6-month spot rate:

$$98 = \frac{100}{(1+y)^{1/2}} \rightarrow y = 4.1233\%$$

- Then, compute the one-year rate from

$$101.505 = \frac{3}{(1+0.041233)^{1/2}} + \frac{103}{1+r} \rightarrow r = 4.4996\ \%$$

Arbitrage if mispriced

- Suppose the 1-year bond price is 95.00 instead of

$$\frac{100}{1+0.044996} = 95.6942$$

- BUY CHEAP, SELL EXPENSIVE:

  - buy the 1-year bond
  - go short 100/103 of the portfolio that replicates it:

    sell short 100/103 units of the coupon bond;

    buy $0.03 \cdot 100/103$ units of the 6-month bond.

- this results in initial profit of *95.6942*

- After 6 months: have to pay $3 \cdot 100/103$, and receive the same amount
- After 1 year: have to pay 100 and receive 100
- Total profit: *95.6942 – 95.00* → arbitrage!

Forward rates

- r_k = annualized spot rate for k periods from now
- Annualized forward rate between the i-th and j-th period, compounding n times a year:

$$\left(1 + r_j/n\right)^j = (1 + r_i/n)^i \left(1 + f_{i,j}/n\right)^{j-i}$$

- EXAMPLE: The 1-year zero c. bond trades at 95, and the 2-year z.c. bond trades at 89, compounding done once a year.

$95 \cdot (1 + r_1) = 100$ $\rightarrow$ r_1 = 5.2632%

$89 \cdot (1 + r_2)^2 = 100$ $\rightarrow$ r_2 = 5.9998%

$1.052632 \cdot \left(1 + f_{1,2}\right) = (1.059998)^2$ $\rightarrow$ $f_{1,2}$ = 6.7416%

- Suppose you believe $f_{1,2}$ is too high:

- buy one 2-year bond and sell short 89/95 units of the 1-year bond, at zero cost.

- After 1 year: have to pay 89/95·100 = 93.6842
- After 2 years: receive 100 for the second year return of 6.7416%
- If, after 1 year, the 1-year spot rate is, indeed, less than 6.7416%, sell the 1-year bond and receive more than 93.6842, and make arbitrage profit.

Pricing Options with Mathematical Models

8. Model independent pricing relations: forwards, futures and swaps

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Pricing forward contracts

- Consider a forward contract on asset S, starting at t, with payoff at T equal to

  S(T) – F(t)

- Here, F(t) is the **forward price**, decided at t and paid at T.
- QUESTION: What is the value of F(t) that makes time t value of the contract equal to zero?
- Suppose $1.00 invested/borrowed at risk-free rate at time t results in payoff $B(t,T) at time T.

- CLAIM: There is no arbitrage if and only if

$$F(t) = B(t,T)S(t),$$

that is, if the forward price is equal to the time T value of one share worth invested at the risk-free rate at time t.

- Suppose first $F(t) > B(t,T)S(t)$:

 - At t: borrow S(t) to buy one share, and go short in the forward contract;

- At T: deliver the share, receive $F(t)$, which is more than enough to cover the debt of $B(t,T)S(t)$. Arbitrage!

- Suppose now $F(t) < B(t,T)S(t)$:

 - At t: sell short one share, invest S(t) risk-free, long the forward contract.

- At T: have more than enough in savings to pay for $F(t)$ for one share and close the short position.

- Suppose S pays deterministic dividends between t and T, with present value $D(t)$. Then,

$$F(\mathrm{t}) = \mathrm{B}(\mathrm{t},\mathrm{T})(\mathrm{S}(\mathrm{t}) - \mathrm{D}(\mathrm{t}))\ .$$

- Suppose dividends will be paid between t and T, continuously at a constant rate q. Then,

$$F(\mathrm{t}) = \mathrm{B}(\mathrm{t},\mathrm{T})\ e^{-q(T-t)}\ \mathrm{S}(\mathrm{t})\ .$$

EXAMPLE: S(t)=100, a dividend of 5.65 is paid in 6 months. The 1-year continuous interest rate is 10%, and the 6-month continuous annualized rate is 7.41%.The price of the 1-year forward contract should be

$$F(\mathrm{t}) = e^{0.1}\,(100 - e^{-0.0741/2} \cdot 5.65) = 104.5$$

Suppose that the price is instead $F(\mathrm{t}) = 104$.

- At t: long the forward, sell one share, buy the 6-month bond in the amount of $e^{-0.0741/2} \cdot 5.65 = 5.4445$; invest the remaining balance, 100-5.4445=94.5555, in the 1-year bond.

- At 6 months from t : receive 5.65 from the 6-month bond and pay the dividend of 5.65.

- At 1 year from t : receive $e^{0.1} \cdot 94.5555 = 104.5$ from the 1-year bond; pay $F(\mathrm{t}) = 104$ for one share, and deliver the share to cover the short position; keep 104.5-104 = 0.5, as profit. Arbitrage!

EXAMPLE: **Forward contract on foreign currency.** Let S(t) denote the current price in dollars of one unit of the foreign currency. We denote by r_f (r) the foreign (domestic) risk-free rate, with continuous compounding. The foreign interest is equivalent to continuously paid dividends, so we guess that

$$F(\mathrm{t}) = e^{(r-r_f)(T-t)}\,\mathrm{S(t)}\,.$$

If, for example, $F(\mathrm{t}) < e^{(r-r_f)(T-t)}\,\mathrm{S(t)}$:

- At time t: long the forward, borrow $e^{-r_f(T-t)}$ units of foreign currency and invest its value in dollars $e^{-r_f(T-t)}\,\mathrm{S(t)}$ at rate r.

- At time T: use part of the amount $e^{(r-r_f)(T-t)}\,\mathrm{S(t)} > F(t)$ from the domestic risk-free investment to pay $F(t)$ for one unit of foreign currency in the forward contract, and deliver that unit to cover the foreign debt. There is still extra money left. Arbitrage! Similarly if $e^{(r-r_f)(T-t)}\,\mathrm{S(t)} < F(t)$.

Futures

- Main difference relative to forwards: **marked to market** daily.
- The daily profit/loss is deposited to/taken out of the **margin account**:

Total profit/loss for a contract starting at t, ignoring the margin interest rate, using F(T)=S(T),

= [F(t+1)-F(t)] + … + [F(T)-F(T-1)] = S(T)-F(t)

- CLAIM: If the interest rate is deterministic, futures price F(t) is equal to the corresponding forward price.
- REPLICATION: At t=0, go long $e^{-r(T-1)}$ futures; at t=1, increase to $e^{-r(T-2)}$ futures, …, at t=T-1, increase to 1 future contract.

Profit/loss in period (k,k+1) = $[F(k+1)-F(k)]e^{-r(T-(k+1))}$, the time T value of which is= $[F(k+1)-F(k)]$. Thus, time T profit/loss is S(T)-F(0), the same as for a forward contract.

Swaps pricing

- The payoff of the party receiving the floating rate L and paying the fixed rate R at time T_i is

$$C_i := \Delta T[L(T_{i-1}, T_i) - R]$$

- LIBOR rate is, by definition,

$$L(T_{i-1}, T_i) := \frac{1 - P(T_{i-1}, T_i)}{\Delta T P(T_{i-1}, T_i)} \quad ,$$

implying

$$C_i = \frac{1}{P(T_{i-1}, T_i)} - (1 + R\Delta T)$$

- The value at time $t < T_0$ of the constant amount $(1 + R\Delta T)$ paid at time $T_i > t$ is $(1 + R\Delta T)P(t, T_i)$. (Why?)

Swaps pricing (continued)

- As for the first term, we claim that the value at time $t < T_0$ of the payoff $1/P(T_{i-1}, T_i)$ paid at time T_i is equal to $P(t, T_{i-1})$. Indeed, if we invest $P(t, T_{i-1})$ at time t in buying a bond with maturity T_{i-1}, we get 1 dollar at time T_{i-1}, with which we can buy exactly $1/P(T_{i-1}, T_i)$ of bonds with maturity T_i, and hence collect $1/P(T_{i-1}, T_i)$ at time T_i. Altogether, the price at time t is

$$C_i(t) = P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)$$

Therefore, the price $S(t)$ of the swap at time t is

$$S(t) = \sum_{i=1}^{n} C_i(t) = \sum_{i=1}^{n} [P(t, T_{i-1}) - (1 + R\Delta T)P(t, T_i)]$$

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^{n} P(t, T_i) - P(t, T_n)$$

Swaps pricing (continued)

$$S(t) = P(t, T_0) - R\Delta T \sum_{i=1}^{n} P(t, T_i) - P(t, T_n) \quad .$$

- The **swap rate** R is the fixed rate such that the cost of entering the swap at the initial time 0 is equal to zero. We get

$$R = \frac{P(0, T_0) - P(0, T_n)}{\Delta T \sum_{i=1}^{n} P(0, T_i)}$$

Example

- Consider a swap investor who receives a fixed rate of 10% in semiannually paid coupons and pays the six-month LIBOR. The swap still has 9 months left to maturity. At the last resetting date, the six-month LIBOR was 6%. The continuous three-month and nine-month rates are 5% and 7%, respectively. Nominal principal is 10,000. In the above notation, T_0 is 3 months in the past, T_1 is 3 months from now, and T_2 is 9 months from now, $\Delta T = 0.5$. We want to find the value of the investor's position today.

- The payoff 3 months from now, on one unit of the notional principal, is

$$C_1 = \frac{1}{P(T_0, T_1)} - (1 + R\Delta T)$$

and 9 months from now it is

$$C_2 = \frac{1}{P(T_1, T_2)} - (1 + R\Delta T)$$

Example (continued)

- The first payoff's price is obtained by multiplying it by the price of the 3-month bond, and is equal to

$$C_1(t) = \left[\frac{1}{P(T_0, T_1)} - (1 + R\Delta T)\right] e^{-0.05 \cdot 0.25}$$

- The bond price $P = P(T_0, T_1)$ can be found from $P(1 + \Delta T L) = 1$, where $L = 0.06$ is the LIBOR rate. We get $P = 1/1.03$, and $C_1(t) = -0.0198$.

- The price of C_2 is found to be

$$C_2(t) = P(t, T_1) - (1 + R\Delta T)P(t, T_2) = e^{-0.05 \cdot 0.25} - (1 + 0.1 \cdot 0.5)e^{-0.07 \cdot 0.75} = -0.0087$$

Example (continued)

- Altogether, the value of the swap for the short position is

$$C(t) = C_1(t) + C_2(t) = -0.0285$$

- Since our investor is long the swap, his value is

$$10,000 \cdot 0.0285 = 285$$

Pricing Options with Mathematical Models

9. Model independent pricing relations: options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

- Notation:

- European call and put prices at time t, $c(t), p(t)$
- American call and put prices at time t, $C(t), P(t)$

- RELATION 1: $c(t) \leq C(t) \leq S(t)$
- RELATION 2: $p(t) \leq P(t) \leq K$
- RELATION 3: $p(t) \leq Ke^{-r(T-t)}$
- RELATION 4: $c(t) \geq S(t) - Ke^{-r(T-t)}$, if S pays no dividends

- Suppose not: $c(t) + Ke^{-r(T-t)} < S(t)$; sell short one share and have more than enough money to buy one call and invest $Ke^{-r(T-t)}$ at rate r.

At T: If $S(T) > K$, exercise the option by buying S(T) for K;

If $S(T) \leq K$, buy stock from your invested cash.

- RELATION 5: $p(t) \geq Ke^{-r(T-t)} - S(t)$

- RELATION 6: $c(t) = C(t)$, if S pays no dividends.

- Suppose not: $c(t) < C(t)$

1. At t : Sell $C(t)$ and have more than enough to buy c(t);
2. If C exercised at τ : have to pay $S(\tau) - K$, which is possible by selling c, because $c(t) \geq S(t) - Ke^{-r(T-t)} \geq S(t) - K$;
3. If C never exercised, there is no obligation to cover.

Arbitrage!

- COROLLARY: An American call on an asset that pays no dividends should not be exercised early.

- Indeed, it is better to sell it than to exercise it: $C(t) \geq S(t) - K$.

- What if there are dividends?
- What about the American put option?

- RELATION 7, **Put-Call Parity**:

$$c(t) + Ke^{-r(T-t)} = p(t) + S(t)$$

1. Portfolio A: buy c(t) and invest discounted K at risk-free rate;
2. Portfolio B: buy put and one share.

- If $S(T) > K$, both portfolios worth $S(T)$ at time T.
- If $S(T) \leq K$, both portfolios worth K at time T.

- RELATION 8:

$$\mathrm{S(t)} - \mathrm{K} \leq C(t) - P(t) \leq S(t) - Ke^{-r(T-t)}$$

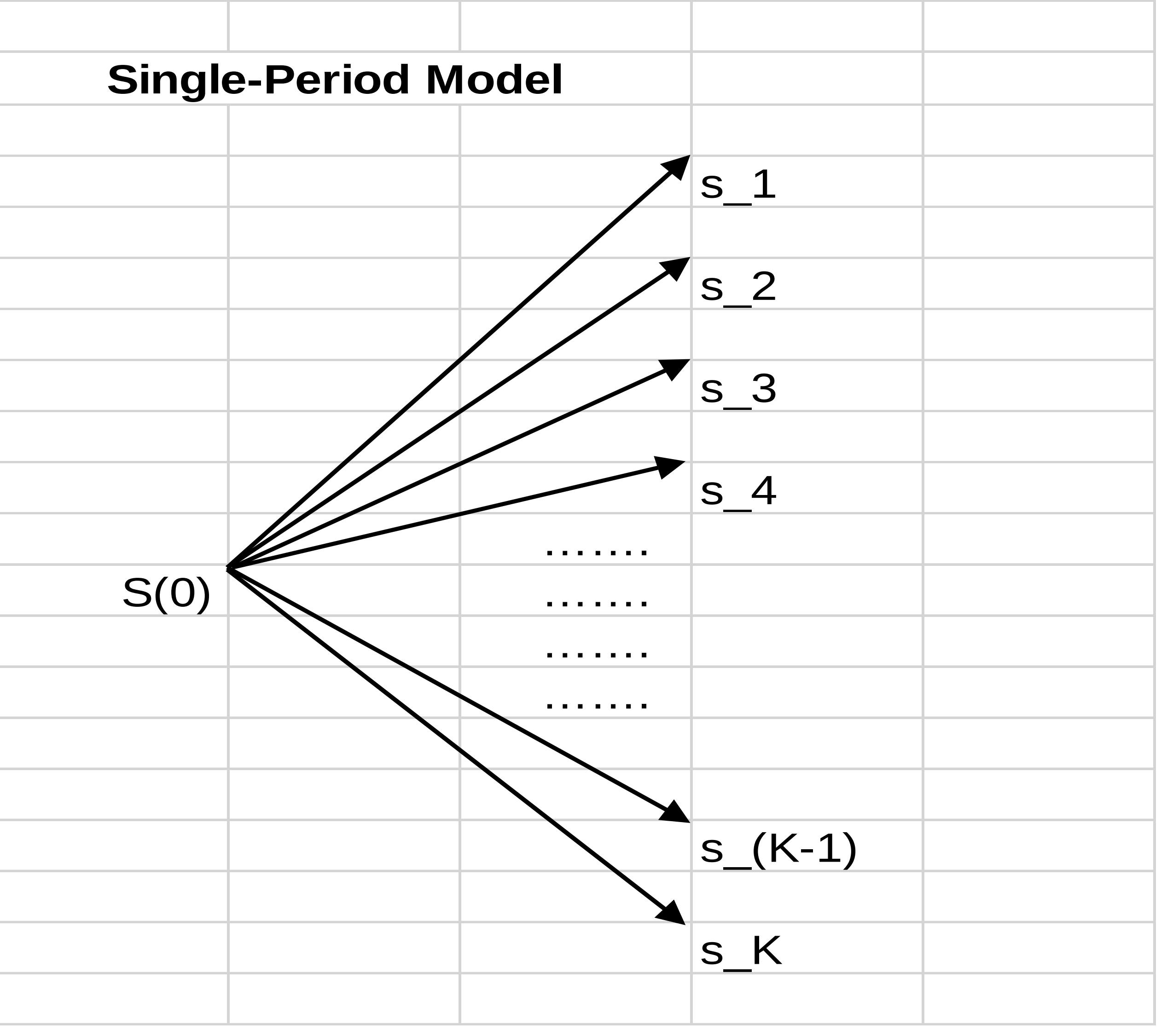
- The RH side follows from put-call parity and $P(t) \geq p(t), C(t) = c(t)$.
- For the LHS, suppose not: $\mathrm{S(t)} + \mathrm{P(t)} > C(t) + K$.

1. At t : Sell the LHS and have more than enough to buy the RHS;
2. If P exercised at τ : use the invested cash to pay K for $S(\tau)$;
3. If P never exercised, exercise C at maturity.

