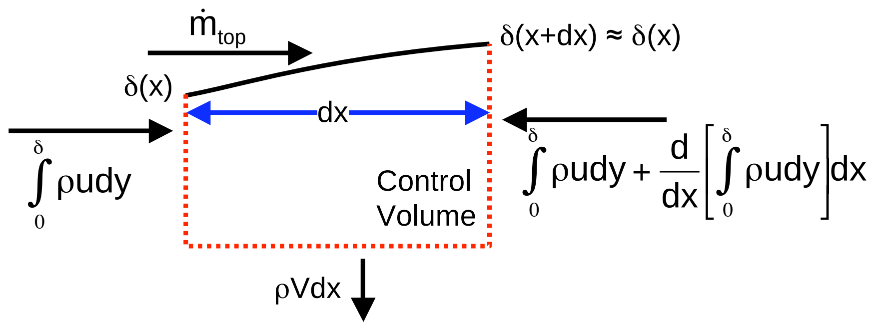
ME19b. **SOLUTIONS.** **Feb. 15** **Due Feb. 22, 2007**

PROBLEM B12.

A laminar boundary layer forms on a porous flat plate. Fluid is removed through the porous plate at a rate of $v(x) = -V$. Using approximate boundary layer methods, assume a similarity solution of the velocity profile $u(y)/U = F(\eta)$. Find the relation between the skin friction coefficient $c_f = f(V, U, \frac{d\delta}{dx}, \alpha)$.



Drawing a control volume from x to $x + dx$ and extending to the edge of the boundary layer ($\delta(x)$ and $\delta(x + dx) \approx \delta(x)$, respectively), continuity states that the mass flow out of the top of the control volume $\dot{m}_{\text{top}}$ is:

$$\dot{m}_{\text{top}} = \frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx + \rho V dx$$

The net change of momentum in the x-direction $\dot{M}_x$ is:

$$\dot{M}_x = \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] dx - [\dot{m}_{top} U]$$

which reduces by substituting $\dot{m}_{top}$ to:

$$\begin{aligned}
\dot{M}_x &= \frac{d}{dx}\left[\int_0^\delta \rho u^2 dy\right] dx - \left[\frac{d}{dx}\left[\int_0^\delta \rho u dy\right] dx + \rho V dx\right] U \\
&= \rho U^2 \left[-\frac{d}{dx}\int_0^\delta \frac{u(y)}{U}\left(1 - \frac{u(y)}{U}\right) dy - \frac{V}{U}\right] dx \\
&= \rho U^2 \left[-\alpha \frac{d\delta}{dx} - \frac{V}{U}\right] dx
\end{aligned}$$

where the constant α:

$$\alpha = \int_0^1 F(\eta)\,(1 - F(\eta))\, d\eta$$

and $\eta = \frac{y}{\delta}$. Note that there is no x-direction momentum flux through the bottom. The force balance in the x-direction is given for the net force on the control volume in the x-direction F_x as:

$$\dot{M}_x = F_x = -\tau_w dx - \frac{dP}{dx}\delta(x) dx$$

However, Bernouilli's equation applied to external flow yields:

$$\frac{dP}{dx} = -\rho U \frac{dU}{dx} = 0$$

By momentum conservation, we know that the change in momentum should be the same as the net force applied on the control volume. Rearranging and simplifying this gives:

$$\tau_w = \rho U^2 \left[\alpha \frac{d\delta}{dx} + \frac{V}{U}\right]$$

The skin friction coefficient C_f is given as:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} = 2\left[\alpha \frac{d\delta}{dx} + \frac{V}{U}\right]$$

PROBLEM B13.

The Karman Momentum Integral Equation is :

$$\frac{\tau_w}{\rho} = \frac{d}{dx}\left(U^2 \delta_M\right) + \delta_D U \frac{dU}{dx}$$

According to the definition of $\beta = \frac{d(u/U)}{d(y/\delta)}$ and $\tau_w = \mu \frac{du}{dy}$, the left hand side of the K.M.I.E. can be expressed as:

$$\frac{\tau_w}{\rho} = \frac{\nu U \beta}{\delta}$$

The definitions of α and γ give:

$$\delta_M = \alpha\delta$$

$$\delta_D = \gamma\delta$$

Substitute into the K.M.I.E.:

$$\begin{aligned}
\frac{\nu U\beta}{\delta} &= \frac{d}{dx}\left(U^2\alpha\delta\right) + \gamma\delta U\frac{dU}{dx} \\
&= \alpha U^2 \frac{dU}{dx} + 2\alpha\delta U\frac{d\delta}{dx} + \gamma\delta U\frac{dU}{dx} \\
\frac{\nu\beta}{\alpha\delta} &= U\frac{d\delta}{dx} + \left(2 + \frac{\gamma}{\alpha}\right)\delta\frac{dU}{dx}
\end{aligned}$$

