SS 214, HMWK 5, Due Monday, Feb 27, 2006

1. Using the PDE approach, provide a complete proof that when maximizing power utility (x^γ/γ) of terminal wealth the optimal proportion is given by

$$\frac{\mu - r}{(1-\gamma)\sigma^2}$$

Hint: Guess that $V(t,x) = f(t)x^\gamma$ for some function $f(t)$.

2. Repeat Problem 1 using the duality approach. Hint: For a given power p, we have, using martingale property

$$E_t[Z^p(T)] = E_t\left[e^{-p^2\theta^2 T/2 - p\theta W(T)}e^{p(p-1)\theta^2 T/2}\right] = e^{-p^2\theta^2 t/2 - p\theta W(t)}e^{p(p-1)\theta^2 T/2}$$

This will enable you to find the "discounted" wealth process $Z(t)\bar{X}(t)$ as the conditional expectation of $Z(T)\bar{X}(T)$. Then find $d(Z(t)\bar{X}(t))$ to see what the portfolio is.

3. Repeat Problem 1 when also there is power utility from consumption.

4. In the presence of consumption, argue that

$$E\left[Z(T)\bar{X}(T) + \int_0^T Z(u)\bar{c}(u)du\right] = x \quad .$$

You may assume that the admissible strategies are such that the local martingales in your computation are martingales. (Actually, the converse is also true in the complete model, if a random variable $X(T)$ and a non-decreasing process c, with $c(0) = 0$ satisfy the above "budget constraint", there exists a portfolio strategy such that $X(T)$ is its finite wealth and c is the consumption.)

OPTIONAL PROBLEMS, YOU DON'T HAVE TO DO THEM:

5. (i) Read sections 4-10 in the Lecture Notes. (We will have discussed most of the results in class.).

(ii) Read the paper available at http://www.hss.caltech.edu/ cvitanic/PAPERS/tb.pdf. It is an application of portfolio optimization techniques.

6. Consider a Black-Scholes model in which the constant risk-premium value is denoted $\theta = (\mu - r)/\sigma$. For simplicity, assume $r = 0$. Introduce the process Z by

$$dZ(t) = -\theta Z(t)dW(t), \quad Z(0) = 1.$$

a. Show, using the duality method, that the optimal terminal wealth for the problem of maximizing

$$P[X(T) \geq C(T)] = E[\mathbf{1}_{\{X(T) \geq C(T)\}}]$$

for a given claim $C(T)$, under the constraint $X(T) \geq A(T)$, (where $A(T)$ is another claim) is given as an "option" written on $A(T), C(T), Z(T)$ of the type

$$\hat{X}(T) = C(T)\mathbf{1}_{\{\lambda Z(T)[C(T)-A(T)] \leq 1\}} + A(T)\mathbf{1}_{\{\lambda Z(T)[C(T)-A(T)] > 1\}} \quad .$$

Here, $\lambda > 0$ is a constant chosen so that $E[Z(T)\hat{X}(T)] = x$.

b. Suppose now that

$$A(T) = C(T) - k$$

for some constant $k > 0$, and that $X(0) < C(0)$, where $C(t) = E_t[Z(T)C/Z(t)]$ is the replicating process for claim C. Show that the optimal wealth at time t (the price of option $\hat{X}(T)$) satisfies

$$\hat{X}(t) = C(t) - kN\left(\frac{-\theta W(t) + \log(k\lambda) + \frac{1}{2}|\theta|^2 T}{|\theta|\sqrt{T-t}}\right) \quad .$$

c. Continuing part b, show also that the optimal portfolio (amount in stock) that replicates this payoff satisfies

$$\hat{\pi}(t) = \pi_C(t) + k\frac{\theta}{|\theta|\sigma\sqrt{T-t}}n\left(N^{-1}\left(\frac{C(t) - \hat{X}(t)}{k}\right)\right),$$

where π_C is the replicating portfolio for claim $C(T)$. Here, N and n are the normal cumulative distribution function and density function, respectively.