Chapter 2. Derivation of the Equations of Open Channel Flow

2.1 General Considerations

Of interest is water flowing in a channel with a free surface, which is usually referred to as open channel flow. The channel could be a man-made canal or a natural stream. It could also be a segment of a channel network.

In this chapter, none of the water flowing in the channel leaves the channel and no external water enters the channel. Thus, all of the flow is longitudinal; there is no lateral component. Lateral flow will be considered in a later chapter, as will network flows.

The open channel flow equations are derived from the fundamental 3-dimensional equations of fluid mechanics. These differential equations and related concepts are reviewed first below, followed by a definition of the open channel flow problem.

2.1.1 Fundamental fluid mechanics

Formulations in fluid mechanics are usually based on an Eulerian approach, which uses control volumes. A control volume is a fixed region in space through which the fluid passes, as shown in Figure 2.1. Each location x, y, z is associated with an infitesimal control volume that surrounds it. A function F of x, y, z and t (t for time) associates the property F with the fluid particle that is passing through the infitesimal control volume (located at x, y, z) at the time instant t. The function F is not attached to fluid particles. This concept applies to the components of a velocity vector as well: u, v and w in the x, y, and z directions, respectively. As functions of x, y, z and t, components u, v and w define the velocity of a fluid particle that is passing through the infitesimal control volume (located at x, y, z) at time instant t; see Figure 2.1.

An assumption made here is that the water is incompressible and of uniform density. In such a case, the mass of fluid inside an infitesimal control volume does not change with time as the fluid flows through the control volume. This condition can be expressed in terms of velocity derivatives as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,. \qquad (2.1)$$

This equation is known as the continuity equation. Incompressibility is a good assumption for water flowing in open channels, but density variations can occur due to non-uniform temperature, salt concentration, etc. Density variation is not considered here.

In an Eulerian approach, derivatives of functions such as F with respect to time must distinguish between the time rate of change of F observed at the infitesimal control volume as the particles pass through (denoted by $\frac{\partial F}{\partial t}$, the Eulerian time derivative), or the time rate of change of F for a particular particle as it passes through the control volume (denoted by $\frac{DF}{Dt}$, the material time derivative). These two time derivatives are related by

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} , \qquad (2.2)$$

where the extra terms on the right are the convective part of the material time derivative. Particle acceleration is expressed in a similar way; for the components in the x, y and z directions:

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \qquad (2.3a)$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \qquad (2.3b)$$

$$\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} . \qquad (2.3c)$$

The material derivative for the acceleration vector is appropriate for equations of motion since they require actual accelerations of water particles.

In fluid mechanics, equations of motion are referred to as momentum equations. For the fluid inside the infitesimal control volume at a particular instant of time, the product of its mass and acceleration equals the resultant force acting on the fluid inside the control volume, which is a vector equation. This resultant force is due to the weight of the fluid and to spatial variations in the internal stresses in the fluid. The x, y and z components of the weight vector per unit volume can be expressed as $-\rho g\frac{\partial G}{\partial x}$, $-\rho g\frac{\partial G}{\partial y}$ and $-\rho g\frac{\partial G}{\partial z}$, respectively, where G is the vertical distance above some horizontal reference plane, ρ is the fluid density, and g is the gravitational acceleration (acting vertically downward).. Note that $\rho g G$ is the gravitational potential energy per unit volume. Components of the internal stress tensor acting on the sides of the differential control volume are normal stresses σ_x, σ_y and σ_z and shear stresses τ_{xy}, τ_{yz} and τ_{xz}; see Figure 2.2.

The x, y and z components of the differential momentum equations are

$$\rho\frac{Du}{Dt} = -\rho g\frac{\partial G}{\partial x} + \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \qquad (2.4a)$$

$$\rho\frac{Dv}{Dt} = -\rho g\frac{\partial G}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \qquad (2.4b)$$

$$\rho \frac{Dw}{Dt} = -\rho g \frac{\partial G}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} \ . \qquad (2.4c)$$

In the order listed, these are equations of motion in the x, y and z directions, respectively, for a water particle at some instant of time t. Each group of three terms containing derivatives of σ and τ is the resultant force due to spatial variations in these internal stresses. Equation (2.4) is written in per-unit-volume form.

Constitutive equations for fluids involve the deviatoric stresses, those that exist in addition to the pressure: the shear stresses τ_{xy}, τ_{yz} and τ_{xz}, plus σ_x', σ_y' and σ_z' where

$$\sigma_x' = \sigma_x - p; \quad \sigma_y' = \sigma_y - p; \quad \sigma_z' = \sigma_z - p \qquad (2.5)$$

and where the pressure p is the average of the normal stresses:

$$p = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \ . \qquad (2.6)$$

Through a viscosity tensor, these six deviatoric stresses are related to the six strain rates: the normal strain rates $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$ and $\frac{\partial w}{\partial z}$ and the shear strain rates $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$ and $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$. The constitutive relations will not be given here because they are not needed in the derivation of the open channel flow equations. Instead, it will just be mentioned that any product involving a deviatoric stress component and the corresponding strain rate, such as $\sigma_x' \frac{\partial u}{\partial x}$ or $\tau_{yz}(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$, represents a rate of energy dissipation.

Stresses acting on a domain of fluid at its boundary are referred to as tractions; the x, y and z components of the traction vector are denoted by t_x, t_y and t_z. At a point on a boundary, with a tangent plane whose outward normal direction is denoted by the unit vector $\vec{n}$, the tractions are related to the internal stresses by the following expressions:

$$t_x = n_x\sigma_x + n_y\tau_{xy} + n_z\tau_{xz} \qquad (2.7a)$$

$$t_y = n_x\tau_{xy} + n_y\sigma_y + n_z\tau_{yz} \qquad (2.7b)$$

$$t_z = n_x\tau_{xz} + n_y\tau_{yz} + n_z\sigma_z \ , \qquad (2.7c)$$

where n_x, n_y and n_z are the direction cosines of $\vec{n}$.

Solution of the equations of fluid mechanics involves satisfying boundary conditions. As an example of a boundary condition, the velocity component normal to an impervious boundary would be specified to be zero. For a viscous fluid, the tangential component would be zero as well, which is a no slip condition. Tractions from equation (2.7) can also be specified as boundary conditions. For example, neglecting the effects of air currents, the tractions acting on a free surface would be zero. Non-zero values of velocity and traction can also be employed.

Depending on the value Reynolds number, a fluid in motion can develop flow structures at very small length scales, a phenomenon called turbulence. A simple and approximate way to account for turbulence, which is sufficient in many applications, is to interpret u, v and w as averages of the velocity components over time at a time scale just large enough to smooth out the turbulent fluctuations. The same is done for the elements of the stress tensor and boundary traction vector. Values used for viscosity are equivalent values that account for overall energy dissipation, including that due to turbulence.

2.1.2 Open Channel Flow

Whereas control volumes of infitesimal size are useful for developing differential relations, a finite-sized control volume forms the starting point for deriving the equations of open channel flow. Such a control volume is depicted in Figure 2.3, and consists of the region of the channel located between the fixed cross-sections 1 and 2 that is occupied by water. This region, denoted by Ω, is bounded by four surfaces: the free surface A_s, the floor and sides of the channel A_b, and the two cross-sections A_1 and A_2. The free surface boundary can move, so in this sense, the control volume is not a region fixed in space. The symbols Ω, A_s, A_b, A_1 and A_2 all denote domains, either volume or surface. An additional domain, that of an interior cross-section, is denoted by A without a subscript. The intersection of an interior cross-section A and the domain A_b is the 1-dimensional line domain L_b. Because of the movable free surface, the extents of all these domains are functions of time.