Pricing Options with Mathematical Models

10. Discrete-time models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.



$$P(S(T) = s_i) = p_i$$

- Risk-free asset, bank account:

$$B(0) = 1, B(1) = 1 + r$$

- Initial wealth:

$$X(0) = x$$

- Number of shares in asset i: δ_i

- End-of-period wealth:

$$X(1) = \delta_0 B(1) + \delta_1 S_1(1) + \cdots + \delta_N S_N(1)$$

- Budget constraint, self-financing condition:

$$X(0) = \delta_0 B(0) + \delta_1 S_1(0) + \cdots + \delta_N S_N(0)$$

- **Profit/loss, P&L**, or the **gains** of a portfolio strategy:
  G(1) = X(1) - X(0)
- Discounted version of process Y: $\overline{Y}(t) = Y(t)/B(t)$
- Change in price: $\Delta S_i(1) = \mathrm{S}_i(1) - \mathrm{S}_i(0)$
- We have

$$\mathrm{G}(1) = \delta_0 r + \delta_1 \Delta S_1(1) + \cdots + \delta_N \Delta S_N(1)$$

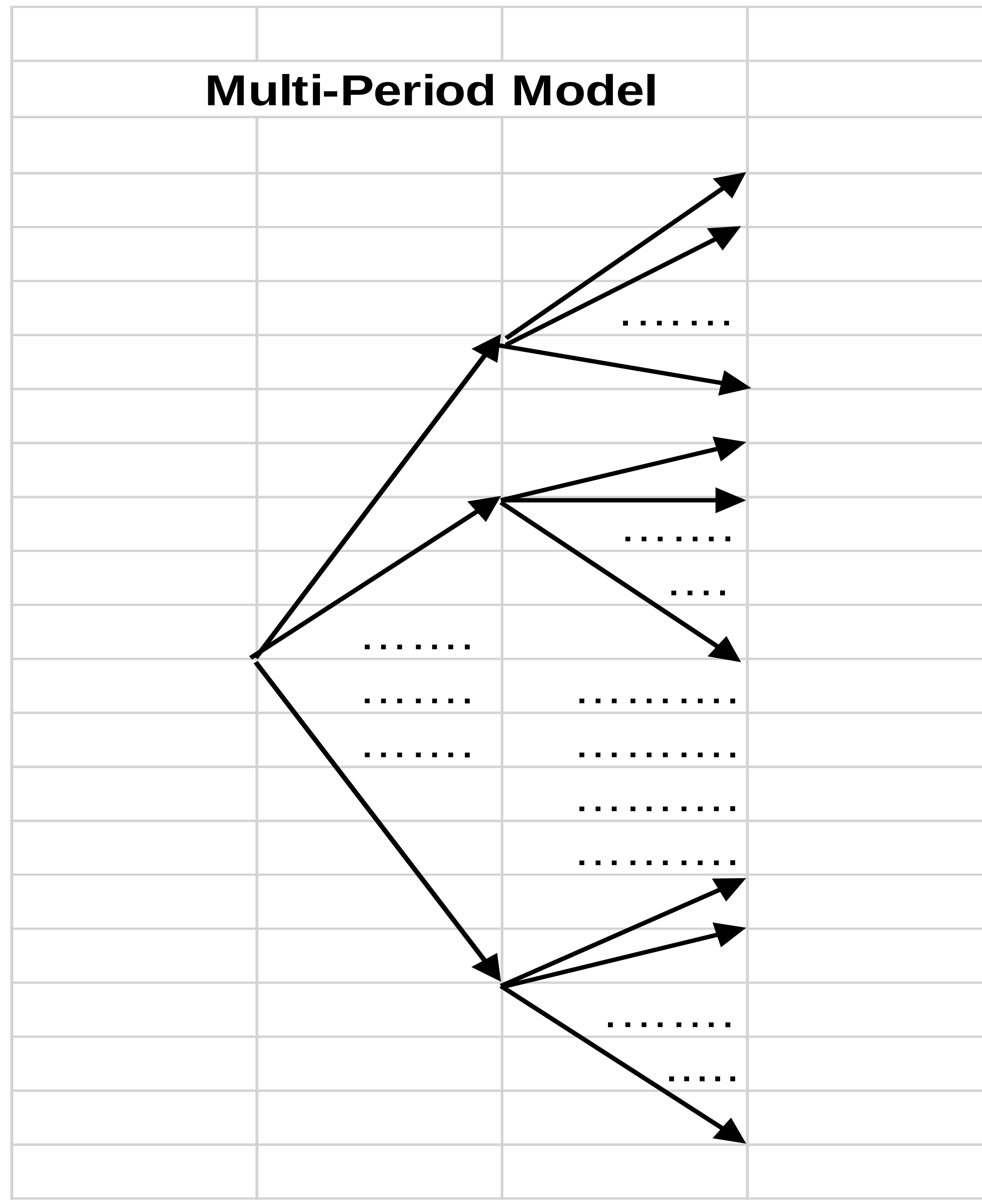
$$X(1) = X(0) + G(1)$$

Denoting

$$\Delta\bar{S}_i(1) = \bar{S}_i(1) - \mathrm{S}_i(0), \qquad \overline{G}(1) = \delta_1 \Delta\bar{S}_1(1) + \cdots + \delta_N \Delta\bar{S}_N(1)$$

one can verify that

$$\bar{X}(1) = X(0) + \overline{G}(1)$$



Multi-Period Model

- Risk-free asset, bank account:

$$B(0) = 1, B(t) = \big(1 + r(t)\big)B(t-1)$$

- Number of shares in asset i during the period $[t-1, t)$: δ_i(t)

- Wealth process:

$$X(t) = \delta_0(t)B(t) + \delta_1(t)S_1(t) + \cdots + \delta_N(t)S_N(t)$$

- Self-financing condition:

$$X(t) = \delta_0(t+1)B(t) + \delta_1(t+1)S_1(t) + \cdots + \delta_N(t+1)S_N(t)$$

- Change in price: $\Delta S_i(t) = \mathrm{S}_i(\mathrm{t}) - \mathrm{S}_i(t-1)$

- $\mathrm{G}(\mathrm{t}) = \sum_{s=1}^{t} \delta_0(s)\, \Delta B(s) + \sum_{s=1}^{t} \delta_1(s)\, \Delta \mathrm{S}_1(s) + \cdots + \sum_{s=1}^{t} \delta_N(s)\, \Delta \mathrm{S}_N(s)$

- It can be checked that $X(t) = X(0) + G(t)$

Denoting

$$\Delta \bar{S}_i(t) = \bar{S}_i(\mathrm{t}) - \bar{S}_i(\mathrm{t}-1),$$

$$\overline{G}(t) = \sum_{s=1}^{t} \delta_1(s)\, \Delta \bar{S}_1(s) + \cdots + \sum_{s=1}^{t} \delta_N(s)\, \Delta \bar{S}_N(s)$$

one can verify that

$$\bar{X}(t) = X(0) + \bar{G}(t)$$

For example, with one risky asset and two periods:

- Change in price: $\Delta S_i(t) = S_i(\mathrm{t}) - S_i(t-1)$
- $G(2) = \delta_0(1)\big(B(1) - B(0)\big) + \delta_0(2)\big(B(2) - B(1)\big)$
  $+\delta_1(1)\big(S(1) - S(0)\big) + \delta_1(2)\big(S(2) - S(1)\big)$
- Using self-financing

$$\delta_0(1)B(1) + \delta_1(1)S(1) = \delta_0(2)B(1) + \delta_1(2)S(1)$$

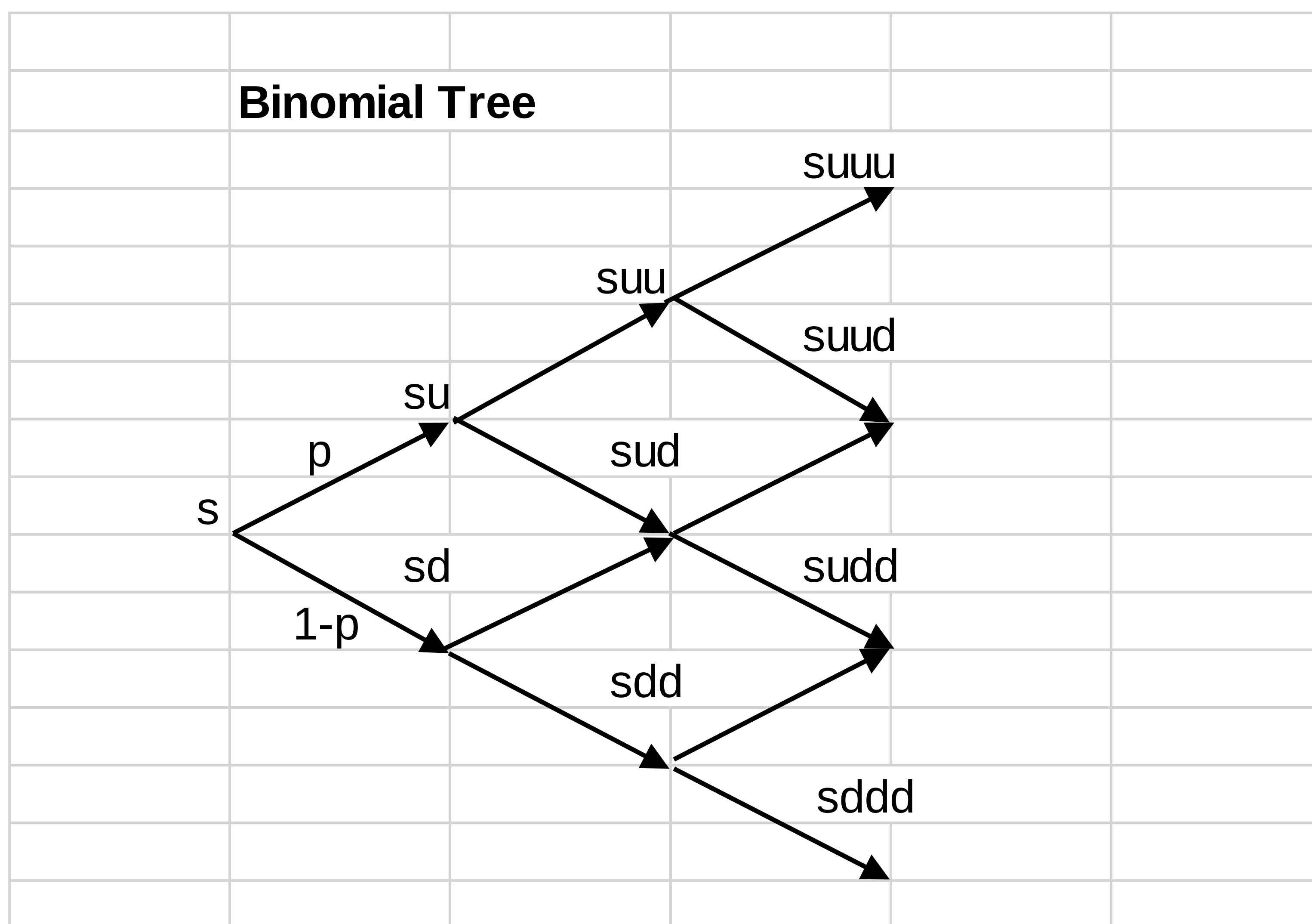
we get

- $G(2) = \delta_0(2)B(2) + \delta_1(2)S(2) - \delta_0(1)B(0) - \delta_1(1)S(0)$
- This is the same as

G(2) = X(2)-X(0)

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P\big(S(t+1) = u\,S(t)\big)$, $1 - p = P\big(S(t+1) = d\,S(t)\big)$
- u > 1+r > d



Pricing Options with Mathematical Models

11. Risk-neutral pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Martingale property

- Insurance pricing:

$$C(t) = E_t[\, e^{-r(T-t)} C(T)\,]$$

where E_t is expectation given the information up to time t.

- For a stock, this would mean:

$$e^{-rt} S(t) = E_t[\, e^{-rT} S(T)\,]$$

- If so, we say that $M(t) = e^{-rt} S(t)$ is a **martingale process**:

$$M(t) = E_t[\, M(T)\,]$$

Martingale probabilities (measures)

- Typically, the stock price process will not be a martingale under the actual (physical) probabilities, but it may be a martingale under some other probabilities.
- Those are called **martingale**, or **risk-neutral**, or **pricing probabilities**.
- Such probabilities are typically denoted Q, q_i, sometimes $P^*,\ p_i^*$.
- We write:

$$e^{-rt}S(t) = E_t^Q\left[\, e^{-rT}S(T) \,\right],$$

or $$e^{-rt}S(t) = E_t^*\left[\, e^{-rT}S(T) \,\right]$$

Risk-neutral pricing formula

- Thus, we expect to have, for some risk-neutral probability Q

$$\boldsymbol{Price\ of\ claim\ today}$$
$$\boldsymbol{= expected\ value, under\ Q, of\ the\ claim's\ discounted\ future\ payoff}$$

or

$$\mathrm{C}(t) = E_t^Q\ [\ e^{-r(T-t)}C(T)\ ]$$

if $C(T)$ is paid at T, and the continuously compounded risk-free rate r is constant.

- How to justify this formula?
- Which Q? Are there any? How many?

Example: A Single Period Binomial model

- r=0.005, S(0)=100, $s^u = 101, s^d = 99$, that is, u=1.01, d=0.99.
- The payoff is an European Call Option, with payoff

$$\max\{S(1) - 100, 0\}$$

- It will be \$1 if the stock goes up and \$0 if the stock goes down. Looking for the **replicating portfolio**, we solve

$$\delta_0(1 + 0.005) + \delta_1 101 = 1\,;$$
$$\delta_0(1 + 0.005) + \delta_1 99 = 0.$$

We get

$$\delta_0 = -49.254,\ \delta_1 = 0.5$$

Example continued

- $\delta_0 = -49.254$, $\delta_1 = 0.5$
- This means borrow 49.254, and buy one share of the stock. This costs

$$C(0) = 0.5 \times 100 - 49.254 = 0.746$$

This is the no arbitrage price:

1) Suppose the price is higher, say 1.00. Sell the option for 1.00, invest 1-0.746 at the risk-free rate; use 0.746 to set up the replicating strategy; have 1 if stock goes up, and 0 if it goes down, exactly what you need. Arbitrage.
2) Suppose the price is lower, say 0.50. Buy the option for 0.50, sell short half a share for 50, invest 49.254 at the risk-free rate; This leaves you with extra 0.246 today. If stock goes up you make 1.00 from the option; together with 49.254×1.005, this covers 101/2 to close your short position. If stock goes down, use 49.254×1.005 to cover 99/2 when closing your short position. Arbitrage.

Martingale pricing

- Suppose the discounted wealth process $\bar{X}$ is a martingale under Q, and suppose it replicates C(T), so that X(T)=C(T). By the martingale property,

$$\bar{X}(t) = E_t^Q \bar{X}(T) = E_t^Q \bar{C}(T)$$

For example, if discounting is continuous at a constant rate r, this gives

$$X(t) = E_t^Q [\, e^{-r(T-t)} C(T) \,]$$

This is the cost of replication at time t, therefore, for any such probability Q,

the price/value of the claim at time t is equal to the expectation, under Q, of the discounted future payoff of the claim.

