ME19b. SOLUTIONS. Feb. 22 Due Mar. 1, 2007

PROBLEM B16.

For a pipe of radius R (and diameter D, so that $D = 2R$), the friction factor, f, is defined as

$$f = \frac{8\tau_w}{\rho V^2}$$

where V is the average velocity through the pipe,

$$V = \frac{Q}{\pi R^2}$$

where

$$\begin{aligned} Q &= 2\pi \int_0^R \bar{u}(r) r dr \\ &= 2\pi \int_0^R \bar{u}(y)\,(R-y)\,dy \end{aligned}$$

is the volume flow rate. Of note is the fact that the velocity profile is given in y, the distance from the outside of the pipe and the radius r is a measure of the distance from the center out. The average velocity is thus

$$\begin{aligned} V &= \frac{2}{R^2}\int_0^R \bar{u}(y)\,(R-y)\,dy \\ &= \frac{2}{R^2}u_\tau \int_0^R u^*(y^*)\,(R-y)\,dy \\ &= \frac{2}{R^2}u_\tau \int_0^R 8.7\left(\frac{u_\tau y}{\nu}\right)^{\frac{1}{7}}(R-y)\,dy \\ &= \frac{(2)(8.7)}{R^2}u_\tau \left[\frac{7}{8}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} R y^{\frac{8}{7}} - \frac{7}{15}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} y^{\frac{15}{7}}\right]_0^R \\ &= \frac{(2)(8.7)}{R^2}u_\tau \left[\frac{49}{120}\left(\frac{u_\tau}{\nu}\right)^{\frac{1}{7}} R^{\frac{15}{7}}\right] \\ &= \frac{(2)(8.7)(49)}{120}u_\tau \left(\frac{u_\tau R}{\nu}\right)^{\frac{1}{7}} \\ &= 7.105\left(\frac{R}{\nu}\right)^{\frac{1}{7}} u_\tau^{\frac{8}{7}} \end{aligned}$$

where

$$u^* = \frac{\bar{u}}{u_\tau} = 8.7\,(y^*)^{\frac{1}{7}} = 8.7\left(\frac{u_\tau y}{\nu}\right)^{\frac{1}{7}}$$

Substituting $u_\tau = \sqrt{\frac{\tau_w}{\rho}}$ and $f = \frac{8\tau_w}{\rho V^2}$ into the average velocity yields

$$\begin{aligned} V &= 7.105\sqrt{\frac{\tau_w}{\rho}}\left[\left(\sqrt{\frac{\tau_w}{\rho}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \\ &= 7.105\sqrt{\frac{fV^2}{8}}\left[\left(\sqrt{\frac{fV^2}{8}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \\ \frac{1}{7.105} &= \sqrt{\frac{f}{8}}\left[\left(\sqrt{\frac{fV^2}{8}}\right)\frac{R}{\nu}\right]^{\frac{1}{7}} \end{aligned}$$

$$
\begin{aligned}
\frac{\sqrt{8}}{7.105} &= \sqrt{f}\left[\left(\sqrt{\frac{f}{8}}\right)\frac{VR}{\nu}\right]^{\frac{1}{7}} \\
\frac{\sqrt{8}\,(8)^{\frac{1}{14}}}{7.105} &= \sqrt{f}\left[\left(\sqrt{f}\right)\frac{VD}{2\nu}\right]^{\frac{1}{7}} \\
\frac{\sqrt{8}\,(8)^{\frac{1}{14}}\,(2)^{\frac{1}{7}}}{7.105} &= f^{\frac{8}{14}}\left(\frac{VD}{\nu}\right)^{\frac{1}{7}} \\
f^{\frac{4}{7}} &= \frac{\sqrt{8}\,(8)^{\frac{1}{14}}\,(2)^{\frac{1}{7}}}{7.105}\mathrm{Re}^{-\frac{1}{7}} \\
f &\simeq 0.308\ \mathrm{Re}^{-\frac{1}{4}},
\end{aligned}
$$

where

$$
\mathrm{Re} = \frac{VD}{\nu}
$$

For $\mathrm{Re} = 1 \times 10^6$, the equation yields a friction factor for smooth pipes of $f = .00974$ whereas the graph gives a friction factor of $f = 0.0117$ thus the equation under-predicts the friction factor by approximately 17%.

PROBLEM B17.