The domain symbols Ω, A_1, A_2, A and L_b will serve a dual purpose; the actual meaning should be clear by the context. Ω will also denote the volume of the control volume, i.e., the volume of water within the domain Ω. A_1, A_2 and A will also denote the areas of the respective cross-sectional domains. L_b will also denote the length of the intersection of A and A_b (the wetted perimeter of a cross-section).

Figure 2.3 also shows the coordinate system employed and other geometrical variables. The x axis is parallel to the slope of the channel bottom, making an angle θ with horizontal, and θ is taken to be constant. The z axis extends across the channel in the horizontal plane, and y is perpendicular to x and z so that the three axes form a right-handed system. So, if the channel slope is not steep, y will be mostly vertically upward. Cross-sections 1 and 2, located at $x = x_1$ and $x = x_2$, are parallel to the y, z plane, as is the interior cross-sectional domain A. Along the boundaries of Ω, the normal direction is denoted by the unit vector $\vec{n}$, positive being outward, with direction cosines n_x, n_y and n_z. For a prismatic channel, in which the shape of the cross-section does not vary as a function of x, $n_x = 0$ along A_b.

With respect to a horizontal reference plane, the elevation of a point along the x axis is G_0, and the elevation of some fixed point x, y, z is G, given by

$$G = G_0 + y \cos\theta \; . \qquad (2.8)$$

In terms of θ, the derivatives of G in the weight component terms of equation (2.4) are $\frac{\partial G}{\partial x} = -\sin\theta$; $\frac{\partial G}{\partial y} = \cos\theta$; and $\frac{\partial G}{\partial z} = 0$.

The equations of open channel flow are based on an assumption of the velocity distribution over a cross-section being primarily in the x direction. Although the two velocity components v and w exist, they are assumed to be much smaller than u. This leads to a simplified, 1-dimensional flow theory in which the velocity parameter appearing in the open channel flow equations is the average of u over a cross-section, denoted by V and given by

$$V = \frac{1}{A}\int_A u \, dA \; . \qquad (2.9)$$

The channel flow rate Q, with units of volume of water per unit time, is

$$Q = VA \, . \qquad (2.10)$$

Q, V and A are functions of x and t.

As will be shown later, other assumptions lead to the condition that the water pressure p varies hydrostatically in the y direction and is constant in the cross-channel direction z, the latter implying that the water surface elevation y_s in the z direction is constant. Two of these assumptions, that the accelerations $\frac{Dv}{Dt}$ and $\frac{Dw}{Dt}$ are small, are consistent with v and w being small. If v is small (compared to u), then the water surface elevation y_s will vary slowly in the x direction. Such a flow situation is referred to as gradually varying, for which the water pressure can be taken to be hydrostatic. At the other end of the flow spectrum is rapidly varying flow, in which steep water surface gradients exist in the x direction. An example of the latter is the flow in the vicinity of a hydraulic jump (Figure 2.4).

The open channel flow equations can be written either for time varying flow or steady flow. Time differentiation appears in the former but not the latter. The equations can be written for cross-sections 1 and 2 a finite distance $x_2 - x_1$ apart (the finite control volume in Figure 2.3), which will be referred to as the algebraic form. By taking the limit as $x_2 - x_1$ approaches zero, the algebraic form can be turned into a differential form, which contains x differentiation. Thus, the open channel flow equations for steady flow and a finite control volume contain no differentiation, while the differential form written for unsteady flow contains differentiation with respect to both x and t.

In the following sections, the open channel flow equations based on continuity (Section 2.2), momentum (Section 2.3) and energy (Section 2.4) are derived. Since the momentum and energy equations have the same origin, one of them plus the continuity equation provide two independent equations from which the flow in the channel can be determined at every cross-section. The flow is characterized by two independent parameters that are functions of x and t, such as V and y_s for one set or Q and A as another.

Development of the momentum and energy equations requires that various correction coefficients be introduced so that V, as defined by equation (2.9), appears explicitly. A discussion of these coefficients is presented in Section 2.5. The equations also require a roughness coefficient to control the boundary shear resistance in the momentum equations and energy dissipation in the energy equations. This roughness coefficient is developed in Section 2.6.

Application of the algebraic form of the open channel flow equations in practice is hampered by the difficulty of evaluating terms that involve the unknown flow parameters between x_1 and x_2. These parameters can be interpolated based on their values at x_1 and x_2, but to reduce errors, the distance $x_2 - x_1$ should be small, which then becomes more or less equivalent to integrating the differential form numerically. However, the algebraic form can be very important for accommodating short extents of rapidly varying flow in a channel, but only in the specialized application of steady flow. In this case, enough of the difficult-to-evaluate terms drop out to make the application possible. This topic is presented in Section 2.7.

All of the derivations in Sections 2.2, 2.3 and 2.4 take the channel segment within the control volume Ω to be straight, which means constant slope and no curves in its horizontal alignment. Applicability of the open channel flow equations to the more general case of a curved channel of varying slope is discussed in Section 2.8. Finally, a junction is considered in Section 2.9.

2.2 Continuity Equations

Integrating the continuity equation (eq. 2.1), over the finite control volume Ω shown in Figure 2.3 and applying Green's theorem results in

$$\int_{\Omega}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)d\Omega=\underbrace{\int_{A_2} u\,dA_2}_{1}-\underbrace{\int_{A_1} u\,dA_1}_{2}+\underbrace{\int_{A_s} v_{sn}\,dA_s}_{3}+\underbrace{\int_{A_b} v_{bn}\,dA_b}_{4}=0, \qquad (2.11)$$

in which

$$v_n = n_x u + n_y v + n_z w \qquad (2.12)$$

has been used for v_{sn} and v_{bn} over A_s and A_b. v_n is the component of water velocity normal to a boundary, positive outward.

Terms 1 and 2 on the right side of equation (2.11) are the flow rates through sections 2 and 1, respectively, denoted by Q_2 and Q_1. Term 3 is the time rate of change of the volume of water within the control volume, $\frac{d\Omega}{dt}$, since the free surface A_s is the only moving boundary of the control volume. Term 4 is the flow rate out of the control volume through the floor and sides of the channel, which is zero since lateral flow is not being considered. Substituting the above as well as equation (2.1) into equation (2.11) results in

$$\frac{d\Omega}{dt} = Q_1 - Q_2\,, \qquad (2.13)$$

which is the algebraic form of the continuity equation for unsteady flow. This equation basically states that the rate of increase (or decrease) of the volume of water within the control volume equals the flow rate of water into or (or out of) the control volume through sections 1 and 2.

The differential form of the continuity equation is obtained by taking the limit as sections 1 and 2 become closer together. Thus, $\frac{d\Omega}{dt}$ in equation (2.13) is replaced by $\frac{\partial A}{\partial t}dx$ and $Q_1 - Q_2$ is replaced by $-\frac{\partial Q}{\partial x}dx$, where A and Q are functions of x and t. Substitution into equation (2.13) and division by dx results in

$$\frac{\partial A}{\partial t} = -\frac{\partial Q}{\partial x}\,, \qquad (2.14)$$

which is the differential form of the continuity equation for unsteady flow.

