Chapter 3. Surface Profiles for Steady Channel Flow

This chapter is concerned with steady flow in open channels – a qualitative examination rather than a numerical computation of the flow parameters along the channel length. The tool most useful for this process is a reformulated momentum or energy equation in terms of water depth, and so the qualitative description of the flow will be in terms of the water surface profile.

Flows considered in this chapter are gradually varying with the possibility of localized places of rapidly vary flow, such as a hydraulic jump. In the examples presented, different water surface profiles are generated in three ways: 1.) by considering various slopes for the channel; 2.) by changing the slope along the channel so as to divide the channel into a number of reaches, each of constant slope; and 3.) by continuously varying the width of a channel within a reach. To simplify the results, the channels with changing slope are all prismatic, and the channels of varying width are all of constant slope. In some cases, similar profiles could be generated by varying Manning's n, but that parameter is assumed to be constant.

3.1 Flow Equations in Terms of Water Depth

The continuity equation for steady flow indicates that the flow rate Q is constant along the channel. For the second equation needed to solve a flow problem, consider the specific energy form of the differential, steady energy equation (2.62''), written below with the expression for the specific energy from equation (2.61) inserted into the derivative:

$$\frac{d\hat{E}}{dx} = \frac{d\left(\alpha \frac{V^2}{2g} + y_s \cos\theta\right)}{dx} = S_0 - S_f \,. \qquad (3.1)$$

In order to be useful for categorizing surface profiles, this equation is modified in several steps in the derivation that follows. For simplification, the coefficient α will be assumed to be constant. Also, the x axis will be located at the channel bottom so that y_s is the water depth. To simplify the notation, y_s will be written as just y.

Using $V = \frac{Q}{A}$ and taking the derivative, equation (3.1) becomes

$$-\frac{\alpha V^2}{gA}\frac{dA}{dx} + \frac{dy}{dx}\cos\theta = S_0\left(1 + y\frac{d\theta}{dx}\right) - S_f \,, \qquad (3.2)$$

where $\sin\theta = S_0$ has been used. The $y\frac{d\theta}{dx}$ term on the right side is only present if the channel slope angle θ varies with x. Unless the angle change is large, this term can be neglected. For example, if a 2° angle change, which is relatively large, occurs over a length of the channel equal to the water depth y, then $y\frac{d\theta}{dx} = 0.035$. So the effect is small over a relatively short length of the channel. If this slope angle change is more abrupt, then $\frac{d\theta}{dx}$ increases but the length of the channel involved decreases in proportion, so the overall effect would be similar. Henceforth, the $y\frac{d\theta}{dx}$ term is omitted.

Next, $\frac{dA}{dx}$ in equation (3.2) is expanded as

$$\frac{dA}{dx} = B\frac{dy}{dx} + \frac{dA}{dx}\text{(const y)}\ , \qquad (3.3)$$

where B is the width of the channel at the water surface, and the last term is the rate of change of A with respect to x considering only the variation in cross-sectional shape of the channel and not due to any change in water depth. Substitution of equation (3.3) into equation (3.2) and some manipulation leads to

$$\frac{dy}{dx} = \frac{1}{\cos\theta}\frac{S_0 - S_f + \frac{\alpha V^2}{gA}\frac{dA}{dx}\text{(const y)}}{(1-\boldsymbol{F}^2)}\ , \qquad (3.4)$$

which is the desired expression for the slope $\frac{dy}{dx}$ of the water surface profile. In equation (3.4), $\boldsymbol{F}$ is the Froude number given by

$$\boldsymbol{F} = \left(\frac{\alpha V^2}{gD\cos\theta}\right)^{\frac{1}{2}}, \qquad (3.5)$$

and D is the hydraulic depth defined as $\frac{A}{B}$.

The third term in the numerator on the right side of equation (3.4) accounts for a channel being non-prismatic. It's contribution can be thought of as modifying the slope of the channel, so it will be represented by S_{np}, the subscript standing for non-prismatic. Thus,

$$\frac{dy}{dx} = \frac{1}{\cos\theta}\frac{S_0 - S_f + S_{np}}{(1-\boldsymbol{F}^2)}\ . \qquad (3.6)$$

For a prismatic channel, this equation simplifies to

$$\frac{dy}{dx} = \frac{1}{\cos\theta}\frac{S_0 - S_f}{(1-\boldsymbol{F}^2)}\ . \qquad (3.7)$$

If the previous derivation had been done with the momentum equation (eq. 239') instead of the energy equation (eq. 262''), assuming that β is constant, exactly the same results would have been obtained except that β would replace α in equation (3.5). This discrepancy is apparently due to the assumption of α or β being constant in the respective derivations. Results of subsequent derivations in this chapter are obtained using α, although this ambiguity should be kept in mind.

3.2 Subcritical and Supercritical Flow

For channel flow such that $\boldsymbol{F}$ is equal to 1, the denominator in equation (3.6/3.7) becomes zero, and this is referred to as critical flow. For a specified flow rate Q, there is a value of y that produces V and D for which $\boldsymbol{F}$ equals 1. This value of y is called the critical water depth y_c, and the corresponding flow velocity is the critical velocity V_c. The critical depth is a function of Q and the shape of the cross-section. Within the small angle assumption that allows $\cos\theta$ to be replaced by 1, y_c also does not depend on the channel slope angle θ. From equation (3.5), in terms of D, the expression for V_c is

$$V_c = \left(\frac{gD\cos\theta}{\alpha}\right)^{\frac{1}{2}}, \qquad (3.8)$$

where it is recognized that α could be β depending on whether the previous derivation starts with the energy or momentum equation.

Using equations (3.5) and (3.8), $\boldsymbol{F}$ can be written as

$$\boldsymbol{F} = \frac{V}{V_c}\ , \qquad (3.9)$$

the ratio of the actual flow velocity to the critical velocity.

As an example, consider a rectangular channel cross-section. Using equation (3.5) with $\boldsymbol{F} = 1$; $V = \frac{Q}{A}$; $A = BD$ and $D = y_c$, the critical depth can be found as

$$y_c = \left(\frac{\alpha Q^2}{B^2 g\cos\theta}\right)^{\frac{1}{3}} = \left(\frac{\alpha q^2}{g\cos\theta}\right)^{\frac{1}{3}}, \qquad (3.10)$$

where $q = \frac{Q}{B}$ is the flow rate per unit width of the channel. For an arbitrarily shaped channel cross-section, such a closed form solution is not possible, so an iterative solution procedure must be employed to determine y_c.

Water in a channel flowing faster than the critical velocity ($\boldsymbol{F}$ greater than 1; water depth less than y_c) is referred to as supercritical flow. Conversely, water in a channel flowing slower than the critical velocity ($\boldsymbol{F}$ less than 1; water depth greater than y_c) is called subcritical flow.

For flow at a given rate Q in a prismatic channel with water depth equal to the critical depth y_c, the velocity $V = V_c$ is closely related to the speed that a wave in the channel would travel if the water were the same depth. This travelling wave speed, denoted by c, is found by imposing a travelling wave solution into the continuity equation (2.14) and into the linearized, homogenous form of the unsteady differential momentum or energy equation, and then combining. Using the momentum equation (eq. 2.35), the linearized, homogenous form is

$$\rho\frac{\partial Q}{\partial t} = -\rho g A\frac{\partial y_s}{\partial x}\cos\theta\ . \qquad (3.11)$$

The resulting wave speed (details omitted here) is

$$c = (gD\cos\theta)^{\frac{1}{2}}, \qquad (3.12)$$

which is identical to V_c except for the presence of α; see equation (3.8). The reason for mentioning this relationship between critical flow velocity and travelling wave speed is just to give another perspective on the concept of critical flow velocity. Travelling waves are transitory phenomena and this chapter is concerned solely with steady flow. A final note: Because of the hydrostatic pressure condition used in deriving equation (2.35), the wave speed c from equation (3.12) applies when the wave length is long compared to the water depth.

Another aspect of critical flow is revealed by a plot of specific energy $\hat{E}$ as a function of water depth y for a given channel cross-section. $\hat{E}$ is given by equation 2.61. Q is held constant, so as y varies, V also changes to maintain the same value of Q. As y approaches zero, V approaches infinity, which causes $\hat{E}$ to approach infinity. As y approaches infinity, V approaches zero, and $\hat{E}$ again approaches infinity because of y. These two parts of the $\hat{E}$ vs. y curve are asymptotic to the y axis and asymptotic to the $\hat{E} = y\cos\theta$ line, respectively. The plot is shown in Figure 3.1 where it is seen that a minimum value exists for $\hat{E}$, marked C in the figure. This minimum value occurs at the critical water depth y_c, and so the minimum specific energy is denoted by $\hat{E}_c$.