There are two analytical tools available to find the average velocity in this pipe flow. First, the friction factor gives

$$
\begin{aligned}
f &= \frac{D\left(-\frac{dp}{dx}\right)}{\frac{1}{2}\rho V^2} \\
V_f &= \sqrt{\frac{D\left(-\frac{dp}{dx}\right)}{\frac{1}{2}\rho f}} \\
&= \sqrt{\frac{0.5\ m\left(\frac{1\ kg/m\ s^2}{50\ m}\right)}{\frac{1}{2}(1.2\ kg/m^3)f}} \\
&= \sqrt{\frac{1}{60f}}\ m/s
\end{aligned}
$$

Second, the definition of the Reynolds number yields

$$
\begin{aligned}
Re &= \frac{DV}{\nu} \\
V_{\mathrm{Re}} &= \frac{\mathrm{Re}\ \nu}{D} \\
&= \frac{2.3 \times 10^{-6}\ m^2/s}{0.5\ m}\mathrm{Re}\frac{\mathrm{Re}\ \nu}{D} \\
&= 4.6 \times 10^{-6}\ \mathrm{Re}\ m/s
\end{aligned}
$$

Thus there are two equations and three unknowns (f, Re, $V = V_f = V_{\mathrm{Re}}$). To solve the problem, one must either guess the Reynolds number or the friction factor and use the Moody chart to predict the other value. Using guesses for the Reynolds number and the chart to find the friction factor, one iteration method is demonstrated below.

Iteration	Re	f	V_f (m/s)	V_{Re} (m/s)
1	6×10^4	0.02	0.912	0.276
2	2×10^5	0.0155	1.04	0.92
3	3×10^5	0.014	1.09	1.38
4	2.5×10^5	0.015	1.05	1.15
5	2.4×10^5	0.015	1.05	1.104
6	2.3×10^5	0.0151	1.047	1.058

Therefore,

$$V \simeq 1.05 \text{ m/s}$$

PROBLEM B18.

The velocity profile for a turbulent boundary layer of incompressible fluid on a flat plate (where $U =$ constant) is approximated as

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}}$$

Finding α, an expression for δ.

$$\begin{aligned}
\alpha = \frac{\delta_M}{\delta} &= \int_0^1 \frac{u}{U}\left(1 - \frac{u}{U}\right) \mathrm{d}\left(\frac{y}{\delta}\right) \\
&= \int_0^1 \left[\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right] \mathrm{d}\left(\frac{y}{\delta}\right) \\
&= \left[\frac{7}{8}\left(\frac{y}{\delta}\right)^{\frac{8}{7}} - \frac{7}{9}\left(\frac{y}{\delta}\right)^{\frac{9}{7}}\right|_0^1 \\
&= \frac{7}{8} - 79 \\
&= \frac{7}{72} = 0.0972
\end{aligned}$$

From the K.M.I.E.,

$$\begin{aligned}
\tau_W &= \rho \frac{\mathrm{d}\left(U^2 \delta_M\right)}{\mathrm{d}x} + \rho \delta_D U \frac{\mathrm{d}U}{\mathrm{d}x} \\
&= \rho U^2 \frac{\mathrm{d}\left(\alpha \delta\right)}{\mathrm{d}x} \\
&= \rho U^2 \alpha \frac{\mathrm{d}\delta}{\mathrm{d}x}
\end{aligned}$$

where α and U are constants. If the wall shear stress for this turbulent profile is assumed to be given by $\tau_W = 0.023 \rho U^2 \left(\nu / \delta U\right)^{\frac{1}{4}}$,

$$\begin{aligned}
\tau_W = \rho U^2 \alpha \frac{\mathrm{d}\delta}{\mathrm{d}x} &= 0.023 \rho U^2 \left(\nu / \delta U\right)^{\frac{1}{4}} \\
\delta^{\frac{1}{4}} \mathrm{d}\delta &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} \mathrm{d}x \\
\frac{4}{5}\delta^{\frac{5}{4}} &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x + c
\end{aligned}$$

To evaluate c, evaluate expression at $x = x_0$, $\delta = \delta_0$:

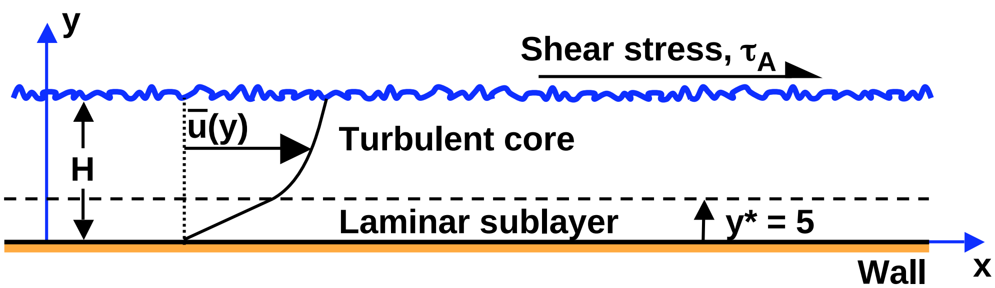
$$c = \frac{4}{5}\delta_0^{\frac{5}{4}} - \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}} x_0$$