Equations (2.13) and (2.14) can be specialized for steady flow by dropping the terms with time derivatives. Thus,

$$Q_1 = Q_2 \qquad (2.15)$$

for the algebraic form and

$$\frac{dQ}{dx} = 0 \qquad (2.16)$$

for the differential form. Both equations indicate that Q is constant along the channel.

2.3 Momentum Equations

The first step in developing the algebraic form of the momentum equation for open channel flow is to integrate the x-direction momentum equation (eq. 2.4a) over the domain Ω of the control volume (Figure 2.3). Results of integrating individual terms or groups of terms are as follows:

$$\int_{\Omega} \frac{\partial u}{\partial t} d\Omega = \underbrace{\frac{\partial}{\partial t}\int_{\Omega} u\, d\Omega}_{1} - \int_{A_s} u\, v_{sn} dA_s \qquad (2.17)$$

$$\int_{\Omega} \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) d\Omega =$$

$$\underbrace{\int_{A_2} u^2 dA_2}_{2} - \underbrace{\int_{A_1} u^2 dA_1}_{3} + \int_{A_s} u\, v_{sn} dA_s + \underbrace{\int_{A_b} u\, v_{bn} dA_b}_{4} \qquad (2.18)$$

$$\int_{\Omega} \rho g \sin\theta \, d\Omega = \underbrace{\rho g \Omega \sin\theta}_{5} \qquad (2.19)$$

$$\int_{\Omega} \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) d\Omega = \underbrace{\int_{A_2} t_x dA_2}_{6} + \underbrace{\int_{A_1} t_x dA_1}_{7} + \underbrace{\int_{A_b} t_x dA_b}_{8}, \qquad (2.20)$$

where the Leibniz integration rule is used in equation (2.17), and Green's theorem is used in equations (2.18) and (2.20). Term 4 is zero in the absence of lateral flow. Term 5 is the component of the water weight that acts along the channel slope θ. In Terms 6, 7 and 8, t_x is the x component of the traction vector on the boundaries of Ω; the free surface is assumed to be traction free. The traction t_x is related to the internal stresses by equation (2.7a). Since the normal direction is parallel to x on A_1 andA_2, $t_x = \sigma_X$ on A_2 $(n_x = 1)$ and $t_x =- \sigma_X$ on A_1 $(n_x =- 1)$. For a prismatic channel, $t_x = n_y \tau_{xy} + n_z \tau_{xz}$ on A_b since $n_x = 0$, which is entirely a shear traction.

The numbered terms in equations (2.17) and (2.18) can be expressed as:

$$\text{Term 1} = \frac{\partial}{\partial t} \int_{x_1}^{x_2} Q \, dx \qquad (2.21a)$$

$$\text{Term 2} = \beta_1 Q_1 V_1 \qquad (2.21b)$$

$$\text{Term 3} = \beta_2 Q_2 V_2 \qquad (2.21c)$$

$$\text{Term 4} = 0 \qquad (2.21d)$$

where

$$\beta = \frac{1}{V^2 A} \int_A u^2 dA \,. \qquad (2.22)$$

The coefficient β is a correction factor so that Terms 2 and 3 can be written in terms of the section velocity V; see Section 2.5 for a discussion of β. Terms 6, 7 and 8 on the right side of equation (2.20) are the x-direction forces acting on the water occupying the control volume that are exerted by the water outside section 1, by the water outside section 2, and by the floor and sides of the channel, respectively, and will be denoted by F_2, F_1 and F_b.

Substitution of the above into the momentum equation (eq. 2.4a) after integration, noting that the two unnumbered integrals on the right side of equations (2.17) and (2.18) cancel, results in the algebraic form of the momentum equation for 1-dimensional open channel flow:

$$\rho \frac{\partial}{\partial t} \int_{x_1}^{x_2} Q \, dx + \rho \beta_2 Q_2 V_2 - \rho \beta_1 Q_1 V_1 = \rho g \Omega \sin\theta + F_2 + F_1 + F_b \,. \qquad (2.23)$$

This equation equates the time rate of change of the momentum of the water occupying the control volume at some time t to the resultant force acting on this volume of water at the same time. All terms represent quantities in the x direction.

In order to develop expressions for F_1, F_2 and F_b, the other two momentum equations (eqs. 2.4b and 2.4c) will be considered after making some further assumptions. These assumptions are: neglect particle accelerations in the y and z directions (set the left sides of equations 2.4b and 2.4c to zero); take σ_x, σ_y and σ_z equal to the water pressure p (i.e., $\sigma'_x = \sigma'_y = \sigma'_z = 0$); neglect τ_{yz}; and neglect the x variations of τ_{xy} and τ_{xz}. Equation (2.4b) thus simplifies to

$$\frac{\partial p}{\partial y} = -\rho g \cos\theta\,, \qquad (2.24)$$

which corresponds to hydrostatic pressure. Integration gives

$$p = \rho g\,(y_s - y)\cos\theta\,. \qquad (2.25)$$

using $p = 0$ at $y = y_s$. Equation (2.4c) simplifies to

$$\frac{\partial p}{\partial z} = 0\,; \qquad (2.26)$$

thus, p does not vary with z, implying that the free-surface elevation is constant over the cross-section, i.e., y_s is a function only of x and t.

Using various expressions and definitions above,

$$F_2 = -\rho g \int_{A_2} (y_{s2} - y)\cos\theta\, dA_2 = -\rho g A_2 (y_{s2} - \bar{y}_2)\cos\theta \qquad (2.27)$$

$$F_1 = \rho g \int_{A_1} (y_{s1} - y)\cos\theta\, dA_1 = \rho g A_1 (y_{s1} - \bar{y}_1)\cos\theta\,, \qquad (2.28)$$

which are simple hydrostatic forces, and where $\bar{y}$ is y at the centroid of the cross-section. Also,

$$F_b = -\rho g \int_{A_b} n_x (y_s - y)\cos\theta\, dA_b + F_{shear}\,, \qquad (2.29)$$

which has been decomposed into a part that reacts against the water pressure and a part due to shear tractions.

Making the substitutions into equation (2.23) results in

$$\begin{aligned}\rho\frac{\partial}{\partial t}\int_{x_1}^{x_2} Q\, dx + \rho\beta_2 Q_2 V_2 - \rho\beta_1 Q_1 V_1 \\ = \rho g \Omega \sin\theta - \rho g A_2 (y_{s2} - \bar{y}_2)\cos\theta + \rho g A_1 (y_{s1} - \bar{y}_1)\cos\theta \\ - \rho g \int_{A_b} n_x (y_s - y)\cos\theta\, dA_b + F_{shear}\,, \qquad (2.30)\end{aligned}$$

which is the algebraic form of the momentum equation incorporating the hydrostatic pressure condition.