That the minimum point of the $\hat{E}$ vs. y curve corresponds to critical flow can be shown by setting the derivative of $\hat{E}$ with respect to y to zero:

$$\frac{d\hat{E}}{dy} = \frac{d\left(\alpha\dfrac{V^2}{2g} + y\cos\theta\right)}{dy} = 0\,. \qquad (3.13)$$

Using $V = \frac{Q}{A}$ and taking the derivative while assuming α is constant and again neglecting the $y\frac{d\theta}{dx}$ term, equation (3.13) becomes

$$-\frac{\alpha V^2}{gA}\frac{dA}{dy} + \cos\theta = 0\,. \qquad (3.14)$$

Since $\frac{1}{A}\frac{dA}{dy} = \frac{B}{A} = \frac{1}{D}$, comparison to equation (3.8) leads to $V = V_c$, which indicates the minimum point corresponds to critical flow.

A similar result occurs for the specific force $\hat{F}$ (given by equation 2.32), and a plot of $\hat{F}$ vs. y is shown in Figure 3.2. The plot looks similar to $\hat{E}$ vs. y except the part extending upward to the right does not have an asymptote.

$\hat{F}$ has a minimum at y_c as shown by setting the derivative of $\hat{F}$ to zero:

$$\frac{d\hat{F}}{dy} = \frac{d\left(\beta\dfrac{Q^2}{gA} + A(y-\bar{y})\cos\theta\right)}{dy} = 0\,, \qquad (3.15)$$

where $\frac{Q}{A}$ has been substituted for V in the first term of the numerator. Assuming β is constant and neglecting the $y\frac{d\theta}{dx}$ term, equation (3.15) becomes

$$-\frac{\beta Q^2}{g}\frac{1}{A^2}\frac{dA}{dy} + \cos\theta\,\frac{d(Ay)}{dy} - \cos\theta\,\frac{d(A\bar{y})}{dy} = 0\,. \qquad (3.16)$$

In the first term $\frac{dA}{dy}$ equals B; the second term equals $(A + By)\cos\theta$; and the third term equals $By\cos\theta$; thus:

$$-\frac{\beta V^2 A}{gD} + A\cos\theta = 0\,, \qquad (3.17)$$

which leads to $V = V_c$ except for a discrepancy between α and β. However, as mentioned earlier, V_c from equation (3.8) could alternatively have been defined using β.

3.3 Channel Slope Classification

A channel's slope is classified as mild, critical, steep, horizontal or adverse. The mild, critical and steep designations apply to a positive θ (water flowing downhill), and an adverse slope has a negative θ (water flowing uphill). The distinction between mild, critical and steep slopes is based on whether the water depth under uniform flow conditions (the normal depth y_n, since uniform flow is also referred to as normal flow) is greater than, equal to or smaller than the critical depth y_c, respectively. Note that the concept of uniform flow does not apply to horizontal and adverse slopes.

As explained in the previous section, for a given flow rate Q and channel cross-sectional shape, the critical depth y_c is the one that makes the Froude number $\boldsymbol{F}$ equal to 1. The normal depth y_n depends on Q and the cross-sectional shape, roughness (the Manning n value) and slope of the channel, and it can be determined from an iterative solution involving equation (2.73) and $V = \frac{Q}{A}$. For the given value of Q, the slope is mild if $y_n > y_c$, critical if $y_n = y_c$, and steep if $y_n < y_c$.

Another way to view slope classification is that a slope is mild if uniform flow is subcritical and steep if uniform flow is supercritical. However, it is important to recognize that subcritical flow can occur for any slope, including horizontal and adverse; the same is true for supercritical flow.

Because the relations between both y_n and y_c and the flow parameters are nonlinear, it is possible that a slope classified as mild, critical or steep under one value of Q could have a different slope designation under another value of Q. The same is true for different Manning n values.

3.4 Basic Flow Profiles

An actual channel can have varying cross-section, slope and roughness along its length; hence, the profile of the water surface as well as transitions between subcritical and supercritical flow can be quite complicated. In order to provide some insight into flow behavior, it is instructive to consider a prismatic channel segment that has constant slope and roughness. However, except for uniform flow, analytical solutions to equation (3.7) are not possible due to the nonlinear terms in the equation. It is possible, though, to identify 13 basic types of solutions to equation (3.7). These basic solution types are referred to as profile types because they describe the profile of the water surface, although the classification contains information about the subcritical or supercritical nature of the flow as well.

Before presenting the 13 basic solution types, an alternative form of equation (3.7) that can provide additional description of the solution types is derived. V in equation (2.73) is replaced by V_n and then that equation is combined with equation (2.74) to produce

$$S_f = S_0 \left(\frac{V}{V_n}\right)^2 , \qquad (3.18)$$

where V_n is the water velocity when the depth is y_n. Substitution of equations (3.9) and (3.18) into equation (3.7) gives the desired result:

$$\frac{dy}{dx} = \frac{S_0}{\cos\theta} \frac{1 - \left(\frac{V}{V_n}\right)^2}{1 - \left(\frac{V}{V_c}\right)^2} . \qquad (3.19)$$

Flow profiles are classified according to the channel slope (S for steep, C for critical, M for mild, H for horizontal and A for adverse) and as 1, 2 or 3 depending on the relation of the water depth y to the normal and critical depths y_n and y_c. The basic profile types are summarized in Table 3.1 and plotted in Figure 3.3, where the water surfaces for normal flow (NDL, normal depth line) and critical flow (CDL, critical depth line) are shown for reference. The NDL and the CDL shown are parallel to the x axis; the NDL for the horizontal slope lies at an infinite elevation and one for the adverse slope is not defined. None of the water surface profiles crosses the NDL or CDL. One consequence of the latter is that each profile is entirely subcritical or entirely supercritical. The last column in Table 3.1 is deduced from equation (3.19); note that a slope $\frac{dy}{dx}$ of the water surface profile equal to $\frac{S_0}{\cos\theta}$ means the profile is horizontal since $\frac{S_0}{\cos\theta} = \tan\theta$.

The convention used to construct the plots in Figure 3.3 is to expand the plots in the y direction and compress them in the x direction to reveal the details. The plots are also rotated so that the slopes of the channels relative to each other are clear. For the water surface profiles whose slopes approach the horizontal, the profiles are distorted to maintain this feature.

Slope	Profile Type	Relation of y to y_n and y_c	Sign of $S_0 - S_f$	Sign of $1 - \boldsymbol{F}^2$	Sign of $\frac{dy}{dx}$	Additional
Steep ($y_c > y_n$)	S1	$y \geq y_c$	+	+	+	$\frac{dy}{dx} > \frac{S_0}{\cos\theta}$
	S2	$y_c \geq y > y_n$	+	−	−	
	S3	$y < y_n$	−	−	+	$\frac{dy}{dx} < \frac{S_0}{\cos\theta}$
Critical ($y_n = y_c$)	C1	$y > y_n, y_c$	+	+	+	$\frac{dy}{dx} = \frac{S_0}{\cos\theta}$
	C2	$y = y_n, y_c$	?	?	?	$\frac{dy}{dx} = \frac{0}{0}$
	C3	$y < y_n, y_c$	−	−	+	$\frac{dy}{dx} = \frac{S_0}{\cos\theta}$
Mild ($y_n > y_c$)	M1	$y > y_n$	+	+	+	$\frac{dy}{dx} < \frac{S_0}{\cos\theta}$
	M2	$y_n > y \geq y_c$	−	+	−	
	M3	$y \leq y_c$	−	−	+	$\frac{dy}{dx} > \frac{S_0}{\cos\theta}$
Horizontal ($y_n = \infty$)	H2	$y_n > y \geq y_c$	−	+	−	
	H3	$y \leq y_c$	−	−	+	
Adverse (no y_n)	A2	$y \geq y_c$	−	+	−	
	A3	$y \leq y_c$	−	−	+	

Table 3.1 Summary of basic profile types for a prismatic channel

Each of the plotted profiles in Figure 3.3 can be thought of as the solution to equation (3.7) or (3.19) that passes through some selected point whose y coordinate falls in the desired depth range. For example, for M2, choose a point in the mild slope diagram that lies between the NDL and the CDL and then solve equation (3.7) or (3.19) subject to the condition that the solution passes through the selected point.