$$\begin{aligned}
\frac{4}{5}\left(\delta^{\frac{5}{4}}-\delta_0^{\frac{5}{4}}\right) &= \frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0) \\
\delta^{\frac{5}{4}} &= \frac{5}{4}\left(\frac{0.023}{\alpha}\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0)+\frac{4}{5}\delta_0^{\frac{5}{4}}\right) \\
\delta &= \left[\frac{5}{4}\left(\frac{0.023}{\alpha}\right)\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0)+\delta_0^{\frac{5}{4}}\right]^{\frac{4}{5}} \\
&= \left[0.296\left(\frac{\nu}{U}\right)^{\frac{1}{4}}(x-x_0)+\delta_0^{\frac{5}{4}}\right]^{\frac{4}{5}}
\end{aligned}$$

This solution will be valid for $x \geq x_0$, i.e., within the turbulent boundary layer.

PROBLEM B19.

Either by considering the momentum theorem applied to the control volume or by observing the similarity



to Couette flow, one can conclude that the shear stress, σ_{xy}, is a constant throughout the flow. Hence $\sigma_{xy} = \tau_A$.

- In the *turbulent core*, since the viscous stresses are negligible

$$\begin{aligned}
\sigma_{xy} = \tau_A &= -\rho\overline{u'v'} \\
&= \rho\kappa^2 y^2 \left(\frac{d\overline{u}}{dy}\right)^2
\end{aligned}$$

  by Prandtle's mixing length theory. Hence

$$\begin{aligned}
\kappa y \frac{d\overline{u}}{dy} &= \sqrt{\frac{\tau_A}{\rho}} = u_\tau = \text{constant} \\
\frac{\overline{u}}{u_\tau} &= \frac{1}{\kappa}\ln y + C
\end{aligned}$$

  where C is an integration constant.

- Within the *laminar sublayer*, we now know $u_\tau = \sqrt{\frac{\tau_A}{\rho}}$

$$u^* = \frac{\overline{u}}{u_\tau} = y^* = \frac{u_\tau y}{\nu}$$

  Therefore at the edge of the laminar sublayer

$$\begin{aligned}
y^* &= 5 \\
\frac{\overline{u}}{u_\tau} &= 5 \\
y &= \frac{5\nu}{u_\tau}
\end{aligned}$$

Hence to find the constant, C

$$C = 5 - \frac{1}{\kappa} \ln\left(\frac{5\nu}{u_\tau}\right)$$

Therefore in the turbulent core

$$\begin{aligned}
\frac{\overline{u}}{u_\tau} &= \frac{1}{\kappa} \ln\left(\frac{u_\tau y}{5\nu}\right) + 5 \\
(\overline{u})_{y=H} &= \sqrt{\frac{\tau_A}{\rho}} \left[\frac{1}{\kappa} \ln\left(\frac{H}{5\nu}\sqrt{\frac{\tau_A}{\rho}}\right) + 5\right]
\end{aligned}$$

PROBLEM B20.

The pipe has an internal diameter $d = 5\ cm$, flow rate $Q = 0.05\ m^3/s$ and roughness ranging from $\epsilon = 0.05\ mm$ to $\epsilon = 1.0\ mm$. Water has a kinematic viscosity of $\nu = 10^{-6}\ m^s/s$ The average velocity of flow through the pipe,

$$U = \frac{Q}{\pi(d/2)^2} = \frac{0.05\ m^3/s}{\pi(0.025\ m)^2} = 25.5\ m/s \tag{1}$$

The Reynolds number of the flow is

$$\text{Re} = \frac{dU}{\nu} = \frac{(0.05\ m)(25.5\ m/s)}{10^{-6}\ m^s/s} = 1.27 \times 10^6 \tag{2}$$

The relative roughness of the new pipe

$$\frac{\epsilon}{d} = \frac{5 \times 10^{-5}\ m}{0.05\ m} = 0.001 \tag{3}$$

From the Moody friction factor chart at $\frac{\epsilon}{d} = 0.001$ and $\text{Re} = 1.27 \times 10^6$, the friction factor $f = 0.02$
The relative roughness of the old pipe

$$\frac{\epsilon}{d} = \frac{0.001\ m}{0.05\ m} = 0.02 \tag{4}$$

From the Moody friction factor chart at $\frac{\epsilon}{d} = 0.02$ and $\text{Re} = 1.27 \times 10^6$, the friction factor $f \approx 0.048$
Since the pressure difference is linearly related to the friction factor,

$$\frac{p_{\text{old}}}{p_{\text{new}}} = \frac{0.048}{0.02} = 2.4 \tag{5}$$