As mentioned in Section 2.1.2, use of the algebraic form is hampered by the difficulty of evaluating terms that involve the unknown flow parameters between x_1 and x_2, which for equation (2.30) includes the two integral terms as well as $\rho g \Omega \sin\theta$ and F_{shear}. Application is usually limited to cases where the two integrals drop out, which requires steady flow in a prismatic channel. The limitation to a prismatic channel allows the pressure part of F_b to be dropped (the second to last term on the right side of equation 2.30). The result is

$$\rho\beta_2 Q V_2 - \rho\beta_1 Q V_1 \\ = \rho g \Omega \sin\theta - \rho g A_2 (y_{s2} - \bar{y}_2)\cos\theta + \rho g A_1 (y_{s1} - \bar{y}_1)\cos\theta + F_{shear}\,, \qquad (2.31)$$

which applies for steady flow in a prismatic channel without lateral flow.

Equation (2.31) can be written compactly using a quantity defined over a cross-section called specific force, defined as

$$\hat{F} = \frac{1}{g}\beta Q V + A(y_s - \bar{y})\cos\theta\,. \qquad (2.32)$$

After dividing through by ρg and making the substitution, equation (2.31) takes the form

$$\hat{F}_2 - \hat{F}_1 = \Omega \sin\theta + \frac{F_{shear}}{\rho g}\,. \qquad (2.33)$$

For the differential form of the momentum equation, take the limit of equation (2.30) as cross-sections 1 and 2 become closer together. Thus,

$$\rho\frac{\partial Q}{\partial t} + \rho\frac{\partial}{\partial x}(\beta Q V) \\ = \rho g A \sin\theta - \rho g \frac{\partial}{\partial x}\int_A (y_s - y)\cos\theta\, dA - \rho g \int_{L_b} \frac{n_x}{\sqrt{n_y^2 + n_z^2}}(y_s - y)\cos\theta\, dL_b \\ + f_{shear}\,, \qquad (2.34)$$

where f_{shear} is F_{shear} per unit length of the channel. Application of the Leibniz integration rule to the second term on the right side of equation (2.34) results in two terms; one cancels the third term on the right side above and the other appears as the last term in the result below:

$$\rho\frac{\partial Q}{\partial t} + \rho\frac{\partial}{\partial x}(\beta Q V) = \rho g A \sin\theta + f_{shear} - \rho g A \frac{\partial y_s}{\partial x}\cos\theta\,. \qquad (2.35)$$

This is the differential momentum equation for 1-dimensional open channel flow. Hydrostatic pressure is assumed along the channel. A discussion of the f_{shear} term appears in Section 2.6.

A steady version of equation (2.35) can be obtained by omitting the first term

$$\rho\frac{d(\beta Q V)}{dx} = \rho g A \sin\theta + f_{shear} - \rho g A \frac{d y_s}{dx}\cos\theta\,. \qquad (2.36)$$

An alternate form of equation (2.35) that does not contain Q can be obtained using equations (2.10) and (2.14) along with some manipulation. The result is

$$\frac{1}{g}\frac{\partial V}{\partial t}+\frac{\partial H_\beta}{\partial x}=\frac{1}{\rho g A}f_{shear}+(\beta-1)\frac{V}{gA}\frac{\partial A}{\partial t}-\frac{V^2}{2g}\frac{\partial \beta}{\partial x}\ , \qquad (2.37)$$

where

$$H_\beta=\beta\frac{V^2}{2g}+y_s\cos\theta+G_0\ . \qquad (2.38)$$

This equation has less physical correspondence than equation (2.36), but it bears some resemblance to the differential energy equation derived in the next section. For steady state flow, equation (2.37) reduces to

$$\frac{dH_\beta}{dx}=\frac{1}{\rho g A}f_{shear}-\frac{V^2}{2g}\frac{d\beta}{dx}\ . \qquad (2.39)$$

2.4 Energy Equations

To develop the algebraic form of the energy equation, multiply the three momentum equations (eqs. 2.4a, b and c) by u, v and w, respectively, and add them together to form a single equation. With substitution from equation (2.3) and the G terms moved to the left side, the result is

$$\begin{aligned} u\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)+v\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)\\ +w\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)+\rho g\left(u\frac{\partial G}{\partial x}+v\frac{\partial G}{\partial y}+w\frac{\partial G}{\partial z}\right)\\ =u\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right)+v\left(\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}\right)\\ +w\left(\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}\right)\ . \qquad (2.40)\end{aligned}$$

The sum of the first three groups of terms on the left side of equation (2.40) can be expressed in terms of the kinetic energy per unit volume defined as

$$T=½\rho U^2, \qquad (2.41)$$

where $U^2=u^2+v^2+w^2$; and U is the amplitude of the water particle velocity vector. As can be verified by substitution, this three group sum is the material time derivative of T:

$$\frac{DT}{Dt}=\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\ . \qquad (2.42)$$

The last group of terms on the left side of equation (2.40) is the material time derivative of the gravitational potential energy per unit volume, $\rho g G$:

$$\rho g\frac{DG}{Dt}=\rho g\frac{\partial G}{\partial t}+\rho g\left(u\frac{\partial G}{\partial x}+v\frac{\partial G}{\partial y}+w\frac{\partial G}{\partial z}\right) \qquad (2.43)$$

since the Eulerian time derivative $\rho g\frac{\partial G}{\partial t}$ is zero. The right side of equation (2.40) represents the rate of work per unit volume done by the internal stresses.

Equation (2.40) is integrated over the control volume Ω. After substituting the material time derivatives of T and $\rho g G$ and applying Green's theorem to the right side, the result is

$$\int_{\Omega} \frac{D(T + pgG)}{Dt} d\Omega = \dot{W}_{ext} - \dot{\Phi}_{\Omega} \,, \qquad (2.44)$$

where

$$\dot{W}_{ext} = \int_{A_{1,2,b}} (ut_x + vt_y + wt_z) dA_{1,2,b} \qquad (2.45)$$

is the rate of work done by the external tractions on the boundaries of the control volume, and

$$\dot{\Phi}_{\Omega} = \int_{\Omega} \left[\frac{\partial u}{\partial x}\sigma_x + \frac{\partial v}{\partial y}\sigma_y + \frac{\partial w}{\partial z}\sigma_z + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\tau_{xy} + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\tau_{yz} + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\tau_{xz}\right] d\Omega \qquad (2.46)$$

is the rate of energy dissipated over the control volume by the internal stresses. The sum $T + pgG$ on the left side of equation (2.44) is the total energy (kinetic plus gravitational potential) per unit volume.