Single Period Binomial model

- The future wealth value is

$$X(1) = \delta_0(1+r) + \delta_1 S(1)$$

thus, when discounted,

$$\bar{X}(1) = \delta_0 + \delta_1 \bar{S}(1)$$

Therefore, if the discounted (non-dividend paying) stock is a martingale, so is the discounted wealth. For the stock to be a Q-martingale, we need to have

$$S(0) = E^Q \frac{S(1)}{1+r} = \frac{1}{1+r}(q \times s^u + (1-q) \times s^d)$$

Solving for q, we get, with $s^u = S(0)u, s^d = S(0)d$,

$$q = \frac{(1+r)-d}{u-d}, 1-\mathrm{q} = \frac{u-(1+r)}{u-d}$$

Example (the same as above)

$$s^u = 100 \times 1.01, s^d = 100 \times 0.99,$$

$$q = \frac{(1+r) - d}{u - d} = \frac{1.005 - 0.99}{1.01 - 0.99} = 0.75$$

Thus, the price of the call option is

$$\text{C}(0) = E^Q \frac{C(1)}{1+r} = \frac{1}{1+r}\left(q \times C^u + (1-q) \times C^d\right)$$

$$= \frac{1}{1 + 0.005}(0.75 \times (101 - 100) + (1 - 0.75) \times 0)$$

=0.746 (the same as above)

Forwards

- Let D denote the process used for discounting, for example

$$\mathrm{D}(t) = e^{-rt}$$

- .We want the forward price $F(t)$ to be such that the value of the forward contract zero at the initial time t:

$$0 = E_t^Q[\,\{S(T) - F(t)\,\}\frac{D(T)}{D(t)}\,]$$

Since DS is a Q-martingale, we have $E_t^Q[D(T)S(T)] = D(t)S(t)$, and we get

$$F(t) = S(t)\frac{D(t)}{E_t^Q[D(T)]}$$

which, for the above $\mathrm{D}(t)$, is the same as

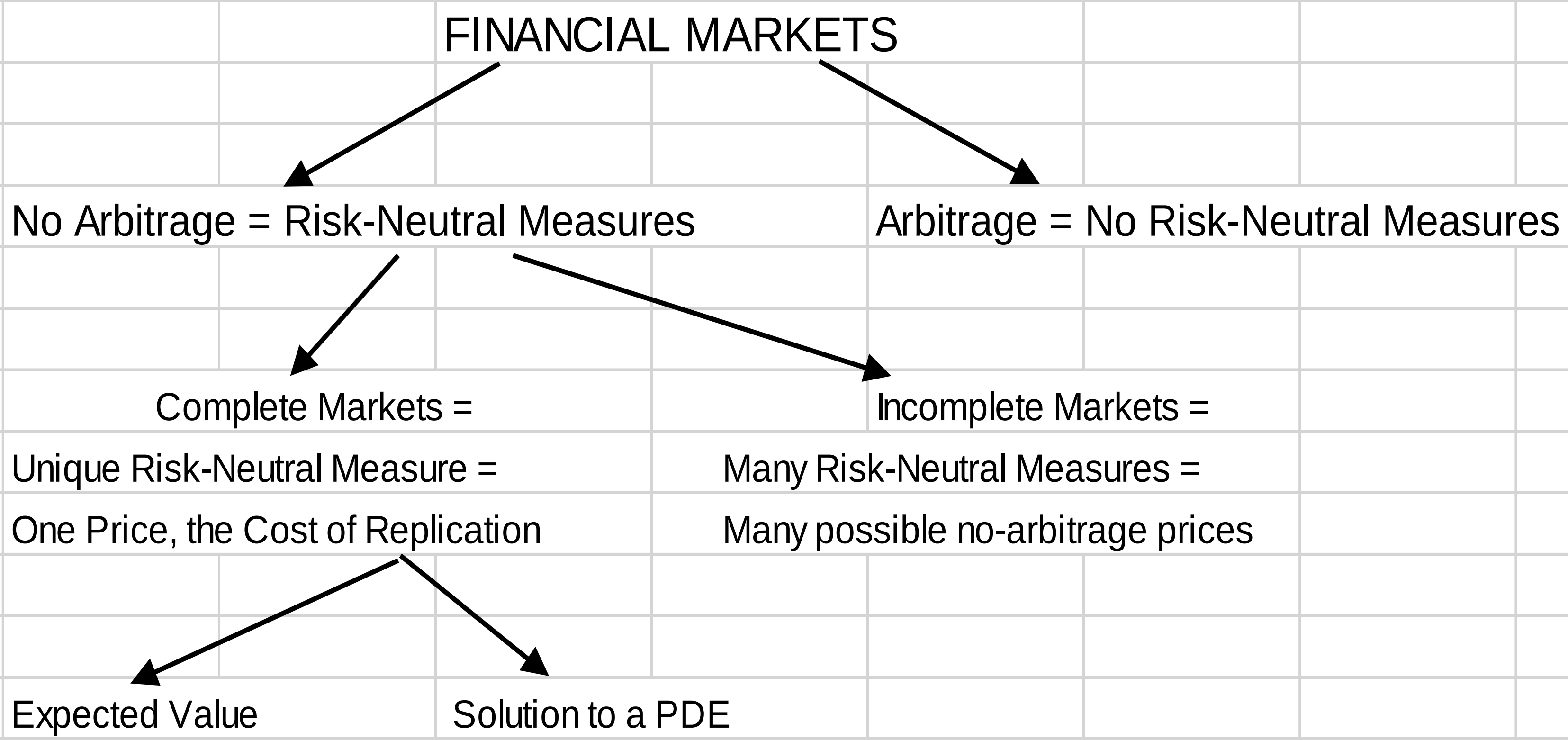
$$F(t) = S(t)e^{r(T-t)} = S(t)B(t,T)$$

Pricing Options with Mathematical Models

12. Fundamental theorems of asset pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Risk-Neutral Pricing



Equivalent martingale measures (EMM's)

Recall

$$q = \frac{(1+r)-d}{u-d}, 1 - \mathrm{q} = \frac{u-(1+r)}{u-d}$$

Thus, q and $1-q$ are strictly between zero and one if and only if

$$d < 1 + r < u$$

Then, the events of non-zero P probability also have non-zero Q probability, and vice-versa. We say that P and Q are **equivalent probability measures**, and Q is called an **equivalent martingale measure** (EMM). Note also that Q is the only EMM.

First fundamental theorem of asset pricing

$$\boldsymbol{No\ arbitrage = existence\ of\ at\ least\ one\ \ EMM}$$

Definition of arbitrage: there exists a strategy such that, for some T,

$X(0) = 0, X(T) \geq 0$ with probability one, and

$$P(X(T) > 0) > 0$$

One direction: suppose there exists an EMM Q, and a strategy with $X(T)$ as above. Then,

$$X(0) = E^Q\,\bar{X}(T) > 0,$$ so, no arbitrage.

Second fundamental theorem of asset pricing

- **Definition of completeness**: a market (model) is complete if every claim can be replicated by trading in the market.

$$\boldsymbol{Completeness\ and\ no\ arbitrage} \\ \boldsymbol{= existence\ of\ exactly\ one\ \ EMM}$$

- In a complete market, every claim has a unique price, equal to the cost of replication, also equal to the expectation under the unique EMM.
- Even in an incomplete market, one assumes that there is one EMM Q (among many), that the market chooses to price all the claims.
- How to compute it?

Example: Binomial tree model is arbitrage free and complete if $d < 1 + r < u$

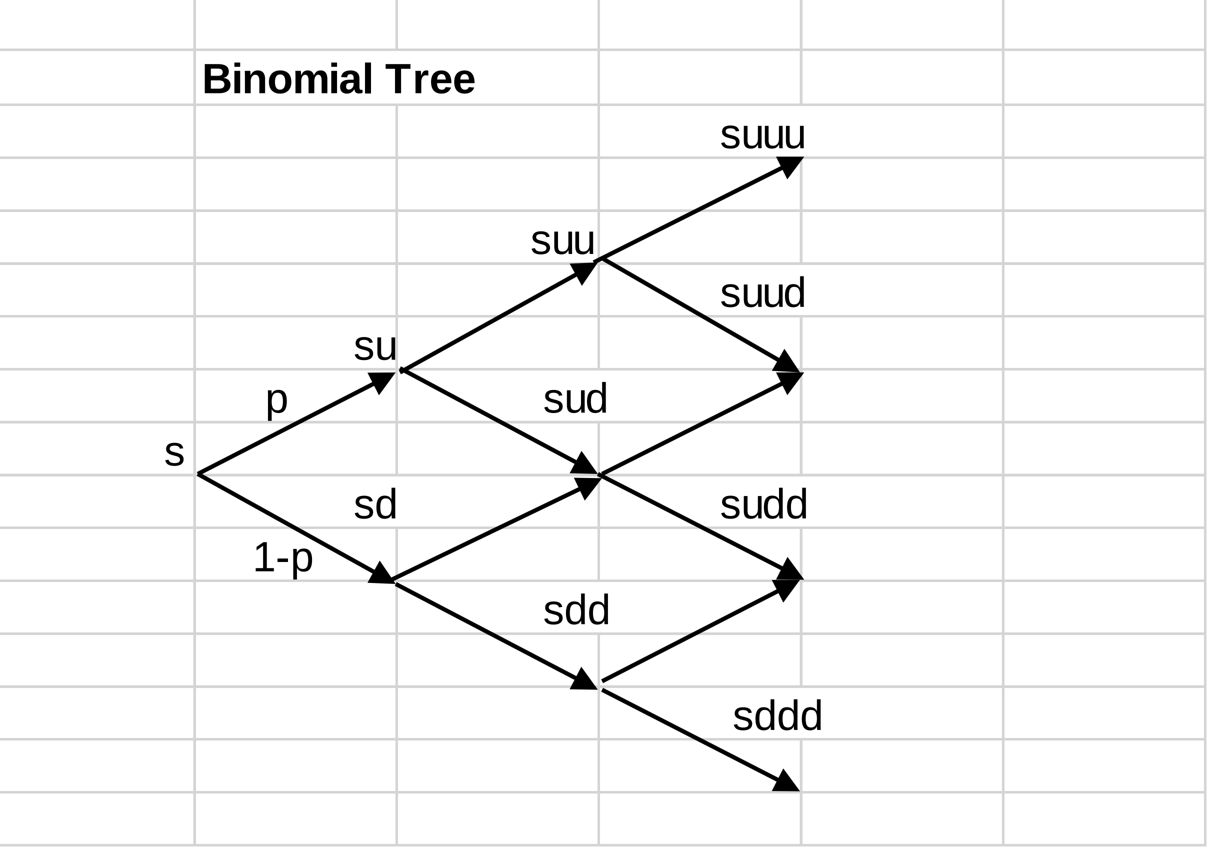
Pricing Options with Mathematical Models

13. Binomial tree pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Binomial Tree (Cox-Ross-Rubinstein) model

- $p = P\big(\,S(t+1) = u\,S(t)\big)$,
- $1 - p = P\big(\,S(t+1) = d\,S(t)\big)$
- u > $e^{r\Delta t}$ > d



Expectation formula

- CLAIM: Given a random variable X whose value will be known at time T, process $\mathrm{M(t)} = E_t[X]$ is a martingale. Indeed, for $s < t$,

$$E_s[M(t)] = E_s E_t\,[\mathrm{X}] = E_s[X] = M(s)$$

where the middle equality is the so- called ***law of iterated expectations***.

- Since

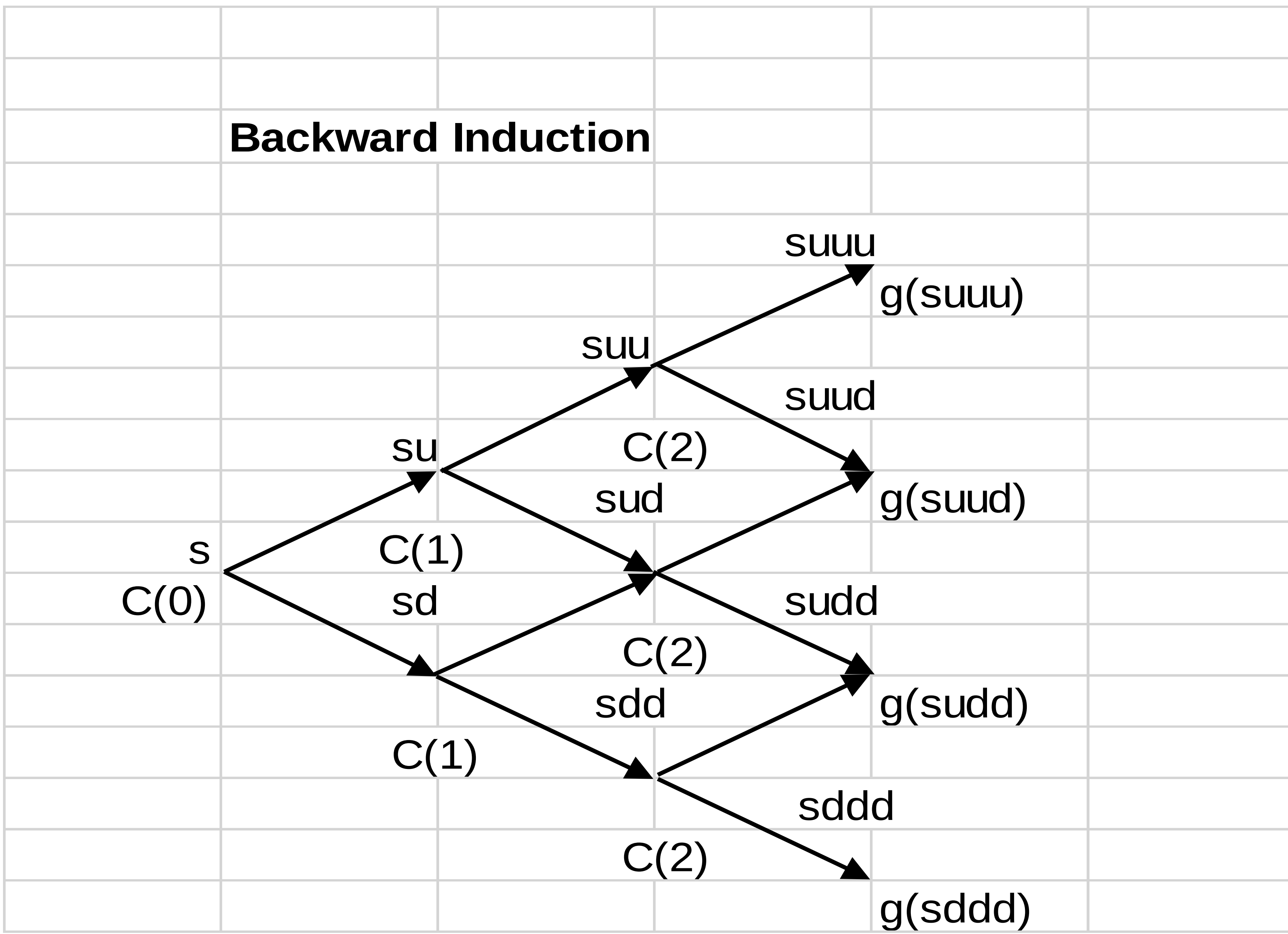
$e^{-rt}\mathrm{C}(t) = E_t^Q\,[\,e^{-rT}C(T)\,]$

we conclude that $e^{-rt}\mathrm{C}(t)$ is a Q-martingale. Therefore,

$e^{-rt}\mathrm{C}(t) = E_t^Q\,[\,e^{-r(t+\Delta t)}C(t+\Delta t)\,]$, and we get the **expectation formula**

$$\mathrm{C}(t) = E_t^Q\left[\,e^{-r\Delta t}C(t+\Delta t)\right]$$
$$= e^{-r\Delta t}[\mathrm{q}\times C^u(t+\Delta t) + (1-q)\times C^d(t+\Delta t)]$$

Pricing path-independent payoff $g(S(T))$



Example: a call option

			European Option Price			
					121.0000	
					21.0000	
				110.0000		
				15.1088		
S(0)	100.0000	S(0)	100.0000		99.0000	
u	1.1000	C(0)	10.8703		0.0000	
d	0.9000					
K	100.0000					
r	0.0500					
p*	0.7564			90.0000		
Delta t	1.0000			0.0000		
					81.0000	
					0.0000	

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$15.1088 = e^{-r\Delta t}[\, q \times 21 + (1 - q) \times 0\,]$$

$$10.87 = e^{-r\Delta t}[\, q \times 15.1088 + (1 - q) \times 0\,]$$

American options

- Consider an American option that pays $g(\tau)$dollars if exercised at time $\tau \leq T$.
- It can be shown that, in a complete market with discrete time and time intervals of length Δt, its no-arbitrage price $A(t)$is given by

$$\mathrm{A}(t) = \max_{t \leq \tau \leq T} E_t^Q[\, e^{-r(\tau - t)} g(\tau) \,]$$

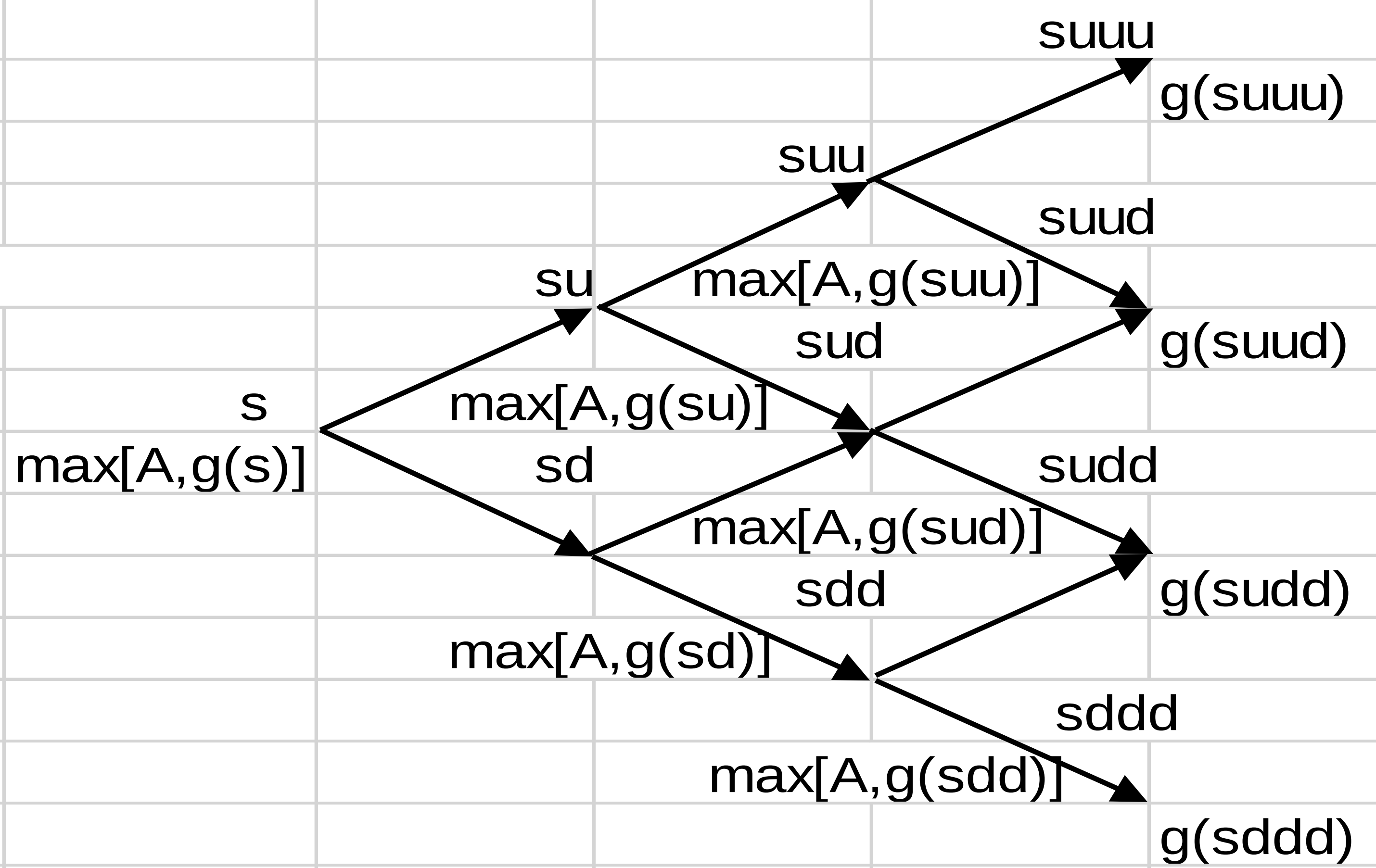
- The expectation formula is given by the dynamic programming principle:

$$\mathrm{A}(t) = \max\ \{g(t)\,, E_t^Q[e^{-r\Delta t} g(t + \Delta t)\,]\,\}$$

or, in the binomial model,

$$\mathrm{A}(t) = \max[g(t),\ e^{-r\Delta t}\{\mathrm{q} \times A^u(t + \Delta t) + (1 - q) \times A^d(t + \Delta t)\}]$$