C1, C2 and C3 are special profiles that can only occur when the channel slope exactly equals the critical slope. For critical slope of the channel, $y_n = y_c$ so the NDL and CDL coincide. Also, $V_n = V_c$, and from equation (3.19), $\frac{dy}{dx} = \frac{S_0}{\cos\theta}$; thus, the C1 and C3 profiles are horizontal. However, the profiles are ill defined at the intersections with the NDL/CDL because, there, $V = V_n = V_c$ and equation (3.19) becomes $\frac{dy}{dx} = \frac{0}{0}$. This is also true along the entire length of the C2 profile, which represents uniform flow in a channel of critical slope. Laboratory experiments that produce a C2 profile show a rough water surface, indicating some instability. However, none of this is much of a concern because one does not expect to have a channel slope exactly equal to the critical slope.

The following explanatory comments are helpful to understand the other profiles.

1. For a profile above the NDL, V is less than V_n and S_f is less than S_0.
2. For a profile below the NDL, V is greater than V_n and S_f is greater than S_0.
3. For a profile above the CDL, V is less than V_c and $\boldsymbol{F}$ is less than 1.
4. For a profile below the CDL, V is greater than V_c and $\boldsymbol{F}$ is greater than 1.
5. For S1 and M1 in the downstream direction and for H2 and A2 in the upstream direction, the water becomes infinitely deeper, so V approaches 0 and $\frac{dy}{dx}$ approaches $\frac{S_0}{\cos\theta}$ (profile approaches horizontal). S1, M1 and A2 have horizontal asymptotes.
6. For S2 and S3 in the downstream direction and M1 and M2 in the upstream direction, V approaches V_n and $\frac{dy}{dx}$ approaches 0, so these profiles approach the NDL asymptotically.
7. For S1 and S2 in the upstream direction and M2, M3, H2, H3 A2 and A3 in the downstream direction, V approaches V_c and $\frac{dy}{dx}$ approaches $\pm\infty$, so these profiles meet the CDL perpendicularly.
8. For profiles S3, M3, H3 and A3 (and also C3) in the upstream direction, the water depth approaches zero, so V approaches ∞. This limit is not physically possible, which is indicated in Figure 3.3 by ending the profiles with dashes.

Profiles S1 and M1 are called backwater curves because they result from something downstream that causes the water to back up. M2, H2 and A2 are called drawdown curves because they result from something downstream that is causing the water to draw down. S2 is also referred to as a drawdown curve.

The places in Figure 3.3 where the various profiles approach and meet the CDL in the direction perpendicular to the CDL obviously involve rapidly varying flow, and so they violate the assumptions behind equations (3.7) and (3.19). These locations occur where the right side of these equations become singular, causing $\frac{dy}{dx}$ to approach $\pm\infty$. Some of these locations, namely where S1, M3, H3 and A3 meet the CDL, indicate the presence of a hydraulic jump. For each one, the jump will occur before the profile intersects the CDL; therefore, the profile is valid up to the jump (assuming it is not too steep even there). The other locations, namely where S2, M2, H2 and A2 meet the CDL, indicate a steep drop of the water surface profile, and there is no hydraulic "drop" phenomenon analogous to a hydraulic jump.

An analytical solution for a water surface profile in the vicinity of where it intersects the CDL can be derived for a channel with rectangular cross-section (not applicable to C1and C3). By combining equation (3.5) and (3.8), $\boldsymbol{F}$ can be expressed as

$$\boldsymbol{F} = \frac{{y_c}^3}{y^3}\ , \qquad (3.20)$$

where equation (3.5) is used with $V = \frac{Q}{A}$, $A = BD$ and $D = y$; and equation (3.8) is used with $V_c = \frac{Q}{A}$, $A = BD$ and $D = y_c$. The expression for $\boldsymbol{F}$ is then substituted into equation (3.7):

$$\frac{dy}{dx} = \frac{y^3}{\cos\theta} \frac{S_0 - S_f}{y^3 - {y_c}^3} \quad . \qquad (3.21)$$

Expanding $y^3 - {y_c}^3$ near y_c and keeping only the lowest order term gives

$$\frac{dy}{dx} = \frac{y_c}{3\cos\theta} \frac{S_0 - S_{fc}}{y - y_c} \quad , \qquad (3.22)$$

where S_{fc} is S_f evaluated at the CDL. The solution is

$$y = y_c + C \cdot x^{\frac{1}{2}} \quad , \qquad (3.23)$$

where

$$C = \left[\frac{2y_c}{3\cos\theta}\left(S_0 - S_{fc}\right)\right]^{\frac{1}{2}} . \qquad (3.24)$$

Equations (3.23) and (3.24) apply directly to S1 close to its intersection with the CDL, with x measured from this intersection. Expressions for the other profiles require sign adjustments only. Thus, the steep parts of the profiles vary as $|x|^{\frac{1}{2}}$, where $x = 0$ is the point at which the water surface profile intersects the CDL.

3.5 Sequenced Flow for Prismatic Channels of Constant Roughness

The channels examined in this section have discrete slope changes, and therefore the water surface profiles become sequences of the basic profile types presented in the previous section. The idea is to construct the solution using the basic profile types shown in Figure 3.3, as well as the uniform flow profiles. Often, only pieces cut out from the basic profile types are used. As will be shown, sometimes construction of a complete solution is not possible without utilizing a hydraulic jump.

Each channel is prismatic and has constant roughness throughout its entire length. The flow rate Q is assumed to be known. In order to add more variety to the water surface profiles, some examples utilize a sluice gate, which allows water to flow underneath it when raised, and some of the channels truncate at an overfall. All channels have infinite extent in both the upstream and downstream directions, except when ending at an overfall.

A condition to be satisfied by the solutions presented here is that far enough downstream, the water surface profile should be equal to or asymptotic to the NDL. This implies that the channel is free to drain at uniform flow and not subject to backing up from some obstruction far downstream. This condition is not used when a channel ends in an outfall; however, the water is still assumed to be able to freely drain away.

The water surface profiles in the figures of this section are stretched vertically and compressed horizontally in order to exaggerate the slope of the channel and the slope of the water surface profile. This means that a horizontal line will remain horizontal and that any line parallel to the y axis (which includes a line designating a channel cross-section) will appear to be vertical. Although reaches of different slope will actually have different directions for their y axes; they will all appear to be vertical. All diagrams will show the CDL for each reach and, except for horizontal and adverse channel slopes, the NDL for each reach as well. Since the channels are prismatic, the CDL will be at the same elevation relative to the bottom of the channel along the entire length of the channel.

A concept introduced in this section is the control point. A control point is defined as a known depth of water at a specific location along a channel. Once a control point is established, the water surface profile can be constructed in both the upstream and downstream directions from the control point, either qualitatively or by numerical integration. There are three types of control points: one where the water depth is the critical depth y_c, one where the depth is the normal depth y_n, and one where the depth is known because of the presence of some structure such as a sluice gate. The latter type is referred to as an artificial control point. In the water surface profile diagrams, these control points are designated with a C, N or A, respectively. If a normal depth control point occurs where the channel changes slope, it will lie on either the NDL for the reach upstream of the slope change or the NDL for the reach downstream of the slope change. A water surface profile may have only one control point or it may have more than one. Sometimes, the presence of a control point for a given problem will be obvious; other times it must be established through trial and error.

3.5.1 Four cases of channel slope change

The channels considered in this section contain one point of slope change. Thus, there are two reaches of constant slope. The four cases of channel slope change are as follows:

a. Mild slope to milder slope
b. Mild slope to steeper but still mild slope
c. Steep slope to less steep but still steep slope
d. Steep slope to steeper slope.

Thus, the two slopes are either both mild or both steep. The parts of the channel on either side of the point of slope change will be referred to as the upstream and downstream reaches.

Only one solution is possible for each case, and these are shown in Figure 3.4 in the same order as listed above. For the two cases with mild channel slope, the downstream reach is normal flow, and the upstream reach is M1 or M2 as shown in parts a and b of the figure, respectively. For the two steep channel cases, the upstream reach is normal flow, and the downstream reach is S3 or S2 (parts c and d, respectively). The downstream condition is satisfied in each of the four cases, either exactly with the NDL or asymptotically with S2 or S3. These are the only ways that the drained downstream condition can be met. For all four cases, the control point is at the point of channel slope change on the NDL for the reach where the flow is uniform.

For the two cases where the downstream channel reach is steeper than the upstream reach, the water surface profile drops in the vicinity of the point where the channel changes slope (parts b and d of Figure 3.4). As the amount of slope change between the two reaches increases, the water surface drops more sharply because the M2 or S2 profile shifts over closer to the point where $\frac{dy}{dx} = -\infty$. A steep water surface profile indicates rapidly varying flow and a source of error in the solution.