Various integrals making up equations (2.44), (2.45) and (2.46) are evaluated as follows:

$$\int_{\Omega} \frac{\partial T}{\partial t} d\Omega = \underbrace{\frac{\partial}{\partial t}\int_{\Omega} T d\Omega}_{1} - \int_{A_s} v_{sn} T dA_s \qquad (2.47)$$

$$\int_{\Omega} \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right) d\Omega = \underbrace{\int_{A_2} uTdA_2}_{2} - \underbrace{\int_{A_1} uTdA_1}_{3} + \int_{A_s} v_{sn}TdA_s + \underbrace{\int_{A_b} v_{bn}TdA_b}_{4} \qquad (2.48)$$

$$\int_{\Omega} \left(u\frac{\partial G}{\partial x} + v\frac{\partial G}{\partial y} + w\frac{\partial G}{\partial z}\right) d\Omega = \underbrace{\int_{A_2} uGdA_2}_{5} - \underbrace{\int_{A_1} uGdA_1}_{6} + \underbrace{\int_{A_s} v_{sn}GdA_s}_{7} + \underbrace{\int_{A_b} v_{bn}GdA_b}_{8} \qquad (2.49)$$

$$\int_{A_{1,2,b}} (ut_x + vt_y + wt_z) dA_{1,2,b} = \underbrace{\int_{A_2} ut_x dA_2}_{9} + \underbrace{\int_{A_1} ut_x dA_1}_{10} + \underbrace{\int_{A_b} v_{bn}t_{bn} dA_b}_{11} \,, \qquad (2.50)$$

where the Leibniz integration rule is used in equation (2.47) and Green's theorem is used in equations (2.48) and (2.49). Terms 4 and 8 are zero in the absence of lateral flow. In Terms 9 and 10, $vt_y + wt_z$ has been omitted since the v and w velocity components are assumed to be small compared to u, and additionally, t_y and t_z will be small compared to t_x. For Term 11, the velocity and traction components have been reoriented in the normal and tangential directions, and the tangential velocity is zero because of the no-slip condition, leaving the $v_{bn}t_{bn}$ term. Therefore, in the absence of lateral flow, Term 11 is zero.

Evaluations of the numbered terms are as follows:

$$\text{Term 1} = \frac{\rho}{2}\frac{\partial}{\partial t}\int_{x_1}^{x_2} \beta' Q V dx \qquad (2.51a)$$

$$\text{Term 2} = \frac{\rho}{2}\alpha_2 Q_2 V_2^2 \qquad (2.51b)$$

$$\text{Term 3} = \frac{\rho}{2}\alpha_1 Q_1 V_1^2 \qquad (2.51c)$$

$$\text{Term 4} = 0 \qquad (2.51d)$$

$$\text{Term 5} = \rho g \int_{A_2} u(G_0 + y\cos\theta) dA_2 \qquad (2.51e)$$

$$\text{Term 6} = \rho g \int_{A_1} u(G_0 + y\cos\theta) dA_1 \qquad (2.51f)$$

$$\text{Term 7} = \rho g \int_{x_1}^{x_2} \frac{\partial A}{\partial t}(G_0 + y_s \cos\theta)\, dx \qquad (2.51g)$$

$$\text{Term 8} = 0 \qquad (2.51h)$$

$$\text{Term 9} = -\rho g \int_{A_2} u(y_s - y)\cos\theta\, dA_2 \qquad (2.51i)$$

$$\text{Term10} = \rho g \int_{A_1} u(y_s - y)\cos\theta\, dA_1 \qquad (2.51j)$$

$$\text{Term 11} = 0, \qquad (2.51k)$$

where

$$\beta' = \frac{1}{V^2 A}\int_A U^2 dA \qquad (2.52)$$

$$\alpha = \frac{1}{V^3 A}\int_A uU^2 dA \qquad (2.53)$$

are velocity coefficients so that Terms 1, 2 and 3 can be expressed in terms of the average section velocity V. See Section 2.5 for a discussion of β' and α. Hydrostatic pressure from equation (2.25) has been used for t_x in Terms 9 and 10.

Substituting the above into equation (2.44), noting that the two unnumbered integrals on the right side of equations (2.47) and (2.48) cancel, leads to

$$\underbrace{\frac{\rho}{2}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\beta' QV dx}_{1} + \underbrace{\rho g\int_{x_1}^{x_2}\frac{\partial A}{\partial t}(G_0 + y_s \cos\theta)\, dx}_{7} + \underbrace{\frac{\rho}{2}\alpha_2 Q_2 V_2^2}_{2} - \underbrace{\frac{\rho}{2}\alpha_1 Q_1 V_1^2}_{3}$$

$$+\underbrace{\rho g Q_2(G_0 + y_s\cos\theta)}_{5,\,9} - \underbrace{\rho g Q_1(G_0 + y_s\cos\theta)}_{6,\,10} = -\dot{\Phi}_\Omega\,. \qquad (2.54)$$

Introducing the concept of water head and dividing through by ρg, equation (2.54) becomes

$$\frac{1}{2g}\frac{\partial}{\partial t}\int_{x_1}^{x_2}\beta' QV dx + \int_{x_1}^{x_2}\frac{\partial A}{\partial t}(G_0 + y_s\cos\theta)\, dx + Q_2 H_{\alpha 2} - Q_1 H_{\alpha 1} = -\frac{\dot{\Phi}_\Omega}{\rho g}\,, \qquad (2.55)$$

where

$$H_\alpha = \alpha\frac{V^2}{2g} + G_0 + y_s\cos\theta\,. \qquad (2.56)$$

The water head H_α is a measure of the sum of kinetic and potential energy in units of length, i.e., energy per unit weight per unit volume.

Equation (2.55) is the algebraic form of the energy equation for unsteady open-channel flow. As mentioned in Section 2.1.2, use of the algebraic form is difficult due to the terms that involve the unknown flow parameters between x_1 and x_2, which for equation (2.55) includes the two integral terms as well as $-\frac{\dot{\Phi}_\Omega}{\rho g}$. Application is usually limited to cases where the two integrals drop out, which requires steady flow. For this case

$$H_{\alpha 2} - H_{\alpha 1} = -\frac{\dot{\Phi}_\Omega}{\rho g Q}\,. \qquad (2.57)$$

To derive the differential form of the energy equation, the limit of equation (2.55) is taken as x_1 and x_2 become closer together. The result is

$$\frac{1}{2g}\frac{\partial}{\partial t}\beta' QV + \frac{\partial A}{\partial t}(G_0 + y_s\cos\theta) + \frac{\partial}{\partial x}QH_\alpha = -\frac{\dot{\Phi}_A}{\rho g}\,, \qquad (2.58)$$

where $\dot{\Phi}_A$is the rate of energy dissipation per unit length of the channel. An alternate form that \does not contain Q on the left side can be obtained using $Q = VA$ and equation (2.14) along with some manipulation, resulting in

$$\frac{\beta'}{g}\frac{\partial V}{\partial t} - (\alpha - \beta')\frac{V}{2gA}\frac{\partial A}{\partial t} + \frac{V}{2g}\frac{\partial \beta'}{\partial t} + \frac{\partial H_\alpha}{\partial x} = -\frac{\dot{\Phi}_A}{\rho g Q}\,. \qquad (2.59)$$

A steady state version can be obtained by omitting the first three terms of equation (2.59):

$$\frac{dH_\alpha}{dx} = -\frac{\dot{\Phi}_A}{\rho g Q}\,. \qquad (2.60)$$

The steady state equation (2.60) is sometimes written in terms of a quantity called specific energy defined as

$$\hat{E} = \alpha\frac{V^2}{2g} + y_s\cos\theta\,, \qquad (2.61)$$

which is the total energy per unit weight per unit volume with the potential energy measured with respect to the x axis. Equation (2.60) becomes

$$\frac{d\hat{E}}{dx} = S_0 - \frac{\dot{\Phi}_A}{\rho g Q}\,, \qquad (2.62)$$

where $S_0 =-\frac{dG_0}{dx} = \sin\theta$. S_0 is usually referred to as the slope of the channel, but this is strictly true for only small θ. S_0 is positive when the channel elevation drops with increasing x.