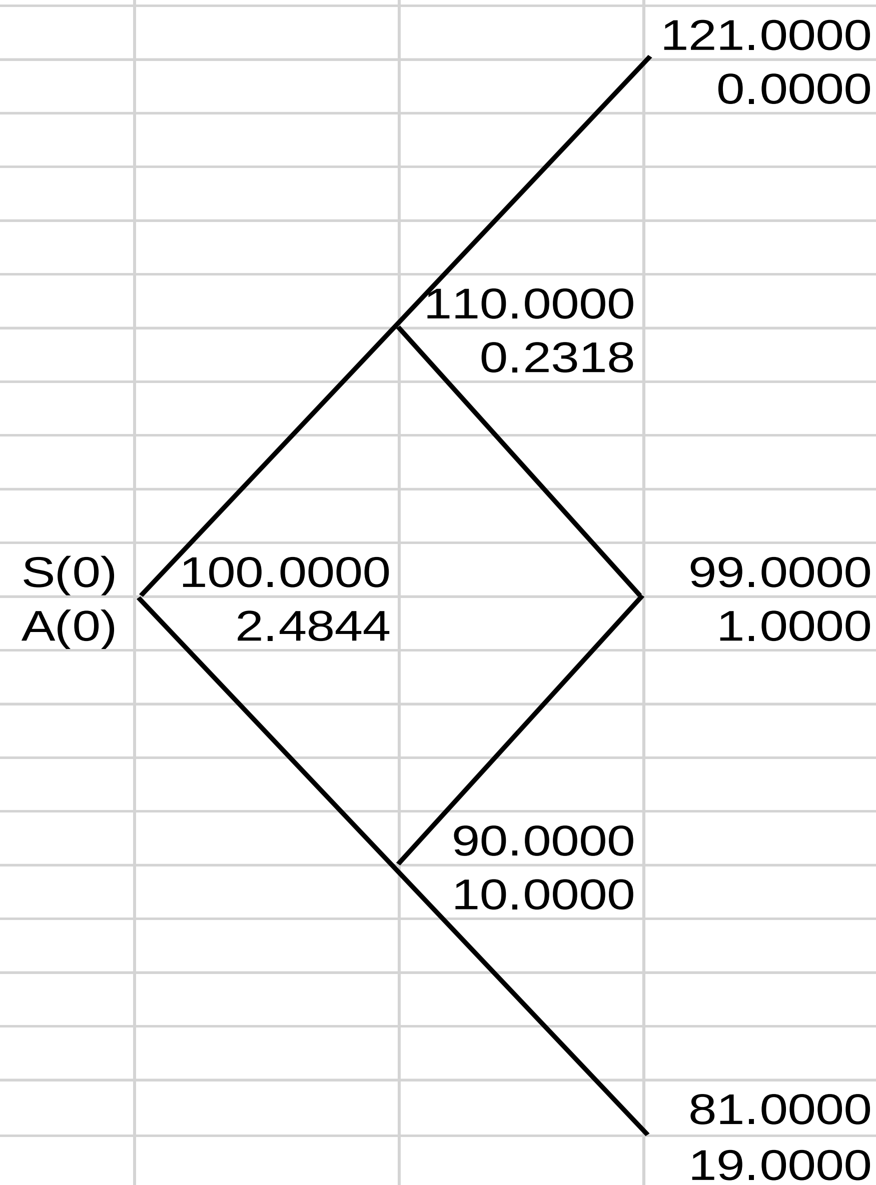
Backward Induction for American Options



Example: a put option

American Option Price

S(0)	100.0000
u	1.1000
d	0.9000
K	100.0000
r	0.0500
p*	0.7564
Delta t	1.0000



$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.7564$$

$$10 = \max\{10, e^{-r\Delta t}[\, q \times 1 + (1 - q) \times 19]\,\}$$
$$= \max\{10, 5.1229\}$$

$$0.2318 = \max\{0, e^{-r\Delta t}[\, q \times 0 + (1 - q) \times 1\,]\,\}$$

$$2.4844 = \max\{0, e^{-r\Delta t}[\, q \times 0.2318 + (1 - q) \times 10\,]\}$$

Pricing Options with Mathematical Models

14. Brownian motion process

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

History

- Brown, 1800's
- Bachelier, 1900
- Einstein 1905, 1906
- Wiener, Levy, 1920's, 30's
- Ito, 1940's
- Samuelson, 1960's
- Merton, Black, Scholes, 1970's

A short introduction to the Merton-Black-Scholes model

- Risk-free asset

$$B(t) = e^{rt}$$

- Stock has a **lognormal distribution**:

$$\log S(t) = log\, S(0) + (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\mathrm{z(t)}$$

where $\mathrm{z(t)}$ is a standard normal random variable. Thus,

$$S(t) = S(0)\ e^{\ (\mu - \frac{1}{2}\sigma^2)t + \sigma\sqrt{t}\mathrm{z(t)}}$$

and it can be shown that

$$ES(t) = S(0)\ e^{\mu t},\ \ \frac{1}{t}Var\left[\log\frac{S(t)}{S(0)}\right] = \sigma^2$$

Discretized Brownian motion

- W(0) = 0
- $\mathrm{W}(t_{k+1}) = \mathrm{W}(t_k) + \sqrt{\Delta t}\, \mathrm{z}(t_k)$

where $\mathrm{z}(t_k)$ are independent standard normal random variables.

Thus,

- $\mathrm{W}(t_l) - \mathrm{W}(t_k) = \sqrt{\Delta t}\ \sum_{i=k}^{l-1} \mathrm{z}(t_i)$

is normally distributed, with zero mean and variance $(l-k)\Delta t = t_l$-t_k

Brownian motion definition

- **(i)** $W(t)-W(s)$ is normally distributed with mean zero and variance $t-s$, for $s<t$.

- **(ii)** The process W has independent increments: for any set of times $0 \le t_1 < t_2 < \cdots < t_n$, the random variables

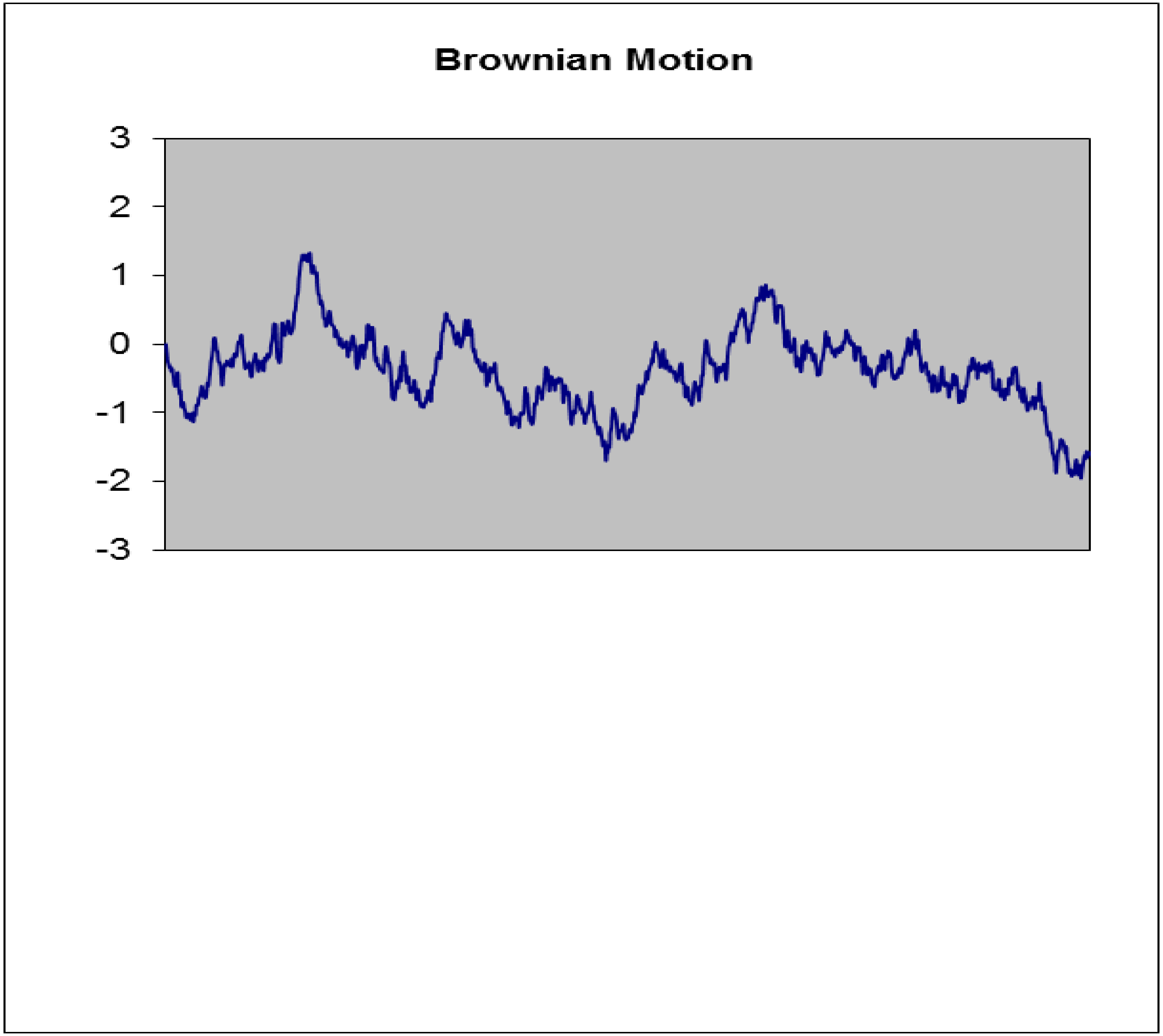
$$W(t_2)-W(t_1), W(t_3)-W(t_2), \ldots, W(t_n)-W(t_{n-1})$$

  are independent.

- **(iii)** $W(0)=0$.

- **(iv)** The sample paths $\{W(t); t \ge 0\}$ are continuous functions of t.

A simulated path of Brownian motion



Brownian motion properties

- Not differentiable: $E\frac{[W(t)-W(s)]^2}{(t-s)^2} = \frac{1}{t-s} \to \infty$ as $(t-s) \to 0$.
- A Markovian process: the distribution of the future value $W(t)$ given information up to time $s < t$ depends only on $W(s)$ and not on the past values.
- Martingale property:

$$E_s W(t) = W(s),\ t > s$$

because

$$\begin{aligned} E_s W(t) &= E[W(t)|W(s)] = E[W(t) - W(s)|W(s)] + E[W(s)|W(s)] \\ &= E[W(t) - W(s)] + W(s) \\ &= W(s) \end{aligned}$$

Pricing Options with Mathematical Models

15. Stochastic integral

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Stochastic Differential Equations

- Modeling a process in time with an Ordinary Differential Equation:

$$\frac{dX(t)}{dt} = \mu(t, X(t))$$

which may be informally written as

$$dX(t) = \mu(t, X(t))dt$$

- We would like to have a Stochastic Differential Equation (SDE):

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW(t)$$

- We will define it in the integral form:

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$$

Stochastic integral, Ito integral

- Fix a process Y adapted to the information given by W, such that

$$E\left[\int_0^t Y^2(u)du\right] < \infty$$

- **Construction:** split interval $[0, t]$ into n subintervals $[t_i, t_{i+1}]$ and consider

$$I_n(t) := \sum_i Y(t_i)[W(t_{i+1}) - W(t_i)]$$

Ito showed that there is a limit $I(t)$,

$$E[(I_n(t) - I(t))^2] \to 0$$

and called the limit the **stochastic integral**:

$$I(t) = \int_0^t Y(s)dW(s)$$

Stochastic integral properties

- Process $I(t) = \int_0^t Y(u)dW(u)$ is a martingale with mean zero, or

$$
\begin{aligned}
E\left[\int_0^t Y(u)dW(u)\right] &= 0 \\
E_s\left[\int_0^t Y(u)dW(u)\right] &= \int_0^s Y(u)dW(u)
\end{aligned}
$$

and the variance is

$$
E\left[\left(\int_0^t Y(u)dW(u)\right)^2\right] = E\left[\int_0^t Y^2(u)du\right]
$$

Reasons why the martingale property

- We have

$$E_s \int_0^t Y(u)dW(u) = E_s \int_0^s Y(u)dW(u) + E_s \int_s^t Y(u)dW(u)$$

$$= \int_0^s Y(u)dW(u) + E_s \int_s^t Y(u)dW(u)$$

- We claim that

$$E_s \int_s^t Y(u)dW(u) = 0$$

For example, for $t_{j+1}, t_j > s$,

$$\begin{aligned} E_s[Y(t_j)(W(t_{j+1}) - W(t_j))] &= E_s E_{t_j}[Y(t_j)(W(t_{j+1}) - W(t_j))] \\ &= E_s \left\{ Y(t_j) E_{t_j}[(W(t_{j+1}) - W(t_j))] \right\} \\ &= 0 \end{aligned}$$

Reasons why the variance

- We have, for example

$$
\begin{aligned}
E\left[Y^2(t_1)(W(t_2) - W(t_1))^2\right] &= E\left[E_{t_1}\left\{Y^2(t_1)(W(t_2) - W(t_1))^2\right\}\right] \\
&= E\left[Y^2(t_1)E\{(W(t_2) - W(t_1))^2\}\right] \\
&= E[Y^2(t_1)(t_2 - t_1)]
\end{aligned}
$$

Here, we used the fact that

$$
E_{t_1}\{(W(t_2) - W(t_1))^2\} = E\{(W(t_2) - W(t_1))^2\}
$$

because $(W(t_2) - W(t_1))$ is independent of the information available up to time t_1.

Pricing Options with Mathematical Models

16. Ito's rule, Ito's lemma

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Ito's rule

- Standard calculus:

$$\frac{d}{dt}f(t,x(t)) = \frac{\partial}{\partial t}f(t,x(t)) + \frac{\partial}{\partial x}f(t,x(t))\frac{d}{dt}x(t)$$

- or, denoting partial derivatives with subscripts,

$$df(t,x(t)) = f_t(t,x(t))dt + f_x(t,x(t))dx(t)$$

- In stochastic calculus, for

$$dX(t) = \mu(t)dt + \sigma(t)dW(t)$$

$$df(t,X(t)) = f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \frac{1}{2}\sigma^2 f_{xx}(t,X(t))dt$$

or

$$df = \left[f_t + \mu f_x + \frac{1}{2}\sigma^2 f_{xx}\right]dt + \sigma f_x dW$$

Reason why – quadratic variation

- Split interval $[0, t]$ into pieces of length Δt.
- Consider the sum of absolute increments to the p-th power

$$Q_p(t, W) := \sum_i |W(t_{i+1}) - W(t_i)|^p$$

- For $p = 2$, its limit is called **quadratic variation** and for Brownian motion we have

$$Q_2(t, W) \to t \ , \quad \text{as } \Delta t \to 0$$

while $Q_1(t, W) \to \infty$.