3.5.2 Channel with slope change from mild to steep

Consider a channel that has a slope change that divides the channel into an upstream reach with a mild slope and a downstream reach with a steep slope. The NDL is above the CDL in the upstream reach and below it in the downstream reach.

The only possible solution is to use profiles M2 and S2 and join them at the section where the channel changes slope, as shown in Figure 3.5. The full extents of both profiles from Figure 3.3 are employed and result in a steep, continuous decrease in the water surface elevation where the two profiles join. Both profiles truncate at the CDL where $\frac{dy}{dx} = -\infty$, which is a violation of the gradually varying flow assumptions. For this case, the control point is on the CDL at the point where the channel changes slope.

It is worth noting that the infinite value of $\frac{dy}{dx}$ does not occur if there is a gradual transition from the mild slope of the channel to the steep slope, as shown in Figure 3.6. This can be shown by the following analysis. Consistent with earlier derivations, terms containing $\frac{d\theta}{dx}$ are neglected.

In Figure 3.6, the CDL maintains its constant elevation relative to the channel bottom through the transition since the channel is prismatic, but there is no NDL in the transition zone because uniform flow does not exist in a reach where the slope varies. However, the equation $S_0 = S_f$ can still be used to compute a depth profile at every point along the transition, which will be referred to as the PDF (parallel depth line). Because satisfying $S_0 = S_f$ means that the numerator in equation (3.7) equals zero, resulting in $\frac{dy}{dx} = 0$, any profile of the water surface that crosses the PDF must be parallel to the channel bottom at that point and $\frac{dy}{dx}$ will switch sign (as long as the denominator of equation 3.7 is not simultaneously equal to zero). The NDL is a special case of the PDL for a channel of constant slope. A water surface profile can cross a PDL, but not an NDL. The PDL is useful because plotting it on a channel profile can help determine where the actual water surface profile lies. In Figure 3.6, the PDL in the transition zone and the NDL of the upstream and downstream reaches make one continuous line.

It is obvious that the CDL and PDL intersect, and this intersection is labeled point P in Figure 3.6. The water surface profile must cross the CDL and PDL simultaneously at this same intersection. If the water surface profile crosses the PDL upstream of point P, it slope changes sign from negative to positive and the water depth will continue to increase. If the profile crosses the CDL upstream of point P, there is no solution to equation 3.7 that allows it to continue downstream.

Point P is the only place where the profile can cross the CDL without $\frac{dy}{dx}$ being infinite. At point P, $\frac{dy}{dx} = \frac{0}{0}$, which can be evaluated using L'Hopital's rule:

$$\lim_{x \to x_P} \frac{dy}{dx} = \frac{\frac{d\left(S_0 - S_f\right)}{dx}_{\lfloor x = x_P \rfloor}}{\frac{d(1 - \boldsymbol{F}^2)}{dx}_{\lfloor x = x_P \rfloor}} \,. \qquad (3.25)$$

Using equations (2.73) and (2.74) as well as $V = \frac{Q}{A}$, $A = BD$ and $\frac{dV}{dx} =- V\frac{1}{D}\frac{dy}{dx}$ for a prismatic channel, the numerator of equation (3.25) can be written as

$$\frac{d\left(S_0 - S_f\right)}{dx}_{\lfloor x = x_P \rfloor} = \left(\frac{dS_0}{dx}\right)_P + S_{0P}\left(\frac{2}{D}\frac{dy}{dx} + \frac{4}{3}\frac{1}{R}\frac{dR}{dx}\right)_P \qquad (3.26)$$

and the denominator can be written as

$$\frac{d(1 - \boldsymbol{F}^2)}{dx}_{\lfloor x = x_P \rfloor} = \frac{1}{D_P}\left(2\frac{dy}{dx} + \frac{dD}{dx}\right)_P , \qquad (3.27)$$

where the subscript P denotes quantities evaluated at the intersection point P. After substitution and some rearrangement, equation (3.25) becomes

$$\left(\frac{dy}{dx}\right)_P \left(2\frac{dy}{dx} + \frac{dD}{dx}\right)_P = D_P \left(\frac{dS_0}{dx}\right)_P + S_{0P}\left(2\frac{dy}{dx} + \frac{4}{3}\frac{D}{R}\frac{dR}{dx}\right)_P , \qquad (3.28)$$

which can be solved for $\left(\frac{dy}{dx}\right)_P$ by iteration. As long as $\left(\frac{dS_0}{dx}\right)_P$ is not infinite (sharp slope change), then $\left(\frac{dy}{dx}\right)_P$ will be finite.

For a rectangular channel, D can be replaced by the water depth y, and if the channel is also wide, R can be approximated by y. Equation (3.28) becomes a quadratic in $\left(\frac{dy}{dx}\right)_P$:

$$3\left(\frac{dy}{dx}\right)_P^2 - \frac{10}{3}S_{0P}\left(\frac{dy}{dx}\right)_P - y_P\left(\frac{dS_0}{dx}\right)_P = 0 , \qquad (3.29)$$

whose solution is

$$\left(\frac{dy}{dx}\right)_P = \frac{5}{9}S_{0P} \pm \sqrt{\frac{25}{81}S_{0P}{}^2 + \frac{y_P}{3}\left(\frac{dS_0}{dx}\right)_P} \qquad (3.30)$$

for a wide rectangular channel. Of interest is the negative solution, which can be obtained by choosing the negative sign for the radical.

As an example evaluation of equation (3.30), assume $\left(\frac{dS_0}{dx}\right)_P$ equals $\frac{12S_{0P}}{y_P}$; that is, the slope of the channel changes at a rate of $12S_{0P}$ for every distance downstream equal to the water depth y_P. Further, if S_{0P} is small then $\left(\frac{dy}{dx}\right)_P$ can be approximated by using only the second term under the radical. Thus, $\left(\frac{dy}{dx}\right)_P$ would be about $2\sqrt{S_{0P}}$. Choosing S_{0P} as 0.0025 results in $\left(\frac{dy}{dx}\right)_P$ equal to 0.05, which is in the range of gradually varying flow.

3.5.3 The hydraulic jump

The channel considered here is of mild and constant slope. At some location along the channel, the flow is specified to be gradually varying and supercritical. Such a situation could arise, for example, downstream of a sluice gate, but the source of such flow is irrelevant to the present discussion. Of interest here is what happens to the flow downstream of the given starting point, which can be viewed as an artificial control point (type A). The solution for this example appears in Figure 3.7.

The only choice to start the water profile is with M3, which will truncate at the CDL where $\frac{dy}{dx} = \infty$. None of the available profiles are able to take water surface up to the NDL; therefore, a hydraulic jump must be introduced.

A hydraulic jump can be analyzed with the algebraic form of the momentum equation (eq. 2.33). For simplicity, the channel slope θ will be set to zero and the roughness is neglected; thus,

$$\hat{F}_1 = \hat{F}_2. \qquad (3.31)$$

Section 1 is upstream of the jump and section 2 is downstream, as shown in part a of Figure 3.8.

In addition to the trivial solution in which the water depths at sections 1 and 2 are equal, equation (3.31) has solutions with unequal water depths. This is easily seen by examining a plot of $\hat{F}$ vs. water depth y as shown in Figure 3.2 and replotted in part b of Figure 3.8. The unequal solutions y_1 and y_2 are indicated, with y_2 being the greater depth. Also evident is that the flow at y_1 is supercritical and the flow at y_2 is subcritical; in other words, $\boldsymbol{F}_1$ is greater than 1and $\boldsymbol{F}_2$ is less than 1. Thus, a hydraulic jump is a rapid transition from supercritical flow to subcritical flow. Through application of the energy equation (eq. 2.57) at sections 1 and 2, the energy dissipated in the jump can be determined. This is also shown in part b using the $\hat{E}$ vs y plot. [Since the slope of the channel is zero, the specific energy $\hat{E}$ can be used to represent the total energy H_α.] The energy dissipated between sections 1 and 2 is denoted by $\Delta\hat{E}$, and, since the channel is frictionless, $\Delta\hat{E}$ must be the energy dissipated in the jump.

The depths y_1 and y_2 are known as conjugate depths. The depth y_1 is also referred to as the initial depth and y_2 as the sequent depth.