2.5 Coefficients β, β' and α

The coefficient β appears in the momentum equations in the term $\rho\beta QV$ (see equation 2.23), which is the rate of x-direction momentum transfer across a cross-section of the channel. If the velocity component u were constant over the cross-section at its average value V, this rate of momentum transfer would be ρQV; thus, β is a correction factor accounting for the variation in u over the cross-section. As β is defined by equation (2.22), it will exceed 1 (or equal 1 for constant u). Since the actual distribution of u will not be known in general, estimates of β based on available information must be used. For uniform, laminar (non-turbulent) flow, a few analytical solutions for u are possible, and these can be used to calculate β using equation (2.22). Two of these solutions are shown in Figure 2.5: one for an infinitely wide cross-section of constant depth and the other for a semicircular cross-section. The resulting values of β are 6/5 and 4/3, respectively. However, actual flows will be turbulent, and as such u will be more nearly constant over the cross-section (dashed lines in Figure 2.5). This means that β will be closer to 1 for turbulent flow than for laminar flow. Realistic ranges for β have been given as 1.03 to 1.07 for man-made cannels and 1.05 to 1.17 for natural streams. In practice, β is often taken as 1, but a higher value such as the range mid-point would seem to be a better choice.

The coefficient β' appears in the energy equations for time varying flow (see equation 2.55) and is defined similarly to β except that the total velocity U is used instead of the x-component u; compare equations (2.22) and (2.52). Under conditions of gradually varying flow, U and u will be similar, and so will β and β'. No information exists that allows β' to be independently estimated, so in practice it is chosen to be equal to β.

The coefficient α appears in the energy equations in the term $\frac{\rho}{2}\alpha QV^2$ (see equation 2.54), which is the rate of kinetic energy transfer across a cross-section of the channel. Similar to the other coefficients, α is a correction factor to account for variations in velocity across the cross-section, in this case, both u and U. Estimates for α must be based on available information as well. For the two uniform, laminar flow solutions shown in Figure 2.5, equation (2.53) gives values for α of 54/35 (infinitely wide cross-section of constant depth) and 2 (semicircular cross-section), which exceed the corresponding β values. These α values will be closer to 1 for turbulent flows, and ranges for α have been given as 1.10 to 1.20 for man-made cannels and 1.15 to 1.50 for natural streams. In practice, α is often taken as 1, but a higher value such as the range midpoint would seem to be a better choice.

2.6 Roughness Parameter

The momentum and energy equations require some parameter that represents the roughness of the floor and sides of actual channels. This is done through the $\frac{1}{\rho g} F_{shear}$ and $\frac{\dot{\Phi}_\Omega}{\rho g}$ terms in the algebraic forms of these equations and through the $\frac{1}{\rho g A} f_{shear}$ and $\frac{\dot{\Phi}_A}{\rho g Q}$ terms in the differential forms.

Consider first the differential forms of the momentum and energy equations for the special case of steady flow. For the energy equation (eq.2.60), the dissipation term $\frac{\dot{\Phi}_A}{\rho g Q}$ represents the negative of the slope of a plot of the head H_α vs. x. Since H_α is associated with energy, the term $\frac{\dot{\Phi}_A}{\rho g Q}$ is referred to as the energy slope S_e; a positive S_e corresponds to a decrease in head H_α with increasing x. The (negative of the) $\frac{1}{\rho g A} f_{shear}$ term in the momentum equation (eq. 2.39) is also associated with energy dissipation, as can be seen by multiplying top and bottom by V: $-\frac{1}{\rho g Q} f_{shear} \cdot V$, so that $-f_{shear} \cdot V$ is comparable to $\dot{\Phi}_A$. The product $-f_{shear} \cdot V$ is the rate of work done by the shear (friction) force through the average water velocity. Note that f_{shear} is positive in the x direction; it's true direction is to oppose the flow. Based on the foregoing, $-\frac{1}{\rho g A} f_{shear}$ is referred to as the friction slope S_f. Thus, S_f and S_e are defined as

$$S_f = -\frac{1}{\rho g A} f_{shear} \qquad (2.63)$$

and

$$S_e = \frac{\dot{\Phi}_A}{\rho g Q} \, . \qquad (2.64)$$

Substitution into equations (2.39) and (2.60) leads to

$$\frac{dH_\beta}{dx} = -S_f - \frac{V^2}{2g}\frac{d\beta}{dx} \qquad (2.39')$$

for the momentum equation and

$$\frac{dH_\alpha}{dx} = -S_e \qquad (2.60')$$

for the energy equation. In terms of specific energy, the energy equation is

$$\frac{d\hat{E}}{dx} = S_0 - S_e. \qquad (2.62')$$

The last three equations apply for steady flow.

A further restriction from steady flow is uniform flow, also referred to as normal flow, in which no flow quantity varies in the x direction. This implies that the channel is prismatic. Equations (2.39') and (2.60'/2.62') reduce to $S_0 = S_f$ and $S_0 = S_e$, respectively, so

$$S_0 = S_f = S_e \qquad (2.65)$$

for uniform flow. The relation $S_0 = S_f$ is basically a statement that the weight force of the water in the x direction is balanced by the shearing force along the floor and sides of the channel.

Assuming uniform flow conditions, the shear force f_{shear} can be expressed as

$$f_{shear} = -\Psi(V, R, \xi, \rho, \mu, k, g)\ , \qquad (2.66)$$

where R is a length scale associated with the channel cross-section, ξ is a dimensionless factor representing the shape of the cross-section, μ is the absolute viscosity of water, k is a length scale associated with the channel roughness (asperity dimension), and Ψ denotes a general functional form. In particular, R is the hydraulic radius, defined as A/L_b, the cross-sectional area divided by the wetted perimeter. Dimensional analysis leads to an equation of dimensionless ratios:

$$\frac{f_{shear}}{\rho V^2 L_b} = -\Psi\left(\frac{k}{R}, \xi, \frac{VR\rho}{\mu}, \frac{V}{\sqrt{gR}}\right)\ , \qquad (2.67)$$

where the second independent variable is Reynolds number **R** and the third independent variable is related to the Froude number **F**. Note that $\frac{f_{shear}}{L_b}$ is the average shear stress over A_b.

To estimate the value of **R** for typical open channel flow, assume V as a few meters per second and R as a few meters, and take the kinematic viscosity of water as $\nu = \frac{\mu}{\rho} = 10^{-6}\ m^2/sec$. This gives an **R** on the order of 10^7. Based on investigations into pressurized pipe flow, such a value for **R** should correspond to open channel flow well beyond the transition to turbulence and in a range where f_{shear}, when normalized as in equation (2.67), is independent of **R**. These investigations also suggest that dependence on the shape factor ξ can be neglected. The effect of the term related to the Froude number is poorly understood but is not felt to be major, so it is neglected. This leaves the normalized f_{shear} dependent only on the roughness parameter $\frac{k}{R}$:

$$\frac{f_{shear}}{\rho V^2 L_b} = -\Psi\left(\frac{k}{R}\right)\ . \qquad (2.68)$$

Equation (2.68) is used to eliminate f_{shear} from the definition of S_f (equation 2.63); S_f is replaced by S_0 according to equation (2.65) and the result is expressed in terms of V as

$$V = C \cdot (RS_0)^{1/2}, \qquad (2.69)$$

where

$$C = \left(\frac{g}{\Psi}\right)^{1/2} \qquad (2.70)$$

is the Chezy roughness coefficient. Investigations into open channel flow on the functional form of Ψ have found that

$$C \propto \left(\frac{k}{R}\right)^{-1/6}. \qquad (2.71)$$

Substitution into equation (2.69) leads to

$$V \propto k^{-1/6} \cdot R^{2/3} \cdot S_0^{1/2}. \qquad (2.72)$$

The roughness length scale k is commonly replaced by two other constants Π and n so that equation (2.72) takes the form

$$V = \frac{\Pi}{n} \cdot R^{2/3} \cdot S_0^{1/2}, \qquad (2.73)$$

where $\Pi = 1$ $m^{1/3}/sec$ for SI units and 1.486 $ft^{1/3}/sec$ for English units, and n is the Manning roughness coefficient (dimensionless). Values for n are obtained from experiments or field measurements; a list of sample values appears in Table 2.1. The rougher the channel, the higher the value for n.