- For a differentiable function f,

$$Q_1(t, f) \to \int_0^t |f'(s)| ds \ , \quad \text{and } Q_2(t, f) \to 0$$

“Proof” of Ito’s rule

- Taylor expansion:

$$f(t+\Delta t, X(t+\Delta t)) - f(t, X(t)) = f_t \Delta t + f_x \Delta X$$

$$+ \frac{1}{2} f_{xx} (\Delta X)^2 \quad + \text{ higher order terms}$$

- The second order term does not disappear:

$$(\Delta X)^2 = (\mu \Delta t + \sigma \Delta W)^2 = \mu^2 (\Delta t)^2 + 2\mu\sigma \Delta W \Delta t + \sigma^2 (\Delta W)^2$$

  In the limit when $\Delta t \to 0$ this gives

$$(dX)^2 = \sigma^2 dt$$

- We get Itô’s rule: $df = f_t dt + f_x dX + \frac{1}{2} f_{xx} \sigma^2 dt$

More on Ito’s rule

- We can write

$$df = f_t dt + f_x dX + \frac{1}{2} f_{xx} dX \cdot dX$$

using the following informal rules:

$$dt \cdot dt = 0, \quad dt \cdot dW = 0, \quad dW \cdot dW = dt$$

- This gives

$$dX \cdot dX = (\mu dt + \sigma dW) \cdot (\mu dt + \sigma dW) = \sigma^2 dt$$

Example: $W^2(t)$

$$\int_0^t W(s)dW(s) = ?$$

Consider function $f(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$. Since

$$dW = 0 \times dt + 1 \times dW$$

we have, by Ito's rule,

$$dW^2(t) = 2W(t)dW(t) + \frac{1}{2} \times 2dt$$

which can be written as

$$W^2(t) - W^2(0) = \int_0^t 2W(s)dW(s) + \int_0^t ds$$

which gives

$$2\int_0^t W(s)dW(s) = W^2(t) - t$$

Exponential of Brownian motion

- The process
$$Y(t) = e^{aW(t)+bt}$$
is a function $Y(t) = f(t, W(t))$ with
$$f(t,x) = e^{ax+bt},\ \ f_t(t,x) = bf(t,x),\ \ f_x(t,x) = af(t,x),\ \ f_{xx}(t,x) = a^2 f(t,x)$$

- Applying Itô's rule we get
$$dY = \left[b + \frac{1}{2}a^2\right] Y dt + aY dW$$

- If $b = -\frac{1}{2}a^2$, so that $Y(t) = e^{aW(t)-\frac{1}{2}a^2 t}$ we have a martingale: $dY = aY dW$, and from $E_s[Y(t)] = Y(s)$ we get
$$E_s[e^{aW(t)}] = e^{aW(s)+\frac{1}{2}a^2(t-s)}$$

Two-dimensional Ito's rule

- Correlated Brownian motions:

$$E[W_X(t)W_Y(t)] = \rho t, \quad dW_X dW_Y = \rho dt$$

- Consider a model with two processes

$$dX = \mu_X dt + \sigma_X dW_X(t), \quad dY = \mu_Y dt + \sigma_Y dW_Y(t)$$

- Ito's rule:

$$df(X(t), Y(t)) = f_x dX + f_y dY + \frac{1}{2} f_{xx} (dX)^2 + \frac{1}{2} f_{yy} (dY)^2 + f_{xy} dX dY$$

$$= f_x dX + f_y dY + [\frac{1}{2} f_{xx} \sigma_X^2 + \frac{1}{2} f_{yy} \sigma_Y^2 + f_{xy} \rho \sigma_X \sigma_Y] dt$$

- **Product Rule:** $d(XY) = XdY + YdX + \rho \sigma_X \sigma_Y dt$

Pricing Options with Mathematical Models

17. Black-Scholes-Merton pricing

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

The model

- The risk-free asset satisfies the ODE

$$dB(t) = rB(t)dt, \quad B(0) = 1$$

implying $B(t) = e^{rt}$.

- The stock satisfies the SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Using Ito's rule we can check that

$$S(u) = S(t)e^{(\mu - \frac{1}{2}\sigma^2)(u-t) + \sigma[W(u) - W(t)]}$$

Black-Scholes-Merton pricing: PDE approach

- We want to find the price of a European path-independent claim with payoff $C(T) = g(S(T))$.
- It is reasonable to guess that the price will be a function $C(t, S(t))$ of the current time and price of the underlying. If so, from Ito's rule,

$$dC = \left[C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + \mu S C_s\right] dt + \sigma S C_s dW$$

- On the other hand, with $\pi(t)$ = amount invested in stock at time t, a self-financing wealth process satisfies:

$$dX(t) = \frac{\pi(t)}{S(t)} dS(t) + \frac{X(t) - \pi(t)}{B(t)} dB(t)$$

$$dX = [rX + (\mu - r)\pi]dt + \sigma\pi dW$$

Replication produces a PDE

- If we want replication, $C(t) = X(t)$, we need the dt terms to be equal, and the dW terms to be equal.
- Comparing the dW terms we get that the number of shares needs to be equal to the so-called **delta of the option**

$$\frac{\pi(t)}{S(t)} = C_s(t, S(t))$$

- Using this and comparing the dt terms we get the Black-Scholes PDE:

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + r(sC_s - C) = 0$$

- subject to the boundary condition,

$$C(T, s) = g(s)$$

The bottom line

- If the PDE has a unique solution $C(t, s)$, it means we can replicate the option by holding delta shares at each time. The option price at time t when the stock price is equal to s is given by $C(t, s)$, and the option delta is the derivative of the option price with respect to the underlying.
- The PDE and the option price do not depend on the mean return rate μ of the underlying!

Black-Scholes formula

- For an European call option $g(s) = (s - K)^+$, the solution of the PDE is given by the Black-Scholes formula:

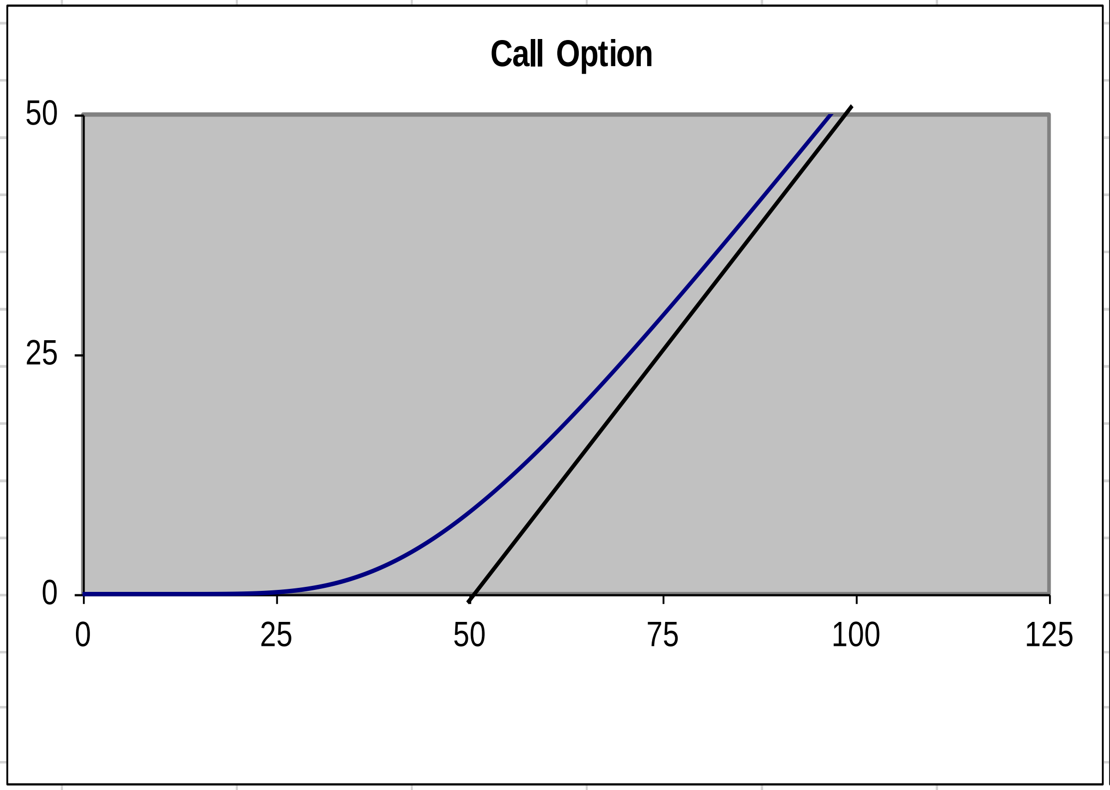
$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

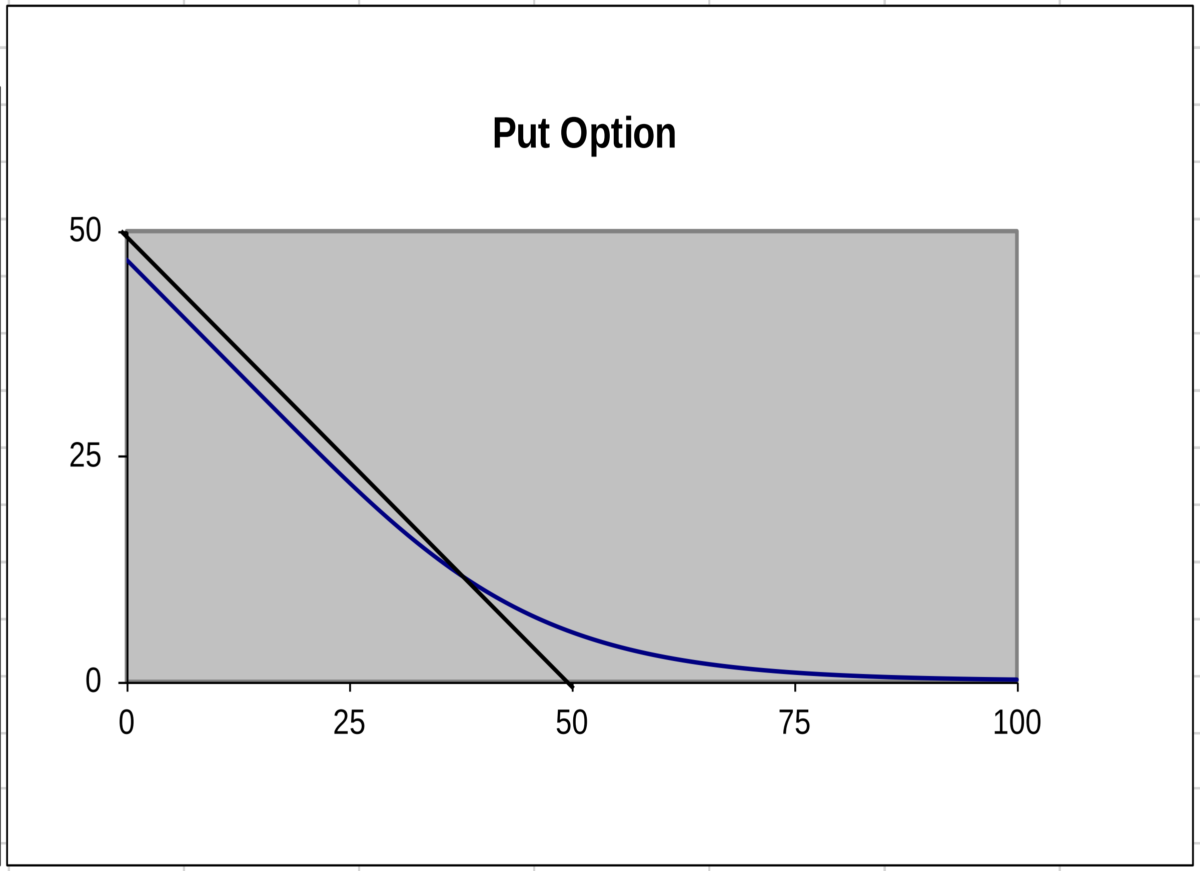
where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned}
d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\
d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\
&= d_1 - \sigma\sqrt{T-t}
\end{aligned}$$

Graphs of the PDE solutions





Pricing Options with Mathematical Models

18. Risk-neutral pricing: Black-Scholes-Merton model

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Risk-neutral probability in B-S-M model

- Let us find the dynamics of S under the risk-neutral probability Q.
- Denote by W^Q the Brownian motion under Q.
- We claim that if we replace μ by r, that is, if the stock satisfies

$$\frac{dS(t)}{S(t)} = rdt + \sigma dW^Q(t)$$

then the discounted stock price is a Q-martingale.

- Indeed, this is because Ito's rule then gives

$$d(e^{-rt}S(t)) = e^{-rt}dS(t) + S(t)d(e^{-rt})$$

$$= e^{-rt}[rS(t)dt + \sigma S(t)dW^Q(t)] - S(t)re^{-rt}dt = 0 \times dt + \sigma \bar{S}(t)dW^Q(t)$$

Girsanov theorem

- In order to have the above dynamics and also

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

  we need to have

$$W^Q(t) = W(t) + \frac{\mu - r}{\sigma} t$$

- The famous **Girsanov theorem** tells us that this is possible: there exists a unique probability Q under which so-defined W^Q is a Brownian motion.
- The discounted wealth process is also a Q-martingale:

$$\begin{aligned} d(e^{-rt}X(t)) &= e^{-rt}(\mu - r)\pi(t)dt + e^{-rt}\sigma\pi(t)dW(t) \\ &= e^{-rt}\sigma\pi(t)dW^Q(t) \end{aligned}$$

Black-Scholes formula as an expected value

- Option prices can then be computed taking expected values under Q.
- To do that, we note that we can write

$$S(T) = S(0)e^{\sigma W^Q(T) + (r - \frac{1}{2}\sigma^2)T}$$

- We have to compute

$$E^Q[e^{-rT}(S(T) - K)^+]$$

$$= E^Q[e^{-rT}S(T)\mathbf{1}_{\{S(T)>K\}}] - Ke^{-rT}E^Q[\mathbf{1}_{\{S(T)>K\}}]$$

Computing the expected values

- For the second term we compute the price of a Digital (Binary) option:

$$E^Q e^{-rT} \mathbf{1}_{\{S(T)>K\}} = e^{-rT} Q(S(T) > K)$$

$$\begin{aligned} Q(S(T) > K) &= Q\left(S(0)e^{(r-\sigma^2/2)T+\sigma W^Q(T)} > K\right) \\ &= Q\left(\frac{W^Q(T)}{\sqrt{T}} > -d_2\right) \\ &= N(d_2) \end{aligned}$$

where the middle equality follows by taking logs and re-arranging.

- The first term is computed using the formula

$$E\left[g\left(\frac{W^Q(T)}{\sqrt{T}}\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-x^2/2} dx$$

Reminder: Black-Scholes formula

$$C(t, S(t)) = S(t)N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$N(x) := P[Z \leq x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(S(t)/K) + (r - \sigma^2/2)(T-t)] \\ &= d_1 - \sigma\sqrt{T-t} \end{aligned}$$

Another way to get the PDE

- Under risk-neutral probability Q, by Ito's rule

$$dC(t,S(t)) = [C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss}]dt + \sigma C_s S(t)dW^Q(t)$$

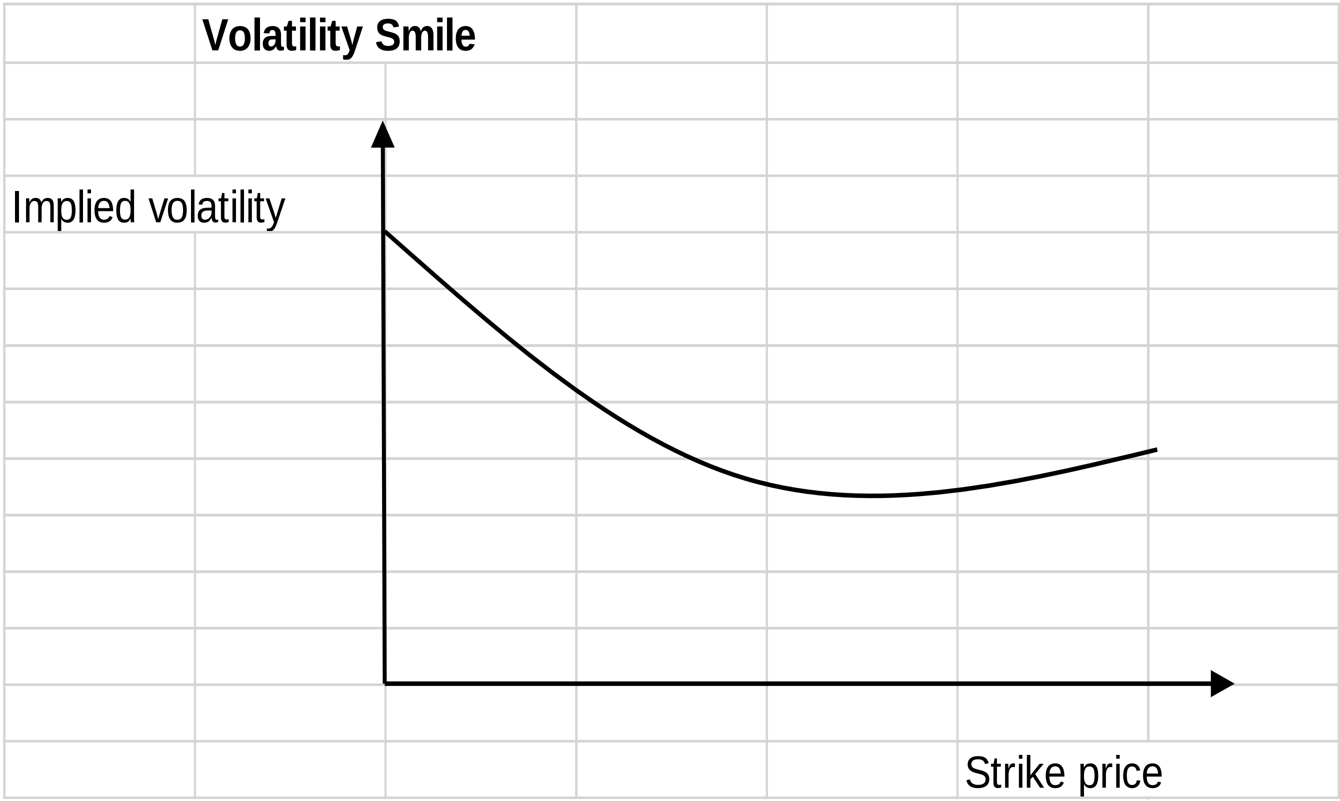
  Then, discounting,

$$d(e^{-rt}C(t,S(t)))$$
$$= e^{-rt}[(C_t + rS(t)C_s + \frac{1}{2}\sigma^2 S^2(t)C_{ss} - rC)]dt$$
$$+e^{-rt}\sigma C_s S(t)dW^Q$$

- This has to be a Q martingale, which means that the dt term has to be zero, resulting in the Black-Scholes PDE.