For a channel of rectangular cross-section, where $\bar{y} = \frac{y}{2}$, and also taking $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$, equation (3.31) can be written as

$$\frac{\beta}{\alpha}\boldsymbol{F}_1{}^2 y_1{}^2 + \frac{1}{2} y_1{}^2 = \frac{\beta}{\alpha}\boldsymbol{F}_2{}^2 y_2{}^2 + \frac{1}{2} y_2{}^2. \qquad (3.32)$$

Then the continuity equation

$$\boldsymbol{F}_1 y_1^{\frac{3}{2}} = \boldsymbol{F}_2 y_2^{\frac{3}{2}}\ , \qquad (3.33)$$

which is in terms of $\boldsymbol{F}$ and y, is used to eliminate $\boldsymbol{F}_2$ or $\boldsymbol{F}_1$. Thus,

$$\frac{y_2}{y_1} = \frac{1}{2}\left[\left(1 + 8\frac{\beta}{\alpha}\boldsymbol{F}_1{}^2\right)^{\frac{1}{2}} - 1\right] \qquad (3.34)$$

or

$$\frac{y_1}{y_2} = \frac{1}{2}\left[\left(1 + 8\frac{\beta}{\alpha}\boldsymbol{F}_2{}^2\right)^{\frac{1}{2}} - 1\right], \qquad (3.35)$$

respectively. Knowing Q and either y_1 or y_2, the other depth can be computed directly.

Recalling the ambiguity between α and β in the definition of $\boldsymbol{F}$, if $\boldsymbol{F}$ had been defined using β instead of α, the $\frac{\beta}{\alpha}$ term in equations (3.34 and 3.35) would disappear.

A hydraulic jump has a form and length L_J that depend on $\boldsymbol{F}_1$. These relations have been determined experimentally and are presented in Figures 3.9 and 3.10.

Another solution to equation (3.30) exists in which y_2 is less than y_1, $\boldsymbol{F}_1$ is less than 1,and $\boldsymbol{F}_2$ is greater than 1. However, this "hydraulic drop" is not physically realizable because it generates energy. The existence of this solution and its energy generation feature can be seen by reversing the flow direction in Figure 3.8.

Returning now to the example flow in Figure 3.7, the M3 profile is connected to the NDL by a hydraulic jump. The water depth y_n for the NDL is the sequent depth y_2, and the initial depth y_1 is found from equation (3.31) (or equation 3.35 for a rectangular channel cross-section) for the given Q. Computing this value of y_1 using a channel slope θ equal to zero and neglecting the roughness is generally considered to be accurate enough, although the actual values of slope and roughness could be employed. The jump begins at the point along M3 where the depth equals the initial depth y_1, and it connects to the NDL a distance L_J downstream of where it begins. Normal flow exists from there.

The notations y_i and y_q will also be used to denote the initial and sequent depths of a hydraulic jump.

3.5.4 Channel with slope change from steep to mild

The channel considered here has a slope change that divides the channel into an upstream reach with a steep slope and a downstream reach with a mild slope. The NDL is below the CDL in the upstream reach and above it in the downstream reach.

The solution for this example requires a hydraulic jump, and depending on the relative slopes of the two reaches, the jump can occur in either reach (Figure 3.11). When the jump is in the downstream reach (part a of the figure), the flow in the upstream reach is uniform and that in the downstream reach begins with the M3 profile. The initial depth y_i is along the M3 and the sequent depth y_q is at the NDL of the downstream reach, which is similar to the jump in the previous section. If there are no conjugate depths in the downstream reach, then the jump occurs either across the point of slope change or entirely in the upstream reach. Considering the latter case (part b of the figure), the flow in the downstream reach is uniform, and the flow in the upstream reach consists of an S1 profile connected by a hydraulic jump to uniform flow further upstream. The initial depth y_i is along the NDL of the upstream reach and the sequent depth y_q is on S1; the latter determines the end point of the jump. The sequent depth y_q is found from equation (3.31) (or equation 3.34 for a rectangular channel cross-section). Regardless of where the jump occurs, it will move further upstream if the slope of either the downstream or upstream reach decreases. For both possible jump locations, the control point is at the point of channel slope change on the NDL for the reach where the flow is entirely uniform (no hydraulic jump).

3.5.5 Channel with a downward or upward step

These cases are more complicated because the channel consists of three reaches. A great many water surface profiles can be produced by choosing different slopes for the reaches and by varying the length of the middle reach. There are too many combinations to examine all of them here. The ones that are considered are arranged so that the middle reach produces either a step down or a step up.

Consider first a channel with upstream and downstream reaches of mild slope and a middle reach of steep slope that produces a downward step. The slopes of the upstream and middle reaches are held constant, while the slope of the downstream reach is assigned four different values to produce a range of water surface profiles. Since the slope of the downstream reach is always mild, it will have uniform flow from its start or after a hydraulic jump.

Figure 3.12 shows the sequence of profiles produced by increasing the slope of the downstream reach, as is evident by the relative positions of its NDL and CDL. The procedure to construct a profile throughout the channel is to assume first that the entire downstream reach is uniform flow, which is the same as assuming that a control point exists at the end of the step on the NDL of the downstream reach. With this assumption, the middle reach will be S1. If this S1 encounters the upstream reach above the NDL of the upstream reach, then the upstream reach will be M1 (part a of the figure), and the assumption for the control point is correct. If S1 of the middle reach encounters the upstream reach between the NDL and CDL of the upstream reach, then the upstream reach is M2 (part b), and the assumption for the control point is also correct. If S1 of the middle reach terminates at the CDL before encountering the upstream reach, the upstream reach is still M2 but the M2 terminates at the CDL at the beginning of the step. In this case, the profile continues into the middle reach with S2 , and there is a critical depth control point on the CDL at the beginning of the step. If the S2 on the left side of the middle reach and the S1 on the right side attain conjugate depths (for the assumed jump length L_J), a hydraulic jump exists in the middle reach (part c). The previous normal depth control point at the end of the step on the NDL of the downstream reach is still valid. If the jump does not occur in the middle reach, it either occurs across the point of slope change at the end of the step or forms entirely in the downstream reach. Considering the latter case, the S2 extends to the end of the step where the profile switches to M3 in the downstream reach. The M3 continues until it and the NDL of the downstream reach attain conjugate depths. A hydraulic jump begins at the initial depth on the M3 and extends up to the NDL of the downstream reach (part d). In this case the only control point is the critical depth one at the start of the step.

If a hydraulic jump begins or ends on an NDL, then determining the conjugate depths is simple because only one of the two conjugate depths will be unknown. However, in part c of Figure 3.12 where the jump occurs in the middle reach, the initial depth y_i is on an S2 curve and the sequent depth y_q is on an S1 curve, and so neither is known. Depths y_i and y_q must be found that satisfy equation (3.31) (or equations 3.34 and 3.35 for a rectangular cross-section) as well as have the point where S2 has depth y_i be a distance L_J upstream of the point where S1 has depth y_q.

For an upward step, consider the upstream and downstream channel slopes to be mild and the middle reach to be adverse. The assumption of uniform flow in the downstream reach, which is the same as assuming that a control point exists at the end of the step on the NDL of the downstream reach, leads to A2 in the middle reach. If this A2 encounters the upstream reach above the NDL of the upstream reach, the profile in the upstream reach is M1. If the encounter is below the NDL, the profile is M2. See Figure 3.13. Since one of these profiles has to be valid, the assumed control point is correct.

The final example of a step in a channel is also an upward step case, and it has the interesting feature of a non-unique solution. The channel has upstream and downstream reaches of steep slope and a middle channel of adverse slope. One solution has normal flow in the upstream reach, A3 in the middle reach, and S2 in the downstream reach, as shown in part a of Figure 3.14. The A3 profile in the middle reach encounters the downstream reach below the elevation of the CDL. For this solution, the control point is on the NDL of the upstream reach at the start of the step. For the second solution, the S2 in the downstream reach is shifted so that it starts from the CDL at the end of the step; this is the new control point. The profile in the middle reach is now A2, which will produce a valid solution as long as a hydraulic jump can form somewhere upstream. In the middle reach the hydraulic jump requires the presence of conjugate depths y_i and y_q along A2 and A3 to form there, or to form in the upper reach it requires conjugate depths y_i and y_q along the NDL and S1 (the extension of A2 from the middle reach). This latter possibility is shown in part b of the figure. This simultaneous occurrence of the two solutions happens within certain ranges of values of the slopes of the reaches and the length of the middle reach. Apparently, the non-uniqueness arises from the "looseness" of the downstream condition, which allows the NDL in the downstream reach to be approached asymptotically.