From equation (2.73), the velocity of flow in any channel can be found as long as the flow is uniform. The flow depth must be known, which determines A and L_b, which determines R. Also, given Q, the normal depth can be found through an iterative process in which Q is replaced by VA.

For non-uniform flow, including the unsteady case, the friction factor n has to be incorporated into the momentum and energy equations of sections 2.3 and 2.4. For the differential forms, an assumption is made that the terms $\frac{1}{\rho g A} f_{shear}$ and $\frac{\dot{\Phi}_A}{\rho g Q}$ at any cross section of the channel are the same as if the flow there were uniform at the same flow parameters of the cross section. Therefore, these terms are replaced by $-S_f$ and S_f, respectively, where S_f is expressed as a nonlinear function of the flow parameters V and R using equation (2.73) with S_f replacing S_0:

$$S_f = V^2 \cdot \frac{n^2}{\Pi^2} \cdot R^{-4/3}. \qquad (2.74)$$

S_e could be used instead of S_f, but convention is to use S_f, so equations (2.60') and (2.62') are now written as

$$\frac{dH_\alpha}{dx} = -S_f \qquad (2.60'')$$

and

$$\frac{d\hat{E}}{dx} = S_0 - S_f\,, \qquad (2.62'')$$

which are steady state forms of the energy equation. Substitutions into the other equations is straight forward.

The algebraic forms of the momentum and energy equations contain the terms $\frac{1}{\rho g}F_{shear}$ and $\frac{\dot{\Phi}_\Omega}{\rho g}$, which can be expressed as

$$\frac{F_{shear}}{\rho g} = \frac{1}{\rho g}\int_{x_1}^{x_2} f_{shear}\,dx \qquad (2.75)$$

and

$$\frac{\dot{\Phi}_\Omega}{\rho g} = \frac{1}{\rho g}\int_{x_1}^{x_2} \dot{\Phi}_A\,dx\ . \qquad (2.76)$$

Substituting for the integrands from equations (2.63) and (2.64), using S_f for the latter, gives

$$\frac{F_{shear}}{\rho g} = -\int_{x_1}^{x_2} AS_f\,dx \qquad (2.77)$$

$$\frac{\dot{\Phi}_\Omega}{\rho g} = \int_{x_1}^{x_2} QS_f\,dx\,, \qquad (2.78)$$

which can be inserted directly into the algebraic forms of the momentum and energy equations from Section 2.3 and 2.4. S_f is again given by equation (2.74).

2.7 Rapidly Varying Flow

All of the continuity equations derived in Section 2.2 apply both to gradually varying flow and to rapidly varying flow since the hydrostatic pressure condition was never imposed during the derivations. However, the differential forms of the momentum and energy equations (equations 2.35/2.37 and 2.58/2.59 and the steady state versions) are only valid in regions where the flow is gradually varying since the hydrostatic pressure condition was incorporated in their derivations.

Consider possible application of the algebraic form of the general momentum and energy equations (eqs.2.30 and 2.55, respectively) to rapidly varying flow. These equations contain many terms that either incorporate the hydrostatic pressure condition or would be impossible to evaluate for rapidly varying flow. So, consider instead equations (2.33) and (2.57) which are written for steady flow and, for the momentum equation, a prismatic channel as well. The hydrostatic pressure condition only has to hold in the vicinity of sections 1 and 2, so the occurrence of rapidly varying flow between these two cross-sections does not rule out the use of these two special-case equations. The weight term $\Omega \sin\theta$ and the shear term $\frac{F_{shear}}{\rho g}$ in the momentum equation (eq. 2.33) and the dissipation term $\frac{\dot{\Phi}_\Omega}{\rho g Q}$ in the energy equation (eq. 2.57), however, depend on what is going on inside the control volume between sections 1 and 2, and so must be dealt with.

Approximations for $\frac{F_{shear}}{\rho g}$ and $\frac{\dot{\Phi}_\Omega}{\rho g Q}$ that apply for gradually varying flow were developed in Section 2.6. For the momentum equation, an assumption will be made that the approximation of equation (2.77) (with S_f given by equation 2.74) can still be used for rapidly varying flow, which seems reasonable since rapid variations affect the flow in the interior of the channel more than on the boundary where F_{shear} acts. It may also be true that F_{shear} does not contribute much to equation (2.33), such as for a short channel segment; in which case, this term could be omitted. For the energy equation, the approximation of equation (2.78) (with S_f given by equation 2.74) would have to be augmented by an extra term representing the energy dissipation caused by the rapid variations in the flow. Thus (for steady flow),

$$\frac{\dot{\Phi}_\Omega}{\rho g Q} = \int_{x_1}^{x_2} S_f \, dx + H_L \,, \qquad (2.78')$$

where the dissipation term H_L is in terms of head loss. For many situations that produce rapidly varying flow, such as a rapid change in cross-section of the channel, expressions using coefficients based on experimental data are available for the extra head loss term.

Substituting equations (2.77) and (2.78') into equations (2.33) and (2.57), respectively, yields the forms for the momentum and energy equations that can be applied for rapidly varying flow within the control volume:

$$\hat{F}_2 - \hat{F}_1 = \Omega \sin\theta - \int_{x_1}^{x_2} A S_f \, dx \qquad (2.33')$$

for the momentum equation and

$$H_{\alpha 2} - H_{\alpha 1} = -\int_{x_1}^{x_2} S_f \, dx - H_L \qquad (2.57')$$

for the energy equation. Equations (2.33') and (2.57') assume that hydrostatic pressure conditions exist at sections 1 and 2, steady flow and, for the momentum equation, a prismatic channel as well. In practice, each of the two integral terms in equations (2.33') and (2.57') is approximated by multiplying the average of its values at sections 1 and 2 by the length of the control volume. This procedure can also be used for the $\Omega \sin\theta$ term in equation (2.33'), or if some *a priori* information is known about the flow profile, it can be used to improve the approximation. Of course, if $\theta = 0$, this term drops out.