Implied volatility

- It is the value of σ that matches the theoretical Black-Scholes price of the option with the observed market price of the option



- In the Black - Scholes model, volatility is the same for all options on the same underlying

- However, this is not the case for implied volatilities: **volatility smile**

Pricing Options with Mathematical Models

19. Variations on Black-Scholes-Merton

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Dividends paid continuously

- Assume the stock pays a dividend at a continuous rate q. Total value of holding one share of stock is

$$G(t) := S(t) + \int_0^t qS(u)du$$

- Therefore, the wealth process of investing in this stock and the bank account is

$$dX = (X - \pi)dB/B + \pi dG/S$$

$$dX(t) = [rX(t) + \pi(t)(\mu + q - r)]dt + \pi(t)\sigma dW(t)$$

- To get the discounted wealth process to be a martingale, that is,

$$dX(t) = rX(t)dt + \pi(t)\sigma dW^Q(t)$$

we need to have

$$W^Q(t) = W(t) + t(\mu + q - r)/\sigma$$

- This makes the stock dynamics

$$dS(t) = S(t)[(r-q)dt + \sigma dW^Q(t)]$$

and the pricing PDE is

$$C_t + \frac{1}{2}\sigma^2 s^2 C_{ss} + (r-q)sC_s - rC = 0$$

- The solution, for the European call option, is obtained by replacing the underlying price s with $se^{-q(T-t)}$:

$$C(t,s) = se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r - q + \sigma^2/2)(T-t)] \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}[\log(s/K) + (r - q - \sigma^2/2)(T-t)] \end{aligned}$$

Dividends paid discretely

- Assume the stock pays deterministic dividends, and denote the process of discounted dividends by $\bar{D}(t)$.
- Assume that the process

$$S_G(t) = S(t) - \bar{D}(t)$$

satisfies

$$dS_G = S_G[\mu dt + \sigma dW(t)]$$

Then, the option price is obtained by replacing $s = S(t)$ by $S(t) - \bar{D}(t)$.

Options on futures

- Since $F(t) = e^{r(T-t)}S(t)$,

$$dF = F(\mu - r)dt + F\sigma dW$$

- With $W^Q(t) = W(t) + t(\mu - r)/\sigma$, we get

$$dF = F\sigma dW^Q$$

- Thus, the PDE for path independent options is

$$C_t + \frac{1}{2}\sigma^2 f^2 C_{ff} - rC = 0$$

- The solution for the call option is

$$
\begin{aligned}
C(t, f) &= e^{-r(T-t)}[f N(d_1) - K N(d_2)] \\
d_1 &= \frac{1}{\sigma_F \sqrt{T-t}}[\log(f/K) + (\sigma_F^2/2)(T-t)] \\
d_2 &= \frac{1}{\sigma_F \sqrt{T-t}}[\log(f/K) - (\sigma_F^2/2)(T-t)]
\end{aligned}
$$

Pricing Options with Mathematical Models

20. Currency options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Currency options in the B-S-M model

- Consider the payoff, evaluated in the domestic currency, equal to

$$(R(T) - K)^+$$

  where $R(T)$ is the exchange rate, the time T domestic value of one unit of foreign currency.

- Assume that the exchange rate process is given by

$$dR(t) = R(t)[\mu_R dt + \sigma_R dW(t)]$$

- The pricing formula is the same as in the case of a dividend-paying underlying, but with q replaced by r_f, the foreign risk-free rate.

Reasons why

- We trade in the domestic and foreign risk-free accounts.
- The dollar value of one unit of the foreign account is

$$R^*(t) := R(t)e^{r_f \cdot t}$$

$$dR^* = R^* \left[(\mu_R + r_f)dt + \sigma_R dW\right]$$

- The wealth dynamics (in domestic currency) of a portfolio of π dollars in the foreign account and the rest in the domestic account are

$$dX = \frac{X - \pi}{B} dB + \frac{\pi}{R^*} dR^* = [rX + \pi(\mu_R + r_f - r)]dt + \pi\sigma_R dW$$

- This is exactly the same as for dividends with q replaced by r_f.

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f))/\sigma_R$$

Call option formula

- The dollar value of the call option is

$$c(t, R) = Re^{-r_f(T-t)}N(d_1) - Ke^{-r(T-t)}[N(d_2)]$$

where

$$d_1 = \frac{1}{\sigma_R\sqrt{T-t}}[\log(R/K) + (r - r_f + \sigma_R^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_R\sqrt{T-t}}[\log(R/K) + (r - r_f - \sigma_R^2/2)(T-t)] = d_1 - \sigma_R\sqrt{T-t} \; .$$

Example: Quanto options

- - $S(t)$: a domestic equity index

  - Payoff: $S(T) - F$ units of **foreign currency**; quanto forward

- As in the previous slide, we have

$$W^Q(t) = W(t) + t(\mu_R - (r - r_f))/\sigma_R$$

and thus

$$dR(t) = R(t)[(r - r_f)dt + \sigma_R dW^Q(t)]$$

- Assume

$$dS(t) = S(t)[rdt + \sigma_S dZ^Q(t)]$$

where BMP Z^Q has instantaneous correlation ρ with W^Q. We have

$$d(S(t)R(t)) = S(t)R(t)[(2r - r_f + \rho\sigma_R\sigma_S)dt + \sigma_R dW^Q(t) + \sigma_S dZ^Q(t)]$$

- $S(T) - F$ units of foreign currency is the same as $(S(T) - F)R(T)$ units of domestic currency. The domestic value is

$$e^{-rT}(E^Q[S(T)R(T)] - FE^Q[R(T)])$$

- To make it equal to zero

$$F = \frac{E^Q[S(T)R(T)]}{E^Q[R(T)]}$$

- We have

$$E^Q[S(T)R(T)] = S(0)R(0)e^{(2r - r_f + \rho\sigma_S\sigma_R)T}$$
$$E^Q[R(T)] = R(0)e^{(r - r_f)T}$$

- We get

$$F = S(0)e^{(r + \rho\sigma_S\sigma_R)T}$$

- If

$$dX = aXdt + bXdW + cXdZ$$

  then

$$EX(t) = X(0)e^{at}$$

- This is because

$$d(EX(t)) = a \times (EX(t))dt$$

  and the solution to this ODE, that has initial value $X(0)$, is the one above.

Pricing Options with Mathematical Models

21. Exotic options

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Most popular exotic options

- Barrier options: they pay a call/put payoff only if the underlying price reaches a given level (barrier) before maturity. Thus, they depend on the maximum or the minimum price of the underlying during the life of the option.
- Asian options: a call/put written on the average stock price until maturity. Useful when the price of the underlying may be very volatile.
- Compound options: the underlying is another option. Call on a call:

$$E_0^Q e^{-rT_1} \left[\mathrm{BS}(T_1) - K_1\right]^+$$

Example: a forward start option

- A call with the strike price $S(t_1)$, $t_1 > 0$. Note that

$$S(0)\frac{S(T)}{S(t_1)} = S(0)\exp\{\sigma(W^Q(T) - W^Q(t_1)) + (r - \sigma^2/2)(T - t_1)\}$$

- We first compute the value at t_1:

$$\begin{aligned} E_{t_1}^Q\left[e^{-r(T-t_1)}(S(T) - S(t_1))^+\right] &= E_{t_1}^Q\left[e^{-r(T-t_1)}\frac{S(t_1)}{S(0)}\left(\frac{S(0)S(T)}{S(t_1)} - S(0)\right)^+\right] \\ &= \frac{S(t_1)}{S(0)}\mathrm{BS}(T - t_1, S(0)) \quad . \end{aligned}$$

- Today's value

$$E_0^Q\left[e^{-rt_1}\frac{S(t_1)}{S(0)}\mathrm{BS}(T - t_1, S(0))\right] = \mathrm{BS}(T - t_1, S(0))E_0^Q\left[e^{-rt_1}\frac{S(t_1)}{S(0)}\right] = \mathrm{BS}(T - t_1, S(0))$$

Example: a chooser option

- The holder can decide at time t_1 whether the payoff will be a call or a put, with the same strike price and maturity. Thus, the value at time t_1 is, using put-call parity,

$$\max(c(t_1), p(t_1)) = \max(c(t_1), c(t_1) + Ke^{-r(T-t_1)} - S(t_1))$$

$$= c(t_1) + \max(0, Ke^{-r(T-t_1)} - S(t_1))$$

- It is a package of a call option with maturity T and strike price K, and a put option with maturity t_1 and strike price $Ke^{-r(T-t_1)}$.

Pricing Options with Mathematical Models

22. Pricing options on more underlyings

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Two risky assets

$$dS_1 = S_1[\mu_1 dt + \sigma_1 dW_1]$$
$$dS_2 = S_2[\mu_2 dt + \sigma_2 dW_2]$$

Equivalently,

$$\begin{aligned} dS_1 &= S_1[\mu_1 dt + \sigma_1 dB_1] \\ dS_2 &= S_2[\mu_2 dt + \sigma_2 \rho dB_1 + \sigma_2 \sqrt{1-\rho^2} dB_2] \end{aligned}$$

This is because, given two independent Brownian Motions B_1 and B_2, we can set

$$W_1 = B_1, \quad W_2 = \rho B_1 + \sqrt{1-\rho^2} B_2$$

Wealth process

- $$dX = \frac{\pi_1}{S_1} dS_1 + \frac{\pi_2}{S_2} dS_2 + \frac{X - (\pi_1 + \pi_2)}{B} dB \quad .$$

This gives

$$dX = [rX + \pi_1(\mu_1 - r) + \pi_2(\mu_2 - r)]dt + \pi_1\sigma_1 dW_1 + \pi_2\sigma_2 dW_2 \quad .$$

For the discounted wealth process to be a martingale under the risk-neutral probability Q, we need to have

$$dX = rX\,dt + \pi_1\sigma_1 dW_1^Q + \pi_2\sigma_2 dW_2^Q$$

for some Q-Brownian Motions W_i^Q with correlation ρ. For that to be the case, we must have

$$W_i^Q(t) = W_i(t) + t(\mu_i - r)/\sigma_i$$

The pricing PDE with two factors

- Suppose $C(T) = g(S_1(T), S_2(T)$. Using the two-dimensional Ito's rule

$$dC = \left[C_t + rS_1C_{s_1} + rS_2C_{s_2} + \frac{1}{2}\sigma_1^2 S_1^2 C_{s_1 s_1} + \frac{1}{2}\sigma_2^2 S_2^2 C_{s_2 s_2} + \rho\sigma_1\sigma_2 S_1 S_2 C_{s_1 s_2}\right] dt$$
$$+ \sigma_1 S_1 C_{s_1} dW_1^Q + \sigma_2 S_2 C_{s_2} dW_2^Q \quad .$$

Comparing the dt term with the wealth equation, or making the drift of the discounted C equal to zero,

$$C_t + \frac{1}{2}\sigma_1^2 s_1^2 C_{s_1 s_1} + \frac{1}{2}\sigma_2^2 s_2^2 C_{s_2 s_2} + \rho\sigma_1\sigma_2 s_1 s_2 C_{s_1 s_2} + r(s_1 C_{s_1} + s_2 C_{s_2} - C) = 0 \quad .$$

$$C(T, s_1, s_2) = g(s_1, s_2)$$

$$\frac{\pi_1}{S_1} = C_{s_1}, \quad \frac{\pi_2}{S_2} = C_{s_2}$$

Example: exchange option

The payoff is

$$g(S_1(T), S_2(T)) = (S_2(T) - S_1(T))^+ = \max(S_2(T) - S_1(T), 0)$$

Since we have

$$(s_2 - s_1)^+ = s_1 \left(\frac{s_2}{s_1} - 1 \right)^+$$

it is reasonable to expect that the option price will be of the form

$$C(t, s_1, s_2) = s_1 D(t, z)$$

for some function D and a new variable $z = s_2/s_1$. After some computations, we can show that D has to satisfy

$$D_t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)z^2 D_{zz} = 0, \quad D(T, z) = (z - 1)^+$$

Example: exchange option (continued)

This is the Black-Scholes PDE corresponding to the European call option with strike price $K = 1$, interest rate $r = 0$, and volatility

$$\sigma_E = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \quad .$$

Using the Black-Scholes formula for D, and the fact that $C = s_1 D$, we get

$$C(t, s_1, s_2) = s_2 N(d_1) - s_1 N(d_2) \quad ,$$

$$d_1 = \frac{1}{\sigma_E\sqrt{T-t}}[\log(s_2/s_1) + (\sigma_E^2/2)(T-t)]$$

$$d_2 = \frac{1}{\sigma_E\sqrt{T-t}}[\log(s_2/s_1) - (\sigma_E^2/2)(T-t)] = d_1 - \sigma_E\sqrt{T-t} \quad ,$$

Pricing Options with Mathematical Models

23. Stochastic volatility

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Complete markets

$$dS(t) = S(t)[r(t, S(t))dt + \sigma(t, S(t))dW^Q(t)]$$

- We have

$$S(T) = S(t)e^{\int_t^T [r(u,S(u)) - \frac{1}{2}\sigma^2(u,S(u))]du + \int_t^T \sigma(u,S(u))dW^Q(u)}$$

The PDE for the value of the payoff $g(S(T))$ is

$$C_t + \frac{1}{2}\sigma^2(t, s)C_{ss} + r(t, s)[sC_S - C] = 0$$

Constant Elasticity of Variance, CEV model, with $0 < \beta < 1$:

$$\sigma(t, s) = \frac{\sigma}{s^\beta}$$

.

Complete markets (continued)

- If $r(t)$ and $\sigma(t)$ are deterministic functions of time, random variable $\int_t^T \sigma(u)dW(u)$ has normal distribution with zero mean and variance $\int_t^T \sigma^2(u)du$.
- For a payoff $g(S(T))$, the value at time t is obtained by replacing $\sigma^2 \times (T-t)$ with $\int_t^T \sigma^2(u)du$ and replacing $r \times (T-t)$ with $\int_t^T r(u)du$.

Incomplete markets

- Consider two independent BMP's W_1 and W_2, and

$$\begin{aligned} dS(t) &= S(t)[\mu(t)dt + \sigma_1(t,V(t))dW_1(t) + \sigma_2(t,V(t))dW_2(t)] \\ dV(t) &= \alpha(t)dt + \gamma(t)dW_2(t) \end{aligned}$$

- Denote by $\kappa(t)$ *any* (adapted) stochastic process. For each, there is a risk-neutral measure Q_κ. In particular, for any such process κ we can set

$$\begin{aligned} dW_1^{Q_\kappa}(t) &= dW_1(t) + \frac{1}{\sigma_1(t)}[\mu(t) - r(t) - \sigma_2(t)\kappa(t)]dt \\ dW_2^{Q_\kappa}(t) &= dW_2(t) + \kappa(t)dt \end{aligned}$$

- It can be checked that discounted S is then a Q_κ-martingale and

$$dV(t) = [\alpha(t) - \kappa(t)\gamma(t)]dt + \gamma(t)dW_2^{Q_\kappa}(t)$$

-

Incomplete markets (continued)

- For constant κ and constant patrameters, the PDE becomes

$$
\begin{aligned}
& C_t + \frac{1}{2} C_{ss} s^2 (\sigma_1^2 + \sigma_2^2) + \frac{1}{2} C_{vv} \gamma^2 \\
+ \quad & s C_{sv} \gamma \sigma_2 + r(s C_S - C) + C_v(\alpha - \kappa\gamma) = 0
\end{aligned}
$$

- The parameters are **calibrated** to the market, that is, chosen so that the market prices of liquidly traded options are matched to the model prices as well as possible.