3.5.6 Channel with a sluice gate

Of interest here is a channel of mild slope containing a vertical sliding sluice gate (Figure 3.15). The open gate is positioned low enough so that the flow passing under the gate is supercritical. After a short length L_C of channel over which the water depth contracts to a minimum value (at section 4), the flow can be assumed to be gradually varying. This minimum water depth is about 0.6 of the gate opening, and the length L_C is about equal to the gate opening; these values are based on experimental results. The water depth at section 4 is denoted by y_4. Since that water depth is known, an artificial control exists at section 4.

Downstream of section 4, the flow consists of an M3 profile followed by a hydraulic jump that takes the water surface up to the NDL; see part a of Figure 3.15. Upstream of the gate (section 3), the usual procedure to determine the flow profile is to apply the algebraic energy equation (eq. 2.57) between sections 3 and 4, placing section 3 close to the gate to minimize the distance between the two sections and justify neglecting the energy dissipation term (even though the rapidly varying flow in the vicinity of the gate will involve some energy dissipation). However, placing section 3 close to the gate is problematic for two reasons: first, there is a significant downward component of the velocity there, which invalidates the hydrostatic pressure assumption, and, second, the velocity distribution over the cross section is affected by the presence of the gate, making α difficult to evaluate. Figure 3.15 shows section 3 located some distance from the gate where the influence of the gate opening is small; however, the frictional energy loss may now be important enough to include.

Application of the algebraic energy equation gives a water depth y_3 at section 3 above the NDL, so the profile is M1, which means the gate is causing the water to back up behind it. Based on the above discussion, the M1 profile in the vicinity of the gate should be viewed as very approximate.

Now consider reducing the slope of the channel on the downstream side of the gate. The NDL moves to a higher elevation and the initial depth y_1 for the hydraulic jump decreases. Thus, the jump moves upstream. After some point the flow under the gate becomes submerged on the downstream side (part b of Figure 3.15). To analyze this case where the flow entering the jump is submerged, several assumptions are made: 1.) the jump is assumed to start at section 4; 2.) the entire flow rate Q passes section 4 within the original water depth y_4 for unsubmerged flow; 3.) the pressure distribution at section 4 is hydrostatic throughout the actual water depth y_4' at section 4. These assumptions allow the hydraulic jump to be analyzed with a pressure surcharge at the initial depth. Equation (3.31) in expanded form for sections 4 and 2 becomes

$$\frac{1}{g}\beta_4 Q V_4 + A_4'(y_4' - \bar{y}_4')\cos\theta = \frac{1}{g}\beta_2 Q V_2 + A_2(y_2 - \bar{y}_2)\cos\theta\,, \qquad (3.36)$$

where $V_4 = \frac{Q}{A_4}$ and A_4 is the area of the cross-section corresponding to the water depth y_4; and A_4' and $\bar{y}_4'$ are properties of the cross-section corresponding to the actual water depth y_4'. The right side of equation (3.36) is written for the sequent depth, which corresponds to uniform flow. For a rectangular cross-section with $\beta_4 = \beta_2 = \beta$ and $\alpha_4 = \alpha_2 = \alpha$ for simplicity, equation (3.36) becomes

$$\frac{\beta}{\alpha}\boldsymbol{F}_4{}^2 y_4{}^2 + \frac{1}{2}y_4'^2 = \frac{\beta}{\alpha}\boldsymbol{F}_2{}^2 y_2{}^2 + \frac{1}{2}y_2{}^2. \qquad (3.37)$$

Using the continuity equation

$$\boldsymbol{F}_4 y_4^{\frac{3}{2}} = \boldsymbol{F}_2 y_2^{\frac{3}{2}} \qquad (3.38)$$

to eliminate $\boldsymbol{F}_4$ leads to

$$\frac{y_4'}{y_2} = \left[1 - 2\frac{\beta}{\alpha}\boldsymbol{F}_2{}^2\left(\frac{y_2}{y_4} - 1\right)\right]^{\frac{1}{2}}, \qquad (3.39)$$

from which the actual depth y_4' at section 4 can be found.

Submerged flow on the downstream side of the gate affects the water depth on the upstream side of the gate. The algebraic form of the energy equation between sections 3 and 4 is still applicable, but the kinetic energy term at section 4 should be based on the water depth y_4 without submergence, and the potential energy term at section 4 should be based on the actual depth y_4'. This calculation will determine a depth y_3' at section 3 greater what would occur if no submergence was present downstream of the gate (as also shown in part b of Figure 3.15). The profile upstream of the gate is still M1, but the greater depth means more backing up of the water.

If instead of reducing the slope on the downstream side of the gate, consider instead raising the gate to increase the height of the opening. This has a similar effect of moving the hydraulic jump upstream until, after some point, the flow entering the jump becomes submerged. If the gate is raised still further until the flow passing section 3 is subcritical, equation (3.36) remains valid. Thus, a hydraulic jump can exist when the entering flow is subcritical as long as it is sufficiently submerged. When the gate is raised to the level of the NDL, the gate has no effect, and uniform flow is restored throughout the channel.

3.5.7 Channel with an overfall

In these examples the channel ends abruptly with an overfall. The effect of an overfall is the same as that of a downstream reach of steep slope with its S2 profile. Therefore, for a channel of mild slope ending in an overfall, the solution in the channel is M2 with the point at which $\frac{dy}{dx} = -\infty$ in the profile occurring at the brink on the CDL. For a channel of steep slope, the solution is uniform flow all the way to the brink.

For the channel with mild slope, the rapidly varying flow in the steep water surface profile at the brink makes the M2 solution invalid there. Experimental evidence shows that the water actually flows smoothly over the brink with the critical depth occurring some distance back from the brink and with a smaller water depth occurring at the brink. The distance from the section of critical depth to the brink depends on many factors, but it is typically several times the critical depth y_c. For the channel with steep slope, a similar phenomenon occurs but the decrease in water depth at the brink is smaller.

An approximate determination of the water depth at the brink can be made by applying the algebraic momentum equation at section 1 upstream of the brink where the water pressure is hydrostatic and at section 2 at the brink where the water pressure is assumed to be zero (i.e. atmospheric) through the depth; see Figure 3.16. An assumption is made that the underside of the falling stream is accessible to the atmosphere and so the pressure there is zero. With the simplification that the slope and the friction force can be neglected, equation (3.31) can be used with the pressure term in $\hat{F}_2$ omitted. Considering a channel with a rectangular cross-section, equation (3.32) without the $\frac{1}{2}{y_2}^2$ term and with $\boldsymbol{F}_2$ eliminated by equation (3.33) results in

$$\frac{y_2}{y_1} = \frac{2\dfrac{\beta}{\alpha}{\boldsymbol{F}_1}^2}{1 + 2\dfrac{\beta}{\alpha}{\boldsymbol{F}_1}^2} \quad , \qquad (3.40)$$

where $\beta_1 = \beta_2 = \beta$ and $\alpha_1 = \alpha_2 = \alpha$ for simplicity.

For a channel of mild slope , $y_1 = y_c$ and $\boldsymbol{F}_1 = 1$, so $y_2 = 0.64\, y_c$ if $\frac{\beta}{\alpha}$ is taken as 0.90. Since there is actually some residual pressure at the brink, the depth would be a little greater. Experimental results indicate a depth at the brink of about 0.72 y_c. For a channel of steep slope, $y_1 = y_n$, $\boldsymbol{F}_1 > 1$, and the depth at the brink will be a greater fraction of y_n.

The next example consists of a channel with horizontal slope containing a sluice gate upstream of an overfall; see Figure 3.17. The flow emerging from under the gate is supercritical and initially contracts, reaching a minimum depth at the artificial control point A. Because of the horizontal slope, the water surface profile upstream of the gate is H2 (drawdown). The procedure for determining the depth at some section upstream of the gate where the flow is gradually varying is similar to that discussed in Section 3.5.6.

In part a of Figure 3.17, the water surface profile H3 begins at the control point A and arrives at the brink below the elevation of the CDL. If the distance from the gate to the overfall is short enough, this is a valid solution. As the distance to the overfall increases, the H3 profile will encounter the CDL before arriving at the brink, indicating the presence of a hydraulic jump. This jump connects two conjugate depths, the initial one y_i on H3 and the sequent one y_q on an H2 profile that extends to the brink where it arrives at the elevation of the CDL, as shown in part b of the figure. For this case, an additional control point at critical depth is established at the brink. Further increasing the distance to the overfall will move the jump further upstream until the flow entering the jump becomes submerged. This situation can be analyzed following the procedure of Section 3.5.6. The critical depth control point at the brink remains in effect. The flow modification of Figure 3.16 at the brink of the overfall is applicable to the flows in Figure 3.17 but is not shown.

