INTRODUCTION TO THE ECONOMICS AND MATHEMATICS OF FINANCIAL MARKETS

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Part III: Equilibrium Models

PREFACE

Why we wrote the book: The subject of financial markets is fascinating to many people: to those who care about money and investments, to those who care about the well-being of the modern society, to those who like gambling, to those who like applications of mathematics, and so on. We, the authors of this book, care about many of these things (no, not the gambling), but what we care about most is teaching. The main reason for writing this book has been our belief that we can successfully teach the fundamentals of the economic and mathematical aspects of financial markets to almost everyone (again, we are not sure about gamblers). Why are we in this teaching business instead of following the path of many of our former students, the path of making money by pursuing a career in the financial industry? Well, they don't have the pleasure of writing a book for the enthusiastic reader like yourself!

Prerequisites: This text is written in such a way that it can be used at different levels and for different groups of undergraduate and graduate students. After the first, introductory chapter, each chapter starts with sections on the single-period model, goes to multi-period models, and finishes with continuous-time models. The single-period and multi-period models require only basic calculus and an elementary introductory probability/statistics course. Those sections can be taught to third and fourth year undergraduate students in Economics, Business, and similar. They could be taught to Mathematics and Engineering students even at an earlier stage. In order to be able to read continuous-time sections it is helpful to have been exposed to an advanced undergraduate course in probability. Some material needed from such a probability course is briefly reviewed in the last chapter.

Who is it for: The book can also serve as an introductory text for graduate students in Finance, Financial Economics, Financial Engineering and Mathematical Finance. Some material from continuous-time sections is, indeed, usually considered to be graduate material. We try to explain much of that material in an intuitive way, while providing some of the proofs in appendices to the chapters. The book is not meant to compete with numerous excellent graduate books in Financial Mathematics and Financial Economics, which are typically written in a mathematically more formal way, using theorem-proof type of structure. Some of those more advanced books are mentioned in the references, and they present a natural next step in getting to know the subject on a more theoretical and advanced level.

Structure of the book: We have divided the book into three larger parts. Introductory Part I goes over the basic securities and financial market organization, the concept of interest rates, the main mathematical models, and ways to measure in a quantitative way the risk and the reward of trading in the market. Part II deals with option pricing and hedging, and similar material is present in virtually every recent book on financial markets. We choose to emphasize the so-called martingale, probabilistic approach consistently throughout the book, as opposed to the differential equations approach, or other existing approaches. For example, the one proof of the Black-Scholes formula that we provide is done calculating the corresponding expected value. Part III is devoted to one of the favorite subjects of Financial Economics, the equilibrium approach to asset pricing. This part is often omitted from books in the field of financial mathematics, having less direct applications to option pricing and hedging. However, it is this theory that gives a qualitative insight into the behavior of market participants and how the prices are formed in the market.

What can a course cover: We have used parts of the material from the book for teaching various courses at the University of Southern California: undergraduate courses in Economics and Business, a Masters level course in Mathematical Finance, and option and investment courses for MBA students. For example, an undergraduate course for Economics/Business students that emphasizes option pricing could cover (in this order):

- the first three chapters without continuous-time sections; Chapter 10 on bond hedging could also be done immediately after Chapter 2 on interest rates.

- the first two chapters of Part II on no-arbitrage pricing and option pricing, without most of continuous-time sections, but including basic Black-Scholes theory.

- chapters on hedging in Part II, with or without continuous-time sections.

- Mean-Variance section in Chapter 5 on risk; Chapter 13 on CAPM could also be done immediately after that section.

If time remains, or if this is an undergraduate Economics course that emphasizes equilibrium/asset pricing as opposed to option pricing, or if this is a two-semester course, one could also cover:

- discrete-time sections in Chapters 4 on utility;

- discrete-time sections in Part III of the book on equilibrium models.

Courses aimed at more mathematically oriented students could go very quickly through the discrete-time sections, and instead spend more time on continuous-time sections. A one semester course would likely have to make a choice: to focus on no-arbitrage option pricing methods in Part II, or to focus on equilibrium models of Part III.

Web Page for this book, Excel Files:
The web page **http://math.usc.edu/ cvitanic/book.html** will be regularly updated with material related to the book, such as corrections of typos. It also contains Microsoft Excel files, with names like "ch1.xls". That particular file has all the figures from Chapter 1, and all the computations needed to produce them. We use Excel because we wanted the reader to be able to reproduce and modify all the figures in the book. A slight disadvantage of this is that our figures sometimes look less professional than if they had been done by a specialized drawing software. We use only basic features of Excel, except for Monte Carlo simulation for which we use the Visual Basic programming language, incorporated in Excel.

The readers are expected to learn the basic features of Excel on their own, if they are not already familiar with it. At a few places in the book we do give "Excel Tips" that point out what trickier commands have been used for creating a figure. Other, more mathematically oriented software may be more efficient for longer computations such as Monte Carlo, and we leave the choice of the software to be used with some of the homework problems to the instructor or the reader. In particular, we do not use any optimization software or differential equations software, even though the instructor could think of projects using those.