Examples

- Heston's model:

$$dS(t) = S(t)[rdt + \sqrt{V(t)}dW^Q(t)]$$

$$dV(t) = A(B - V(t))dt + \gamma\sqrt{V(t)}dZ^Q(t)$$

for some other risk-neutral Brownian motion Z^Q having correlation ρ with W^Q. Price is a function $C(t, s, v)$ satisfying

$$0 = C_t + \frac{1}{2}v[s^2C_{ss} + \gamma^2C_{vv}] + r(sC_s - C) + A(B - v)C_v + \rho\gamma vsC_{sv}$$

- SABR model:

$$dS(t) = S(t)[rdt + \sigma(t)\frac{1}{S^\beta(t)}dW^Q(t)]$$

$$d\sigma(t) = \alpha\sigma(t)dZ^Q(t)$$

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24. Jump-diffusion models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Merton's jump-diffusion model

- Suppose the jumps arrive according to a Poisson process, that is, at independent exponentially distributed intervals.
- The number $N(t)$ of jumps up to time t is given by Poisson distribution:

$$P[N(t) = k] = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

- The stock price satisfies the following dynamics:

$$dS(t) = S(t)[r - \lambda m]dt + S(t)\sigma dW^Q(t) + dJ(t) \quad ,$$

where m is such that the discounted stock price is a Q-martingale, and dJ is the actual jump size.

Merton's jump-diffusion model (continued)

- More precisely, $dJ(t) = 0$ if there is no jump at time t, and $dJ(t) = S(t)X_i - S(t)$ if the i-th jump occurs at time t, where X_i are iid random variables. Therefore,

$$S(t) = S(0) \cdot X_1 \cdot X_2 \cdot \ldots \cdot X_{N(t)} \cdot e^{(r-\sigma^2/2-\lambda m)t+\sigma W^Q(t)}$$

- The price of payoff $g(S(T))$ is

$$C(0) = \sum_{k=0}^{\infty} E^Q \left[e^{-rT} g(S(T)) \;\middle|\; N(T) = k \right] Q(N(T) = k)$$

which is equal to

$$\textstyle\sum_{k=0}^{\infty} \quad E^Q \left[e^{-rT} g\left(S(0) X_1 \cdot \ldots X_k \cdot e^{(r-\sigma^2/2-\lambda m)T+\sigma W^Q(T)} \right) \right] \times e^{-\lambda T} \frac{(\lambda T)^k}{k!}$$

Merton's jump-diffusion model (continued)

- If X_i's are lognormally distributed, the price of the option can be represented as

$$C(0) = \sum_{k=0}^{\infty} e^{-\tilde{\lambda}T} \frac{(\tilde{\lambda}T)^k}{k!} BS_k$$

for $\tilde{\lambda} = \lambda(1+m)$ and BS_k is the Black and Scholes formula with appropriately chosen $r = r_k$ and $\sigma = \sigma_k$.

Pricing Options with Mathematical Models

25. Static hedging with futures

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Perfect hedge with futures

- There is a futures contract that trades:
  - exactly the asset we want to hedge.
  - with the exact maturity we want to hedge.
- Otherwise, “asset mismatch” or “maturity mismatch”. Then, the solution is “crosshedging”.

Crosshedging

- Hedging payoff $S_1(T)$ with futures F_2 maturing at $U \geq T$: the unknown quantity X, called **basis** is, with δ the number of futures conttracts,

$$X = S_1(T) - \delta F_2(T, U)$$

- The risk is measured as

$$Var_t[X] = Var_t[S_1(T)] + \delta^2 Var_t[F_2(T, U)] - 2\delta Cov_t[S_1(T), F_2(T, U)]$$

- If we want to minimize the variance, taking the derivative with respect to δ and setting it equal to zero gives us the optimal δ:

$$\delta = \frac{Cov_t[S_1(T), F_2(T, U)]}{Var_t[F_2(T, U)]} = \rho \frac{\sigma_S}{\sigma_F} \quad ,$$

where ρ is the correlation between $S_1(T)$ and $F_2(T, U)$, and σ_S^2, σ_F^2 are their variances.

Crosshedging (continued)

- The minimal variance is

$$Var_t[X] = Var_t[S_1(T)] - \frac{Cov_t^2[S_1(T), F_2(T,U)]}{Var_t[F_2(T,U)]} \quad .$$

- In the case of perfect hedge, $S_1 = S_2$, and $U = T$, we have $S_1(T) = F_1(T) = F_2(T,U)$. Then, $\delta = 1$, and, using the fact that $Cov[S,S] = Var[S]$, we see that $Var_t[X] = 0$ and there is no basis risk involved.

Example

- Consider a U.S. firm that will receive 1 million of currency A, six months from now. The firm will hedge by shorting δ units of six-month futures contracts on a highly correlated currency B.

- Suppose that the exchange rates are $Q_A = 0.1$ dollar$/A$ and $Q_2 = 0.2$ dollar$/B$. The exchange rate A/B has to be $Q = Q_A/Q_B = 0.5$. However, this does not mean that the company should short 0.5 million of B futures.

- Suppose historical data gives us $\sigma_{Q_A} = 0.03$, and $\sigma_{Q_B} = 0.02$, and that the correlation between the two is 0.9. Thus, the covariance is equal to $0.9 \cdot 0.02 \cdot 0.03 = 0.00054$. We get

$$\delta = 0.00054/0.0004 = 1.35$$

- Therefore, for each unit of A the U.S. company should short an amount of B equivalent to 1.35 of A, thus $1,000,000 \cdot 1.35/2 = 675,000$ of B.

- The minimal variance (per unit of currency) is 0.000171, quite a bit smaller than the variance of the dollar$/A$ exchange rate, equal to $\sigma^2_{Q_A} = 0.0009$.

Rolling the hedge forward: the story of Metallgesellschaft

- In early 1990s, MG sold huge volume of long-term fixed price forward-type contracts to deliver oil.
- Hedging: rolling over short-term future contracts to receive oil.
- Oil price went down: good for fixed price contracts.
- Bad for futures: large margin calls.
- All contracts closed out at a huge loss.

Pricing Options with Mathematical Models

26. Static hedging with bonds

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Duration

- Suppose the bond price is given by

$$P = \sum_{i=1}^{T} \frac{C_i}{(1+y)^i}$$

- The sensitivity of the price with respect to yield is

$$\frac{\partial P}{\partial y} = -\sum_{i=1}^{T} i \frac{C_i}{(1+y)^{i+1}} = -\frac{P}{1+y} \sum_{i=1}^{T} i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

- (Macaulay) **duration** is defined as

$$D = \sum_{i=1}^{T} i \frac{1}{P} \frac{C_i}{(1+y)^i}$$

- It is an average of the coupon payment times, weighted by the relative size of the coupons; it is equal to maturity T for the zero-coupon bond.

Bond immunization

- A second order measure of yield risk, the **convexity**, is defined as

$$C = \frac{1}{P}\frac{\partial^2 P}{\partial y^2}$$

- Duration is a static version of the delta of an option, and convexity is a static version of the gamma of an option.

- **Hedging future cash payments** that have to be delivered at specified times: use a portfolio of bonds with the same duration and the same convexity as the cash payments. This is called **immunization**. It is based on then following approximation:

$$\Delta P \approx -\frac{D}{1+y}P\Delta y + \frac{1}{2}CP(\Delta y)^2$$

Pricing Options with Mathematical Models

27. Perfect hedging - replication

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Replication in binomial trees

- Consider a call with $S(0) = 55.5625, \sigma = 0.32, r = 0, K = 55, T = 0.08,$

$$\begin{aligned} S^u &= S(0)e^{\sigma\sqrt{T-t}} = 55.5625e^{0.32\sqrt{0.08}} = 60.826 \\ S^d &= S(0)e^{-\sigma\sqrt{T-t}} = 55.5625e^{-0.32\sqrt{0.08}} = 50.7544 \end{aligned}$$

- Option payoff is either $60.826 - 55 = 5.826$ or zero. To replicate it, we solve

$$\begin{aligned} \delta_0 &+ 60.826\delta_1 = 5.826 \\ \delta_0 &+ 50.7544\delta_1 = 0 \end{aligned}$$

- Suppose $S(T) = 68.8125$. Then, the final profit/loss is $\delta_1(68.8125 - 55.5625) + 55 - 68.8125 = -6.1479$

- In general, the number δ_1 of shares of the underlying is given by

$$\delta_1 = \frac{C^u - C^d}{S^u - S^d}$$

Replication in the B-S-M model

- In the B-S-M model the delta of payoff $C(T) = g(S(T))$ is the derivative of its price with respect to the underlying,

$$\Delta_C := \frac{\partial C(t,s)}{\partial s}$$

- For the European call it is

$$\Delta_C = N(d_1)$$

- For the perfect (theoretical) hedge, re-balancing must take place continuously. This requires that the model and its parameters are exactly correct.

A real data example

- A call option on Microsoft stock 20 consecutive days in year 2000, $T = 20/252 = 0.08$ years.
- Daily data: $Y(i) = \log S(i+1) - \log S(i)$
- Sample standard deviation $= \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y}_i)^2} = 0.019978$
- Annual st dev $\sigma = 0.019978 \times \sqrt{252} = 0.317139$
- $K = 55, S(0) = 55.5625$. We set $r = 0$.

A real data example (continued)

- Using B-S-M, $C(0) = 2.2715, \Delta(0) = 0.5629$
- Borrow $\Delta(0)S(0) - C(0)$
- Next day: $S(1) = 55.3750, T - 1 = 19/252$
- Portfolio value: $X(0) = C(0)$ and

$$X(1) = X(0) + \Delta(0)[S(1) - S(0)] = 2.166$$

- $\Delta(1) = 0.5483$, sell $\Delta(0) - \Delta(1)$ shares
- $X(2) = X(1) + \Delta(1)[S(2) - S(1)] = 2.2003$

Replication Experiment

Time	Stock Price	Call Price	Delta	Wealth
0	55.5625	2.2715	0.5629	2.2715
1	55.3750	2.1176	0.5483	2.1660
2	55.4375	2.1009	0.5539	2.2003
3	56.5625	2.7256	0.6481	2.8235
4	59.1250	4.5831	0.8268	4.4841
5	60.3125	5.5702	0.8899	5.4659
6	61.3125	6.4562	0.9313	6.3558
7	60.6250	5.7970	0.9166	5.7155
8	62.6875	7.7346	0.9724	7.6060
9	61.2500	6.3360	0.9507	6.2082
10	63.2500	8.2681	0.9873	8.1095
11	64.1875	9.1933	0.9953	9.0351
12	64.2500	9.2531	0.9972	9.0973
13	65.0000	10.0007	0.9992	9.8452
14	63.0000	8.0025	0.9974	7.8467
15	64.1875	9.1877	0.9997	9.0310
16	65.8125	10.8125	1.0000	10.6556
17	68.2500	13.2500	1.0000	13.0931
18	68.1250	13.1250	1.0000	12.9681
19	68.8125	13.8125	1.0000	13.6556

A real data example (continued)

- Loss in option:

$$S(T) - K = S(T) - 55 = 13.8125$$

- Portfolio value: $X(T) = 13.6556$
- Loss with hedging: $13.8125 - 13.6556 = 0.1569$
- Loss without hedging: $13.8125 - C(0) = 11.5410$

Pricing Options with Mathematical Models

28. Hedging portfolio sensitivities

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Option sensitivities

- In general, a portfolio value X is sensitive to changes in values of all parameters.
- Partial derivatives of the portfolio value with respect to parameters are called greeks

· **Delta**: $\Delta = \frac{\partial}{\partial s} X$

· **Theta**: $\Theta = \frac{\partial}{\partial t} X$

· **Gamma**: $\Gamma = \frac{\partial^2}{\partial s^2} X$

· **Vega**: $\mathcal{V} = \frac{\partial}{\partial \sigma} X$

· **rho**: $\rho = \frac{\partial}{\partial r} X$

Approximation by Taylor expansion

- From Taylor's expansion,

$$X(t+\Delta t, s+\Delta s) = X(t,s) + \frac{\partial X(t,s)}{\partial s}\Delta S + \frac{\partial X(t,s)}{\partial t}\Delta t$$
$$+\frac{1}{2}\frac{\partial^2 X(t,s)}{\partial s^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 X(t,s)}{\partial t^2}\Delta t^2 + \frac{\partial^2 X(t,s)}{\partial s \partial t}\Delta S \Delta t + \cdots$$

  where

$$\Delta S = S(t+\Delta t) - S(t)$$

- Approximation:

$$X(t+\Delta t, s+\Delta s) \approx X(t,s) + \Delta \cdot \Delta S + \Theta \cdot \Delta t + \frac{1}{2}\Gamma \cdot \Delta S^2$$

Approximation by Taylor expansion (continued)

- If the portfolio is delta-neutral, that is, its Δ is zero, then

$$\Delta X \approx \Theta \Delta t + \frac{1}{2}\Gamma \Delta S^2$$

- If Γ is strictly positive, any change in the value of the underlying tends to increase the value of the portfolio.

- If σ and r are stochastic:

$$\begin{aligned} & X(t+\Delta t, s+\Delta s, \sigma+\Delta\sigma, r+\Delta r)) \\ = \; & X(t,s,\sigma,r) + \frac{\partial X}{\partial s}\Delta s + \frac{\partial X}{\partial \sigma}\Delta\sigma + \frac{\partial X}{\partial r}\Delta r + \frac{\partial X}{\partial t}\Delta t \\ & + \frac{1}{2}\frac{\partial^2 X}{\partial s^2}\Delta S^2 + \frac{1}{2}\frac{\partial^2 X}{\partial \sigma^2}\Delta\sigma^2 + \frac{1}{2}\frac{\partial^2 X}{\partial r^2}\Delta r^2 + \cdots \end{aligned}$$

Example

- Denote by Γ the gamma of portfolio X, and by Γ_C the gamma of a contingent claim C. We want to buy/sell n contracts of C in order to make the portfolio gamma neutral, that is,

$$\Gamma + n\Gamma_C = 0$$

- This implies

$$n = -\frac{\Gamma}{\Gamma_C}$$

- However, taking this additional position in C will change the delta of the portfolio. We then buy/sell some shares of the underlying asset in order to make the portfolio delta-neutral. This does not change the gamma, because the underlying asset S has zero gamma. (Why?).

Example (continued)

- As an example, let's say a delta-neutral portfolio X has a gamma $\Gamma = -5,000$. A traded option has $\Delta_C = 0.4$ and $\Gamma_C = 2$. We buy $n = 5,000/2 = 2,500$ option contracts, making the portfolio gamma-neutral. This makes the delta of the portfolio equal to

$$\Delta_X = 2,500 \cdot 0.4 = 1,000 \quad .$$

- Thus, we have to sell 1,000 shares of the underlying asset to keep the portfolio delta-neutral.

Example (continued)

- If we want to make a portfolio vega-neutral, in addition to delta-neutral and gamma-neutral, then it is necessary to hold two different contingent claims written on the underlying asset. In this case we want to have

$$\Gamma + n_1\Gamma_1 + n_2\Gamma_2 = 0$$

$$\mathcal{V} + n_1\mathcal{V}_1 + n_2\mathcal{V}_2 = 0 \quad ,$$

  where $n_i, \Gamma_i, \mathcal{V}_i$ are the number of contracts, the gamma and the vega of claim i.

- This is a system of two equations with two unknowns which can typically be solved. At the end, we still have to adjust the number of shares in the new portfolio in order to make it delta-neutral, similarly as above.

Portfolio insurance

- Idea: A put option with our portfolio as the underlying protects against portfolio losses
- Problem: such options are not traded.
- Solution: a synthetic put - replicating the put payoff by trading.
- However, it led to large losses in the crash of October 1987, due to loss of liquidity.

The Story of Long Term Capital Management

- Merton and Scholes were partners
- Anticipated spreads between various rates to become narrower
- “Russian crisis” pushed the spreads even wider
- LTCM was highly leveraged – margin calls forced it to start selling assets, others also did, their prices went even lower, losses huge in 1998
- Bailed out by government effort
- The reasons: high leverage and unprecedented extreme market moves

Pricing Options with Mathematical Models

29. Introduction to interest rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Bond price as expected value

- We assume that the price of a security is the expected value of the discounted payoff under some risk-neutral probability, now denoted P, not Q.

