SS 214, HMWK 1, Due Wednesday, Jan 18, 2006

1. In a one-period two-state model with $S^u = S_0 u$, $S^d = S_0 d$, what is the relationship between interest rate r and factors u and d so that there is no arbitrage?

2. In the previous model, we have $S_0 = K = \$100.00$, $r = 5\%$, $S_T^u = \$110.00$, $S_T^d = \$90.00$. Find the fair price of the put option $g(S_T) = (K - S_T)^+$ using both a replication argument and the martingale probability measure formula. Construct the strategy that result in arbitrage (positive profit from zero investment) if the price of the put is less than the fair price.

3. Prove the Put-Call parity in the above model (or more generally, if you wish):

$$C_0 - P_0 = S_0 - \bar{K},$$

where C_0, P_0 are call and put prices and $\bar{K}$ is the discounted (present) value of K. You can use either a no-arbitrage argument, or an argument using prices as expected values under a martingale measure.

4. Describe how you would price an option in a two-period, or a multi-period binomial model, in which, given the current stock price, the next period stock can have two values:

$$S_{t+1} = S_t u, \quad \text{or} \quad S_{t+1} = S_t d$$

Hint: Reduce the problem to the single-period pricing, by starting at the maturity date and going backwards.

5. For a self-financing strategy θ show that

$$\bar{W}_t(\theta) = \bar{W}_0(\theta) + \sum_{u=0}^{t-1} \theta_u (\bar{S}_{u+1} - \bar{S}_u),$$

where $\bar{X}$ is a discounted value of X.

(ii) Show that if $\bar{S}_t$ is a P^*-martingale, then $\bar{W}_t$ is also a P^*-martingale (Hint: first show that $\bar{W}_{t+1} - \bar{W}_t = \theta_t(\bar{S}_{t+1} - \bar{S}_t)$).

6. Prove that the process $X(t) := B(t) + \alpha t$ is a submartingale, where $\alpha \geq 0$ and $B(t)$ is standard one-dimensional Brownian motion.

7. Two diffusion processes X and Y satisfy:

$$dX = (2 + 5t + X)dt + 3dB_t^1$$

$$dY = 4Y dt + 8Y dB^1 + 6dB_t^2 \quad ,$$

where B^1 and B^2 are independent Brownian Motions. Use Itô's rule to find the dynamics of the processes

(i) $X^4, \quad e^X, \quad \sin(X)$;

(ii) $X/Y, \quad X^4 \cdot Y, \quad \sin(X) \cdot Y$.

8. (Optional. You don't have to hand this one back.) Read Proposition G on page 30 of the book and Exercise 2.18 (Exercise 2.12. is also related).

9. (Optional. You don't have to hand this one back.). Go through Exercise 2.17.