Notation: superscript *. Some of the sections and problems have a * superscript. These are either more sophisticated in mathematical terms, or they require extensive use of computer software, or they are otherwise somewhat unusual and outside of the main thread of the book. They could be skipped, although we would suggest that the students do most of the problems that require use of computers.

Notation: solved problems with superscript $\tilde{}$. The end-of-chapters problems that are solved in the Student's Manual have a superscript $\tilde{}$ as in $\tilde{\mathbf{1}}\mathbf{4}$, when problem 14 is solved in the manual.

Notation: Greek letters. We use many letters from the Greek alphabet, sometimes both lower and upper case, and we list them here with appropriate pronunciation:

α (alpha), β (beta), γ, Γ (gamma), δ, Δ (delta), ε (epsilon), ζ (zeta), η (eta), θ (theta), λ (lambda), μ (mu), ξ (xi), π, Π (pi), ω, Ω (omega), ρ (rho), σ, Σ (sigma), τ (tau), φ, Φ (phi).

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errors in the working manuscript: Paula Guedes, Frank Denis Hiebsch, Chulhee Lee. Of course, we are solely responsible for any remaining errors.

A Prevailing Theme - Pricing by Expected Values: Before we start with the book's material, we would like to give a quick illustration here in the preface of a connection between a price of a security and the optimal trading strategy of an investor investing in that security. We present it in a simple model, but this connection is present in most market models, and, in fact, the resulting pricing formula is of the form that will follow us through all three parts of this book. We will repeat this type of arguments later in more detail, and we present it this early here only to give the reader a general taste of what the book is about. The reader may want to skip the following derivation, and go directly to equation (0.3).

Consider a security S with today's price $S(0)$, and at a future time 1 its price $S(1)$ either has value s^u with probability p, or value s^d with probability $1-p$. There is also a risk-free security that returns $1+r$ dollars at time 1 for every dollar invested today. We assume that $s^d < (1+r)S(0) < s^u$. Suppose an investor has initial capital x, and has to decide how many shares δ of security S to hold, while depositing the rest of his wealth in the bank account with interest rate r. In other words, his wealth $X(1)$ at time one is

$$X(1) = \delta S(1) + (x - \delta S(0))(1+r) \quad .$$

The investor wants to maximize his expected utility

$$E[U(X(1))] = pU(X^u) + (1-p)U(X^d) \quad ,$$

where U is a so-called utility function, while X^u, X^d is his final wealth in the case $S(1) = s^u$, $S(1) = s^d$, respectively. Substituting for these values, taking the derivative with respect to δ and setting it equal to zero, we get

$$pU'(X^u)[s^u - S(0)(1+r)] + (1-p)U'(X^d)[s^d - S(0)(1+r)] = 0 \quad .$$

The left-hand side can be written as $E[U'(X(1))(S(1) - S(0)(1+r))]$, which, when made equal to zero implies, with arbitrary wealth X replaced by optimal wealth $\hat{X}$,

$$S(0) = E\left[\frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]}\frac{S(1)}{1+r}\right] \quad . \tag{0.1}$$

If we denote

$$Z(1) := \frac{U'(\hat{X}(1))}{E[U'(\hat{X}(1))]} \quad , \tag{0.2}$$

we see that the today's price of our security S is given by

$$S(0) = E\left[Z(1)\frac{S(1)}{1+r}\right] \quad . \tag{0.3}$$

We will see that prices of most securities (with some exceptions, like American options) in the models of this book are of this form: the today's price $S(0)$ is an expected value of the future price $S(1)$, multiplied ("discounted") by a certain random factor. Effectively, we get the today's price as a weighted average of the discounted future price, but with weights which depend on the outcomes of the random variable $Z(1)$. Moreover, in standard option pricing models (having a so-called completeness property) we will not need to use utility functions, since $Z(1)$ will be independent of the investor's utility. The random variable $Z(1)$ is sometimes called **change of measure**, while the ratio $Z(1)/(1+r)$ is called **state-price density**, **stochastic discount factor**, **pricing kernel**, or **marginal rate of substitution**, depending on the context and interpretation. There is another interpretation of this formula, using a new probability; hence the name "change of (probability) measure". For example, if, as in our example above, $Z(1)$ takes two possible values $Z^u(1)$ and $Z^d(1)$ with probabilities $p, 1-p$, respectively, we can define

$$p^* := pZ^u(1) \ , \quad 1 - p^* = (1-p)Z^d(1) \quad .$$

The values of $Z(1)$ are such that that p^* is a probability, and we interpret p^* and $1-p^*$ as modified probabilities of the movements of asset S. Then, we can write (0.3) as

$$S(0) = E^* \left[\frac{S(1)}{1+r} \right] \quad , \tag{0.4}$$

where E^* denotes the expectation under the new probabilities, p^*, $1-p^*$. Thus, the price today is the expected value of the discounted future value, where the expected value is computed under a special, so-called **risk-neutral** probability, usually different from the **real world probability**.

Final word: We hope that we have aroused your interest about the subject of this book. If you turn out to be a very careful reader, we would be thankful if you could inform us of any remaining typos and errors that you find, by sending an e-mail to our current e-mail addresses. Enjoy the book!

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