Often, the momentum and energy equations are applied together to obtain desired information. For a hydraulic jump in a prismatic channel, the flow parameters at sections upstream and downstream of the jump are found by employing the momentum equation, then the energy equation can be used to determine the rate of energy dissipation in the jump. When the flow contains an obstruction such as a bridge pier or sluice gate, the energy equation is used to find the flow parameters upstream and downstream of the obstruction, assuming H_L can be estimated if it is significant. Then the force exerted on the obstruction by the flow can be determined using the momentum equation. Obviously, for such a situation, the channel is not prismatic, so F_b from equation (2.23) must be reintroduced in the form

$$F_b = -F_{ob} + F_{shear} \,, \qquad (2.79)$$

where F_{ob} is the force that the water exerts on the obstruction. ($-F_{ob}$ is the force that the obstruction exerts on the water.) Carrying through with the analysis modifies equation (2.33') to

$$\hat{F}_2 - \hat{F}_1 = \Omega \sin\theta - F_{ob} - \int_{x_1}^{x_2} A S_f \, dx \,. \qquad (2.80)$$

This equation applies for steady flow in a channel with an obstruction, and F_{ob} can be computed once the flow parameters at sections 1 and 2 are determined from the energy equation. The water pressure on the obstruction does not have to be hydrostatic.

2.8 Slope Variation and Curves in Horizontal Alignment

The previous development of the open channel flow equations considered a straight channel, i.e., no changes in slope and no curves in horizontal alignment. However, in practice these same equations are often used for non-straight channels. The x, y and z axes still form a right-angled coordinate system, with the z axis horizontal, but the x axis is allowed to follow a path along the channel that curves in both the horizontal and vertical planes. This topic is the subject of the present discussion.

The first point to be made is that the concept of a straight channel may not always be clear. Consider horizontal alignment. Figure 2.6a shows a channel with an unsymmetrical transition from narrow to wide. Two choices for the x axis are shown: one stays straight and one follows the channel center line. While either could be used, the non-straight alignment has the advantage of the x axis remaining more parallel with the average direction of the water velocity in the channel. In the straight arrangement, the "curving" of the channel is accounted entirely by the cross-sectional shape's dependence on x. In Figure 2.6b, the channel is clearly curved and a straight x axis is not an option.

The concept of a straight channel may also not always be clear with regard to channel slope. Figure 2.7a shows a case where the floor of the channel is uneven, but there is an average slope along which the x axis is directed. In Figure 2.7b, the slope appears to be constant in elevation view, but it actually varies with z as shown in the two cross-sections. But again, the x axis is directed along the average slope. In both of these cases, the channel can be considered "straight" regarding its slope, and the variations in slope are accounted for by the channel cross-sectional shape being a function of x. In Figure 2.7c, there is clearly a transition from one slope to another, and the x axis is redirected in the transition region.

Deviation from straightness in a channel can cause a number of effects that violate assumptions made in the derivation of the open channel equations. Water flowing around a horizontal curve experiences z-component accelerations that can alter the hydrostatic pressure distribution, causing the water surface to vary in the z direction (y_s no longer a function of just x and t). Circulatory currents in the cross-sectional plane can be present, as well as cross waves when the flow is supercritical. Variation of the slope angle θ along the channel produces a y component of acceleration, directly changing the hydrostatic pressure distribution. These effects, which represent errors in the standard solution, are a function of the radii of curvature in the horizontal and vertical planes, the amount of direction change of the channel, and the water velocity. While these effects can be significant in some cases, in many other cases the curves are gentle enough or the water velocity slow enough so that the errors can be ignored.

For a curved channel, the solution will depend somewhat on where the x axis is located within the cross-section because different paths trace out different lengths. The associated error is a function of the radii of curvature in the horizontal and vertical planes and the amount of direction change of the channel. Placing the x axis at the centroid of the water cross-section is the best location to minimize this error, but this is not practical since the y coordinate of the centroid varies with the water depth. The usual practice is to place the x axis on the center plane of the channel at the channel bottom. This is an acceptable choice since it will minimize the part of the error due to curves in the horizontal plane, which is usually the major part.

One potential problem with a curved channel pertains only to the momentum equations. Because these equations are vector equations, their terms must have a consistent direction. However, another form of the momentum equation can be written using angular momentum in which the terms are multiplied by a radial distance, which varies over the cross-section. A rigorous derivation will not be presented here, but an approximation is to multiply the momentum equations by an average radius of curvature, converting the terms to angular form. The radius cancels out, leaving the original equations. Thus, application of the momentum equations to channels with curvature in either the horizontal or vertical plane is not a violation of the momentum principal.

A few of the open channel flow equations do not contain any of the assumptions associated with straightness, and so are equally valid for curved or straight channels. These include three of the continuity equations (eqs. 2.13, 2.15 and 2.16) and the general energy equation (eq.2.44 along with the definitions in eqs. 2.45 and 2.46).

This discussion should help explain why the equations in this chapter are often used for curved channels. However, it should always be remembered that approximations are involved, and that sometimes additional techniques will be required.

2.9 Channel Junctions

Junctions in a channel can be of the merging flow type or of the separating flow type, as shown in Figure 2.8. Some equations can be written for flow through a junction by applying the techniques from Section 2.2, 2.3 and 2.4, resulting in continuity, momentum and energy equations, respectively. For the momentum equation, the presence of the F_b term and the existence of three distinct channel directions mean that this equation is not useful in most situations. Therefore, the presentation here focuses on the continuity and energy equations. The flow is restricted to steady conditions.

As shown in Figure 2.8, the domain Ω includes the water volume between the cross-sections A_1, A_2 and A_3. The boundary A_b includes all other boundaries of the water volume, except the free surface. Thus, the procedures are similar to those of the earlier sections, but with the added cross-sectional domain A_3.

The result for the continuity equation is just the addition of the flow term Q_3 to equation (2.15) as follows:

$$Q_1 \pm Q_3 = Q_2 \,, \qquad (2.81)$$

where the sign in front of Q_3 is positive for merging flow (part a of Figure 2.8) and negative for separating flow (part b). Equation (2.81) is a statement that the rate of water flowing into the junction equals the rate of water flowing out.

For the energy equation, the result is a similarly obvious change to equation (2.57):

$$Q_1 H_{\alpha 1} \pm Q_3 H_{\alpha 3} = Q_2 H_{\alpha 2} + \frac{\dot{\Phi}_J}{\rho g} \,, \qquad (2.82)$$

where the sign choice in front of the $Q_3 H_{\alpha 3}$ term is the same as that mentioned above. Equation (2.82) says that the rate that energy flows into the junction equals the rate at which energy flows out plus the rate that energy is dissipated in the junction. The H_α terms are still given by equation (2.56). The energy dissipation term is the sum of contributions from rapidly varying flow and channel friction in the junction, denoted by J.

In practice, a junction is often short enough so that G_0 and $\cos\theta$ at sections 1, 2 and 3 can be taken to be equal and the energy dissipated by channel friction can be neglected. If, in addition, the water heads $\alpha \frac{V^2}{2g}$ at sections 1, 2 and 3 are small compared to the water depth y_s, which is often the case, then the only terms remaining in equation (2.82) are the flow rates Q and water depths y_s at the three sections and the energy dissipated in the junction by rapidly varying flow. Merging channels dissipate more energy in this way than separating ones, but if this contribution to the equation can also be neglected, then equation (2.82) reduces to the continuity equation when the water depth y_s is the same at all three sections, and so it is satisfied for this condition.

If equal water depths are appropriate for steady flow, it may be possible to make this assumption for unsteady flow as well. Use of equal water depths, when appropriate, greatly simplifies an analysis.