3.5.8 Channel with a hump

The hump considered here is produced by a reach of adverse slope followed by a reach of steep slope, added to the middle of a channel. The channel has an otherwise mild slope in one case and steep slope in the other. For each case, two heights of the hump are considered. See Figures 3.18 and 3.19.

The solution for the channel of mild slope and a low hump is shown in part a of Figure 3.18. A normal depth control point exists at the end of the hump, and the water surface profiles are M1 for the upstream reach, A2 and S1 for the two hump reaches, and uniform flow for the downstream reach. The water depth over the hump has dropped below where it would be for uniform flow if the hump were not present, which is somewhat counterintuitive. The minimum water depth occurs at the peak of the hump, and it is greater than the critical depth. As the hump increases in height, the water depth at the peak of the hump drops further to the critical depth, which becomes the new control point, replacing the normal depth control point at the end of the hump. This critical depth control point remains in effect for further increases in height of the hump. The water surface profile, as shown in part b of the figure, is M1 in the upstream reach, A2 and S2 for the two hump reaches, and M3 in the downstream reach until a hydraulic jump occurs, then uniform flow after that. The water depth corresponding to the M1 profile upstream of the hump also directly increases as the hump height increases. The flow over a hump high enough to establish the critical depth control point is referred to as choked flow. The hump is said to be choking the flow, and this causes a relatively deep backup upstream of the hump which is required to maintain the flow.

For the channel of overall steep slope, the water rises above the normal depth but stays below the critical depth if the hump is low (part a of Figure 3.19). A normal depth control point exists at the beginning of the hump, and the water surface profiles are uniform flow in the upstream reach, A3 and S2 for the two hump reaches, and S3 for the downstream reach. As the hump height increases, the water depth at the peak of the hump rises to the critical depth, establishing a new control point there. This critical depth control point remains in effect for further increases in height of the hump, and so at the peak of the hump, the water depth does not increase beyond the critical depth. This situation is also referred to as choked flow. Upstream of the control point, a hydraulic jump forms, and part b of the figure shows it occurring in the upstream reach. For this location of the jump, the complete water surface profile consists of normal flow followed by the hydraulic jump and then S1 in the upstream reach, A2 and S2 in the two hump reaches, and S3 in the downstream reach. As the height of the hump increases, the water depth at the beginning of the hump increases and the jump moves further upstream.

3.6 Non-prismatic channels: widening and narrowing reaches

3.6.1 Basic Water Surface Profiles

[Note: In this section, there is need for a term that refers to a downward channel slope whether or not it is steep or mild. So, a downward slope is referred to as a favorable slope. Favorable is the opposite of adverse. In addition, the effect of a non-prismatic feature will sometimes be referred to as being favorable or adverse. The former means that the non-prismatic effect has some similarity to increasing the slope of the channel, while the latter means that the non-prismatic effect has some similarity to decreasing the slope of the channel, including into the negative slope range.]

To understand qualitatively the effect on the water surface profile of a channel’s cross-sectional shape changing along the length of the channel, a procedure similar to that employed for prismatic channels will be employed; that is, construct water surface profiles from a set of basic profiles for channels of constant slope. These basic profiles are different from the ones shown in Figure 3.3 since they must satisfy equation (3.6), which includes the extra term in the numerator to account for the changing shape of the channel cross-section. In addition, there is no uniform flow for a non-prismatic channel; the NDL does not exist. The CDL line does exist, but it is not parallel to the channel bottom as it is for a prismatic channel of constant slope; nor is the CDL straight.

A useful tool for constructing water surface profiles for a non-prismatic channel is the PDL (parallel depth line) introduced in Section 3.5.2, which in the present case is the depth profile that makes the numerator on the right side of equation (3.6) equal to zero, i.e., the solution to

$$S_0 - S_f + S_{np} = 0\,. \qquad (3.41)$$

If this equation is satisfied, then $\frac{dy}{dx} = 0$. Thus, an actual water surface is parallel to the channel bottom where it crosses the PDL, and $\frac{dy}{dx}$ changes sign there. These statements are true as long as the denominator of equation (3.6) is not simultaneously equal to zero, i.e., as long as there is no simultaneous intersection of the water surface profile with both the PDL and the CDL.

Plotting the PDL and CDL on the channel profile allows one to divide the region into sectors of positive $\frac{dy}{dx}$ and negative $\frac{dy}{dx}$, like what is done using the NDL and CDL for a prismatic channel. However, the process of finding a set of basic water surface profiles is much more complex than for a prismatic channel; therefore, several simplifications are made.

The first simplification is to drop the S_f term in equation (3.6), which is equivalent to neglecting channel friction. This is usually too severe of an assumption; however, if it is only applied to a relatively short length of the channel, an acceptable approximation can often be achieved. The idea is to neglect channel friction only over the length of the channel where the changes in cross-sectional shape of the channel take place. With $S_f = 0$ in equation (3.41), a PDL exists only when S_0 and S_{np} differ in sign. This includes two cases: a narrowing channel with a favorable slope and a widening channel with an adverse slope.

The second simplification is to limit the channel to a rectangular cross-section so that the only possible variation in cross-sectional shape is the channel's width B. Thirdly, the width variation is constrained to be linear, so B is a linear function of x. These last two simplifications impart a simple profile to the CDL; y_c is given by equation (3.10) and so y_c varies according to $B^{-\frac{2}{3}}$. The last two simplifications also impart a simple profile to the PDL. Using equation (3.41) with $S_f = 0$ and replacing $\frac{dA}{dx}$(const y) by y$\frac{dB}{dx}$ in the expression for S_{np}, the depth y_p along the PDL is given by

$$y_p = \left|\frac{\alpha Q^2}{B^3 S_0 g}\frac{dB}{dx}\right|^{\frac{1}{2}}\,, \qquad (3.41)$$

where $\frac{dB}{dx}$ is constant. Thus y_p varies as $B^{-\frac{3}{2}}$.

In a channel of varying width, the depths corresponding to the PDL and CDL increase in the direction of decreasing channel width, and decrease in the direction of increasing channel width. The theoretical depth limits for these two directions are infinity and zero, respectively. The PDL and CDL cross at some point, and where the channel width is greater than that at the crossing point, the CDL will be on top. Where the channel width is less than that at the crossing point, the PDL will be on top.

Before presenting the basic water surface profiles for the non-prismatic channel, the basic profiles for a prismatic channel with $S_f = 0$ are shown in Figure 3.20. These provide a useful reference. The differences in Figure 3.20 regarding profile classification compared to Figure 3.3 where channel friction is included are as follows:

1. The NDL has dropped to the channel bottom, so there is no need for mild and critical slope categories or the S3 profile of the steep slope category.
2. Since there is only one category of positive slope, the steep slope category is referred to as favorable and the two profiles S1 and S2 are renamed F1 and F2.
3. There is just one profile for the horizontal slope category – a horizontal water surface, which is referred to as H.

No qualitative changes occur to the adverse slope category.

For the rectangular cross-section of varying width, different sets of profiles occur depending on whether the channel widens in the downstream direction (Figure 3.21) or narrows in the downstream direction (Figure 3.22). For the widening channel, S_{np} is positive, and it is negative for the narrowing channel. All results are for $S_f = 0$. The left boundary on the diagrams in Figure 3.21 signifies the cross-section of zero width where the widening channel begins. It is a singular point for the flow where the water depth goes to infinity, which has no physical relevance. The right boundary on the diagrams in Figure 3.22 signifies the cross-section of zero width where the narrowing reach ends. It is also a singular point with no physical relevance.

For the widening reach (Figure 3.21), S_n being positive has the effect of imparting a favorable tendency to the channel as far as the basic profiles are concerned. Thus, for the channels with favorable and horizontal slopes shown in parts a and b of the figure, the basic profiles are similar to those for the prismatic channel of favorable slope in Figure 3.20. Neither of these cases has a PDL. For the channel adverse slope, for which a PDL does exist, there is again one profile on each side of the CDL (part c of Figure 3.21). However, these two profiles both begin and end at the CDL. Compared to the profiles in Figure 3.20, the upstream part of the top profile in part c of Figure 3.21 resembles F1 and the downstream part resembles A2, while the upstream and downstream parts of the bottom profile resemble F2 and A3, respectively. Therefore, these same labels are assigned in part c. In an actual channel whose water surface profile is being constructed through a widening reach of adverse slope, F1 or F2 would occur if the adverse slope is not too steep and the degree of widening is relatively severe. The favorable tendency of the widening overcomes the adverse slope. If the adverse slope is steep and the degree of widening is not severe, the adverse slope controls and the profile would be A2 or A3. Otherwise, the profile could be a middle portion of or even an entire F1/A2 or F2/A3.

