SS 214, HMWK 2, Due Monday, Jan 30, 2006

1. Argue that the solution of the SDE

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t)$$

is given by

$$S(t) = S(0)e^{\int_0^t \sigma(s)dW(s) + \int_0^t (\mu(s) - \sigma^2(s)/2)ds}.$$

Find the SDE satisfied by the process $Y = 1/S$.

2. Read Section 5I in the book.

3. Let $R(t) = (W_1^2(t) + W_2^2(t))^{1/2}$, where (W_1, W_2) is a two-dimensional Brownian motion. Show, using the multi-dimensional Ito's rule, that

$$dR(t) = \frac{W_1(t)}{R(t)} dW_1(t) + \frac{W_2(t)}{R(t)} dW_2(t) + \frac{1}{2R(t)} dt.$$

4. Let $M(t) := \int_0^t Y(u)dW(u)$, where $E\left[\int_0^\infty Y^2(u)du\right] < \infty$. Use Itô's rule to find the differential dQ of the process $Q(t) = M^2(t) - \int_0^t Y^2(u)du$ and to conclude that the process Q is a martingale. In particular, argue that this implies $E[M^2(t)] = E\left[\int_0^t Y^2(u)du\right]$.

5. Find the Black-Scholes formula for the European put option with payoff $(S_T - K)^+$, either by direct calculation of the expectation of its discounted value under the equivalent martingale measure, or by "guessing" the solution and checking that it satisfies the PDE. Next, verify your answer by finding the put price from the call price and the put-call parity relation (that you showed in Hmwk 1).

6. Let $C(t, s)$ be a price function (the value of the replicating portfolio) for the option $g(S(T))$ in the Merton-Black-Scholes model with interest rate r.

a. Express, in terms of the function $C(s, t)$ and its derivatives, the set of the pairs (s, t) for which the replicating portfolio borrows money from the bank. (**Hint:** the number of shares of S in the portfolio is given by $C_s(t, s)$; borrowing will take place if the amount held in the stock is larger than the value of the replicating portfolio.)

b. Suppose now that there are two interest rates in the market: r for lending, and R for borrowing, $R > r$. Can you guess (without proof) what the Black-Scholes partial differential equation would look like in this case?

7. Consider the option with the payoff $g(S(T)) = (S(T))^n$, in the Merton-Black-Scholes model. It can be shown that its price at time t has the form $C(t, s) = f(t, T)s^n$. Find the function $f(t, T)$ by the following two methods:

a. computing the risk-neutral expected value;

b. substituting $C(t, s)$ in the Black-Scholes Partial Differential Equation and its boundary condition; then, using this, getting an ordinary differential equation for $f(t, T)$, and solving it.

8. (Optional) Read Chapter 6.