- For a pure discount bond paying \$1.00 at maturity, in discrete time, with maturity in n periods,

$$P(t,n) = E_t\left[\frac{1}{\Pi_{i=0}^{n-1}(1+r_i)}\right]$$

- In continuous time, with maturity T,

$$P(t,T) = E_t\left[e^{-\int_t^T r(u)du}\right]$$

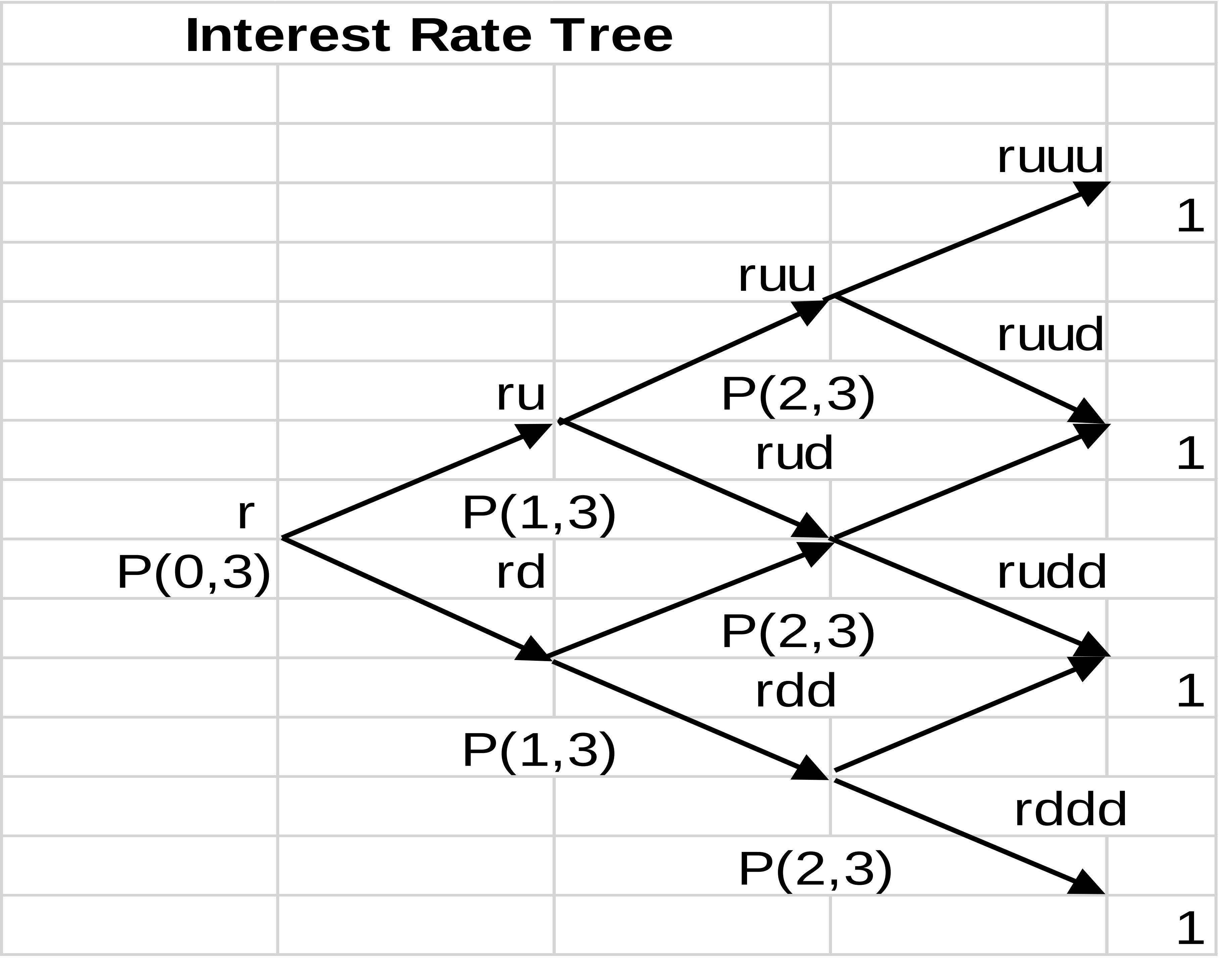
Price of bond option as expected value

- For a European call on the bond, with call's maturity at τ, and $t < \tau < T$:
- In discrete time, (where τ is equal to m periods)

$$C(t) = E_t \left[\frac{1}{\Pi_{i=0}^{m-1}(1+r_i)} \left(E_t \left[\frac{1}{\Pi_{j=m}^{n-1}(1+r_j)} \right] - K \right)^+ \right]$$

- In continuous time

$$C(t) = E_t \left[e^{-\int_t^\tau r(u)du} \left(E_\tau \left[e^{-\int_\tau^T r(u)du} \right] - K \right)^+ \right]$$


Interest Rate Tree
ruuu
1
ruu
ruud
ru
P(2,3)
rud
1
r
P(1,3)
P(0,3)
rd
rudd
P(2,3)
rdd
1
P(1,3)
rddd
P(2,3)
1

Example

- One-year interest rate today is 4%, one year from now it can go up to 5% or go down to 3%.
- A two-year zero-coupon bond with nominal value \$100 trades at \$92.278.
- We want to price an European call option on the bond, with $T = 1$, $K = 96$.
- One year from now, the price of the bond will be, if $r = 5\%$,

$$\frac{100}{1.05} = 95.238$$

and if $r = 3\%$,

$$\frac{100}{1.03} = 97.087$$

Example (continued)

- By no-arbitrage, if p is the risk-neutral probability that the interest rate will be 5% one year from now, we necessarily have

$$92.278 = \frac{1}{1.04}\left(p \cdot 95.238 + (1-p) \cdot 97.087\right)$$

- Then $p = 0.605$ and we can now price the call option:

$$\begin{aligned} & \frac{1}{1.04}\left(p(95.238-96)^{+} + (1-p)(97.087-96)^{+}\right) \\ = \quad & \frac{1}{1.04}(1-0.605)1.087 = 0.413 \end{aligned}$$

Pricing Options with Mathematical Models

30. Continuous-time interest rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Modeling the short rate

- Historically, the first approach was to model the dynamics of r, the **short rate**, using an SDE of the type

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

- It is typically modeled directly under the risk-neutral probability, that is, W now denotes W^Q.

- The models of the short rate with just one Brownian motion are called **one-factor models**.

Vasicek model

- The interest rate satisfies

$$dr = a(b - r)dt + \sigma dW$$

  where a, b and σ are constant.

- This is a well-known Ornstein-Uhlenbeck process in stochastic calculus.
- It is a mean-reverting process.
- Interest rate has normal distribution and can become negative.
- It results in closed-form solutions for bond and option prices.

Cox-Ingersoll-Ross (CIR) model

- The interest rate satisfies

$$dr = a(b - r)dt + \sigma\sqrt{r}dW$$

  where a, b and σ are constant.

- The interest rate cannot become negative.

- The volatility of the interest rate is stochastic, as a function of the interest rate level.

- It also results in closed-form solutions for bond prices.

More one-factor models

- **Ho-Lee model:**

$$dr = b(t)dt + \sigma dW$$

Hull-White model:

$$dr = [b(t) - ar]dt + \sigma dW$$

Black-Derman-Toy model:

$$dr = r[a(t)dt + \sigma dW]$$

- The pricing PDE for the option value with payoff $g(r(T))$ is

$$C_t + \frac{1}{2}\sigma^2 C_{rr} + \mu C_r - rC = 0, \quad C(T, r) = g(r)$$

Affine models of the term structure

- We wish the price of the bond to be of the form, for deterministic functions A, B,
$$P(t,T) = e^{A(t,T)-B(t,T)r(t)}$$
- Since $P(T,T) = 1$, we need to have
$$A(T,T) = 0, \quad B(T,T) = 0$$
- The pricing PDE for the bond price is
$$P_t + \frac{1}{2}\sigma^2 P_{rr} + \mu P_r - rP = 0, \quad P(T,T) = 1$$

Affine models (continued)

- If we assume affine dynamics

$$\mu(t,r) = \alpha(t)r + \beta(t)\ , \quad \sigma^2(t,r) = \gamma(t)r + \delta(t)$$

the PDE becomes

$$A_t - \beta B + \frac{1}{2}\delta B^2 - r\left[1 + B_t + \alpha B - \frac{1}{2}\gamma B^2\right] = 0$$

- Both terms have to be zero, which yields two ODE's

$$B_t + \alpha B - \frac{1}{2}\gamma B^2 = -1\ , \quad A_t = \beta B - \frac{1}{2}\delta B^2$$

that have explicit solutions in the Vasicek and Cox-Ingersoll-Ross models.

Pricing Options with Mathematical Models

31. Forward rate models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Forward rates

- **Forward rate investment**: At time t, $t < S < T$, sell short one S-bond, buy $P(t,S)/P(t,T)$ T-bonds. At time S pay 1 and get $P(t,S)/P(t,T)$ at time T. Continuous rate R:

$$1 \cdot e^{R(t;S,T)(T-S)} = \frac{P(t,S)}{P(t,T)}$$

$$R(t;S,T) = -\frac{\log P(t,T) - \log P(t,S)}{T-S}$$

- **Instantaneous forward rate:**

$$f(t,T) = -\frac{\partial \log P(t,T)}{\partial T}$$

$$P(t,T) = e^{-\int_t^T f(t,u)du}$$

$$f(t,t) = r(t)$$

Heath-Jarrow-Morton (HJM) model

- The objective is to model simultaneously the whole term structure, not just the short-term rate, which is useful for calibration purposes.
- The price of a zero-coupon bond is

$$P(t,T) = E_t\left[e^{-\int_t^T r(u)du}\right] = e^{-\int_t^T f(t,u)du}$$

- HJM model is a model for forward rates:

$$df(t,T) = \alpha(t,T)dt + \sigma^{Tr}(t,T)dW(t)$$

- Here, ^{Tr} denotes "transpose" of a matrix,in case W is multidimensional.

HJM model (continued)

- No arbitrage implies that the drift of the bond price is $r(t)P(t)dt$.
- After some algebra, this implies:

$$\alpha(t,T) = \sigma^{Tr}(t,T)\int_t^T \sigma(t,u)du$$

  so that the term structure of the drift is uniquely determined by the term structure of volatility.

- In calibration, we use the fact that

$$f(0,T) = -\frac{\partial \log P(0,T)}{\partial T}$$

- Then, a model for volatility needs to be chosen.

HJM example

- Flat term structure:

$$\sigma(t,T) \equiv \sigma$$

$$\alpha(t,T) = \sigma \int_t^T \sigma du = \sigma^2(T-t)$$

Therefore,

$$df(t,T) = \sigma^2(T-t)dt + \sigma dW(t)$$

$$f(t,T) = f(0,T) + \sigma^2 t(T - t/2) + \sigma W(t)$$

$$r(t) = f(t,t) = f(0,t) + \sigma^2 t^2/2 + \sigma W(t)$$

$$dr(t) = \left[\frac{\partial}{\partial T} f(0,t) + \sigma^2 t\right] dt + \sigma dW(t)$$

BGM market model

- The forward LIBOR rate:

$$1 \cdot [1 + \Delta T L(t, T_i)] = \frac{P(t, T_{i-1})}{P(t, T_i)}$$

$$L(t, T_i) = \frac{P(t, T_{i-1})}{\Delta T P(t, T_i)} - \frac{1}{\Delta T}$$

- We want to consider it under probability P^{T_i}, called T_i-*forward measure*, under which discounting by T_i-bond results in a martingale:

$$dL(t, T_i) = L(t, T_i)\gamma(t, T_i)dW^{T_i}(t)$$

Pricing a caplet in the market model

- **Caplet** payoff:

$$(L(T_{i-1}, T_i) - R_C)^+$$

The value is given by

$$P(t, T_i) E_t^{T_i} \left[(L(T_{i-1}, T_i) - R_C)^+ \right]$$

Black caplet formula: volatility determined from

$$\sigma^2 = \frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \gamma^2(u, T_i) du$$

A general way to price a caplet

- Denote by $P = P(T_{i-1}, T_i)$, and recall that $L\Delta T = 1/P - 1$. Thus, the caplet payoff at time T_i is given by

$$C(T_i) = \left(\frac{1}{P} - (1 + R_C\Delta T)\right)^+ = \frac{1 + R_C\Delta T}{P}\left(\frac{1}{1 + R_C\Delta T} - P\right)^+ \quad .$$

- Denoting $K = \frac{1}{1+R_C\Delta T}$, the payoff is

$$C(T_i) = \frac{1}{KP}(K - P)^+$$

A general way to price a caplet (continued)

- Note that this payoff is paid at time T_i, but it is known at time T_{i-1}.
- The price at moment T_{i-1} of payoff $C(T_i)$ paid at T_i is $P \times C$. Why?
- This means that the payoff $C(T_i)$ at time T_i is equivalent to the payoff

$$P \times C(T_i) = \frac{1}{K}(K - P)^+$$

at time T_{i-1}, equivalent to $1/K$ put options on the bond maturing at T_i, with option maturity equal to T_{i-1}, and strike price equal to K.

Pricing Options with Mathematical Models

32. Change of numeraire method

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Other pricing probabilities

- Suppose C/S is a martingale under probability P^S, where S is a numeraire security, and C is any other security. Then

$$E_t^S[C(T)/S(T)] = C(t)/S(t)$$

- This is true for any payoff $C(T)$, and the price is

$$C(t) = S(t)E_t^S[C(T)/S(T)]$$

Example

$$C(t) = S(t)E_t^S[C(T)/S(T)]$$

- Compute

$$D(t) = E_t^Q[e^{-r(T-t)}S(T)\mathbf{1}_{\{S(T)>K\}}]$$

- Change to P^S:

$$D(t) = S(t)E_t^S[\mathbf{1}_{\{S(T)>K\}}] = S(t)P_t^S(S(T) > K) = S(t)P_t^S\left(\frac{e^{r(T-T)}}{S(T)} < \frac{e^{r(T-T)}}{K}\right)$$

- Since $M(t) := \frac{e^{r(T-t)}}{S(t)}$ is a P^S-martingale, we have

$$dM(t) = -\sigma M(t)dW^S(t)$$

$$M(T) = M(t)e^{-\frac{1}{2}\sigma^2(T-t)-\sigma[W^S(T)-W^S(t)]}$$

- It is now easy to compute $P_t^S(M(T) < 1/K)$

Black-Scholes-Merton formula for bond options

- Consider zero-coupon bond prices $P(t,T)$ and an asset price process S such that $F(t) = S(t)/P(t,T)$ has deterministic volatility $\sigma_F(t)$. Then, the European call price is

$$C(0) = S(0)N(d_1) - KP(0,T)N(d_2)$$

where

$$d_1 = \frac{1}{\Sigma_F(T)}\left(\log\frac{S(0)}{KP(0,T)} + \frac{1}{2}\Sigma_F^2(T)\right)$$

$$d_2 = \frac{1}{\Sigma_F(T)}\left(\log\frac{S(0)}{KP(0,T)} - \frac{1}{2}\Sigma_F^2(T)\right)$$

$$\Sigma_F(T) := \sqrt{\int_0^T \sigma_F^2(u)du}$$

Bond option example

- **Call option price in the Vasicek model.** Consider a call option with maturity T_1, on a T_2-bond, $T_1 \leq T_2$. The process $F(t) = P(t, T_2)/P(t, T_1)$ is of the form

$$F(t) = e^{A(t,T_2)-A(t,T_1)-(B(t,T_2)-B(t,T_1))r(t)}$$

for some deterministic functions A and B, and

$$\sigma_F(t) = -\sigma[B(t, T_2) - B(t, T_1)]$$

Thus, it is deterministic, and we can use the formula.

Pricing Options with Mathematical Models

33. Introduction to credit risk models

Some of the content of these slides is based on material from the book *Introduction to the Economics and Mathematics of Financial Markets* by Jaksa Cvitanic and Fernando Zapatero.

Structural models

- Denote by V the "value of the firm". We assume the B-S-M model for it:

$$dV = (r - \delta)V\,dt + \sigma_V V\,dW$$

- Denote by E the value of equity, by T the maturity of the debt and by D the nominal value of debt. The value of equity is a call option on the value of the firm:

$$E(T) = \max(V(T) - D, 0)$$

- Bondholders receive

$$\min(V(T), D) = D - \max(D - V(T), 0)$$

Structural models (continued)

- Firm value $V(0)$ not observable. However,

$$E(0) = BS(V(0), \sigma_V)$$

$$\sigma_E E(0) = N(d_1(V_0, \sigma_V))V(0)\sigma_V$$

where the second equality comes from equating the volatilities of E and the call option on V.

Reduced-form, intensity models

- The simplest case: the probability distribution of default is exponential, with intensity λ.
- Denote by A the event that default has not occurred by time T.
- Price of a contingent claim with payoff $C(T)$ (independent of default event):

$$
\begin{aligned}
C(0) = E\left[e^{-rT}C(T)\mathbf{1}_A\right] &= E\left[e^{-rT}C(T)\right]E[\mathbf{1}_A] \\
&= E\left[e^{-rT}C(T)\right]P[A] \\
&= E\left[e^{-rT}C(T)\right]e^{-\lambda T}
\end{aligned}
$$

Reduced-form, intensity models (continued)

- Thus, the price of a defaultable claim is obtained by discounting at a higher rate $r + \lambda$:

$$C(0) = E\left[e^{-(r+\lambda)T} C(T)\right]$$

- If the interest rate is stochastic and the intensity a function of time,

$$C(0) = E\left[e^{-\int_0^T [r(u)+\lambda(u)]du} C(T)\right]$$

- Empirically, implied default probability is much larger than the historical default probability.