For the narrowing reach (Figure 3.22), $S_n$being negative has the effect of imparting an adverse tendency to the channel as far as the basic profiles are concerned. A PDL only exists for a channel of favorable slope as shown in part a of the figure. Each of the two profiles begins and ends at the CDL and the labeling F1/A2 and F2 A3 is similar to that used in part c of Figure 3.21, for the same reason. In an actual channel whose water surface profile is being constructed through a narrowing reach of favorable slope, F1 or F2 would occur if the favorable slope is steep and the degree of narrowing is not too severe. The adverse tendency of the narrowing is overcome by the steepness of the favorable slope. If the favorable slope is not too steep and the degree of narrowing is severe, the narrowing controls and the profile would be A2 or A3. Otherwise, the profile could be a middle portion of or even an entire F1/A2 or F2/A3. For the channels with horizontal and adverse slope shown in parts b and c of Figure 3.22, the profiles are A2 and A3.

It is expected that the qualitative nature of the basic water surface profiles shown in Figures 3.21 and 3.22 will be similar for channels of other cross-sectional shape.

3.6.2 Examples

As a first example, a channel with a constriction is examined. The cross-section of the channel is rectangular and the slope is constant. The constriction is produced by adding a narrowing reach followed by a widening reach to the middle of the channel. Channel friction is neglected for the two reaches comprising the constriction, but is otherwise included. The width of the channel is the same upstream and downstream of the constriction. Two slopes for the channel are considered, one mild and one steep, and each slope case is examined with two degrees of constriction. The basic profiles are selected from those shown in Figure 3.3 for the prismatic reaches and from those in Figures 3.21 and 3.22 for the non-prismatic reaches.

Results are shown in Figure 3.23 for the channel of mild slope and in Figure 3.24 for the channel of steep slope. The water surface profiles closely resemble those in Figures 3.18 and 3.19, respectively, for the channel with a hump. The effect of a constriction is similar to a hump because a narrowing cross-section imparts an adverse slope tendency, and a widening cross-section imparts a favorable slope tendency.

In part a of Figure 3.23 for the mild-sloped channel, the profile is everywhere above the CDL, but it is below where it would be for uniform flow in the channel without the constriction. This counterintuitive behavior is similar to that seen for flow over the hump. As the constriction narrows, the water depth at the throat drops further to the critical depth, which becomes the new control point, replacing the normal depth control point at the end of the constriction. This critical depth control point remains in effect for further narrowing of the constriction (part b of the figure), and the flow is choked; the depth at the throat remains at the critical depth . Downstream of the control point, a hydraulic jump forms, and part b of the figure shows it occurring downstream of the constriction.

For the channel of steep slope, the water in the constriction rises above normal depth but stays below the critical depth if the constriction is not too narrow (part a of Figure 3.24). A normal depth control point exists at the beginning of the constriction. As the constriction narrows, the water depth at the throat rises to the critical depth, establishing a new control point there. This critical depth control point remains in effect for further narrowing of the constriction, and so the water depth does not increase beyond the critical depth at the narrowest part of the constriction (choked flow). Upstream of the control point, a hydraulic jump forms, and part b of the figure shows it occurring upstream of the constriction.

When a widening reach or narrowing reach is used by itself to permanently widen or narrow a channel, the water surface profile will be similar to that of a channel with a downward step or an upward step, respectively. Shown in Figure 3.25 is the water surface profile for another example consisting of a channel of constant mild slope with a widening reach. The rate and amount of widening is sufficient to cause a hydraulic jump downstream. The water surface profile is very similar to that shown in part d of Figure 3.12 caused by a downward step in a channel of mild slope upstream and downstream of the step.

3.7 Using The $\hat{E}$ vs. y Diagram to Interpret Water Surface Profiles

The $\hat{E}$ vs. y diagram can provide insight into the effect of variations in channel geometry such as humps and constrictions on the water surface profile, as well as the role of energy. As a simplification, channel friction will be neglected throughout the entire channel. For any reach having an infinite extent, neglecting friction requires that its slope be horizontal in order for a steady solution to exist. For such a channel, the total energy will be constant at every point along the channel. Water surface profiles can be constructed using the basic profiles in Figures 3.20, 3.21 and 3.22.
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The first example is that of a prismatic channel with a hump, with the hump consisting of a reach of adverse slope followed by a reach of favorable slope. If the elevation of the channel bottom is at $z = 0$ upstream and downstream of the hump, then the total energy of the flowing water along these portions of the channel is its specific energy $\hat{E}$, and the total energy along the hump is $\hat{E} + z$. The total energy is conserved. [In this example, z is used as the vertical elevation of the channel bottom.]

Figure 3.26 shows the water surface profile for subcritical flow in the channel along with the $\hat{E}$ vs. y diagram. As the water passes over the hump, its specific energy decreases by the elevation gained. Thus, from section 1 upstream the hump to section 2 at the peak of the hump, $\hat{E}$ decreases by the height of the hump z_2. The $\hat{E}$ vs. y diagram shows that this corresponds to a decrease in water depth from y_1 to y_2, and that this difference is greater than the amount of elevation gain z_2. Therefore, the elevation of the water surface decreases over the hump. The diagram also shows that the water depth returns to the original value after the hump (section 3). It is also clear from the diagram that there is a maximum height of the hump for which a solution exists, which occurs when the water depth over the peak of the hump reduces to the critical depth y_c. For a hump of greater height, the water does not possess enough energy to pass over. However, when channel friction exists, the water backs up as indicated by part b of Figure 3.18, creating a greater depth with enough energy to pass over the higher hump. The higher the hump, the more the water backs up.

The case for supercritical flow in the channel is shown in Figure 3.27. The $\hat{E}$ vs. y diagram shows that the water depth increases over the hump and makes clear that there is a maximum height of the hump for which a solution exists, and the depth of water over the peak of the hump for this case is the critical depth y_c. For a higher hump, the water does not possess enough energy to pass over. When channel friction exists, the mechanism by which the energy increases is through a hydraulic jump upstream of the hump. Although this dissipates some energy, the jump puts the flow into the subcritical realm so the water can back up, as indicated by part b of Figure 3.19. The higher the hump, the further upstream the jump occurs, and the more the water backs up.

The second example is that of a channel with a constriction, the constriction consisting of a narrowing reach followed by a widening reach. Since the entire channel is horizontal, the total energy is represented by the specific energy with no correction. The channel is prismatic upstream and downstream of the constriction and has the same cross-section.

Figure 3.28 shows the water surface profile for subcritical flow in the channel along with the $\hat{E}$ vs. y diagram. A $\hat{E}$ vs. y curve depends on the shape of a cross-section, and this includes the narrowing of a cross-section caused by a constriction, which shifts the $\hat{E}$ vs. y curve to the right. Two $\hat{E}$ vs. y curves are shown in Figure 3.28, one for sections 1 and 3 (which are the same) and one for section 2 at the throat of the constriction. As the water passes through the constriction, $\hat{E}$ stays the same, but the shifting $\hat{E}$ vs. y curve causes the water depth to decrease from y_1 upstream of the constriction to y_2 at the throat and then back up to y_3 (which equals y_1)

downstream of the constriction, as shown by the $\hat{E}$ vs. y diagram in part b. It is also clear from the diagram that there is a maximum narrowness of the constriction for which a solution exists, which occurs when the water depth at the throat reduces to the critical depth y_c. For greater constriction, the water does not possess enough energy to pass through. However, when channel friction exists, the water backs up as indicated by part b of Figure 3.23, creating a greater depth with enough energy to pass through the narrower constriction.

The case for supercritical flow in the channel with the constriction is shown in Figure 3.29. The $\hat{E}$ vs. y diagram shows that the water depth increases within the constriction and makes clear that there is a maximum amount of constriction for which a solution exists, and the depth of water in the throat for this case is the critical depth y_c. For a narrower constriction, the water does not possess enough energy to pass through. When channel friction exists, the mechanism by which the energy increases is through a hydraulic jump upstream of the constriction, which, although this dissipates some energy, puts the flow into the subcritical realm so it can back up, as indicated by part b of Figure 3.24.

Both of the previous examples with the hump and the constriction demonstrate the choking phenomenon from an energy point of view. This interpretation is valid for channels with friction as well.