Substituting $U = Ax^{\frac{1}{2}}$ and $\delta = Cx^m$ yields:

$$\frac{\nu\beta}{\alpha C x^m} = mACx^{m-\frac{1}{2}} + \left(1 + \frac{\gamma}{2\alpha}\right)ACx^{m-\frac{1}{2}}$$

By matching powers of x,

$$m = \frac{1}{4}$$

and

$$C = \left[\frac{4\nu\beta}{\alpha A(5 + \frac{2\gamma}{\alpha})}\right]^{\frac{1}{2}}$$

PROBLEM B14.

1.) Find the distance from the leading edge of the plate to the point where transition to turbulence begins. From the diagram:

$$Re_{\delta^*,\text{crit}} = \frac{U\delta^*_{\text{crit}}}{\nu} \approx 550$$

Using the Blasius solution to find the position along the plate where this critical displacement thickness occurs.

$$\frac{\delta^*}{x}\sqrt{Re_x} = 1.721$$

$$\begin{aligned}
\Rightarrow x_{\text{crit}} &= \left(\frac{\delta^*_{\text{crit}}}{1.721}\right)^2 \frac{U}{\nu} \\
&= \frac{\nu}{U}\left(\frac{Re_{\delta^*,\text{crit}}}{1.721}\right)^2 \\
&= \frac{10^{-6}}{2}\left(\frac{550}{1.721}\right)^2 = 0.0511 \text{ m}
\end{aligned}$$

2.) Find the frequency of the most unstable disturbance.

At $Re_{\delta^*,\text{crit}}$:

$$\frac{2\pi f \nu}{U^2} = 1.70 \times 10^{-4}$$

$$\Rightarrow f = \frac{1.70 \times 10^{-4}(2)^2}{2\pi(10^{-6})} = 108.2 \text{ Hz}$$

PROBLEM B15.

Consider the momentum flux caused by fluctuations only.

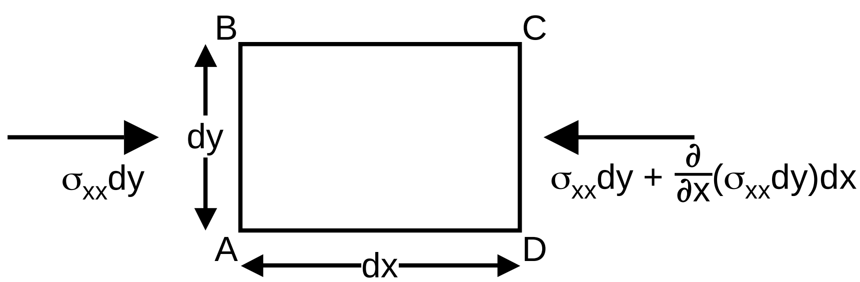


Figure 1: control volume

The fluctuating momentum flux is given by the total mass flux ($\dot{m} = \rho(\overline{u}+u')dy$) multiplied by the fluctuating velocity. Flux of x-momentum in through AB is:

$$\dot{m}u' = \rho u' dy(\overline{u} + u')$$

Flux of x-momentum out through CD is:

$$\overline{m}u' + \frac{\partial}{\partial x}\left[\overline{m}u'\right]dx = \rho u' dy(\overline{u} + u') + \frac{\partial}{\partial x}\left[\rho u' dy\left(\overline{u} + u'\right)\right]dx$$

So, the net flux of x-momentum out through AB and CD is:

$$\rho dx dy \frac{\partial}{\partial x}\left[u'\left(\overline{u} + u'\right)\right]$$

To get the time-average momentum flux caused by this fluctuating momentum flux, the previous equation will be averaged keeping in mind

$$\overline{u'} = 0, \quad \overline{\overline{u}u'} = 0, \quad \overline{u'u'} \neq 0$$

yielding for the average momentum flux caused by the fluctuating velocities

$$dxdy\frac{\partial}{\partial x}(\rho\overline{u'^2})$$

The conceptual forces diagram due to the momentum flux of fluctuations is shown in figure 1. So the net additional normal force is:

$$\begin{aligned}\Sigma F_x &= -\frac{\partial \sigma_{xx}}{\partial x}\\ -dxdy\frac{\partial \sigma_{xx}}{\partial x} &= dxdy\frac{\partial}{\partial x}(\rho\overline{u'^2})\end{aligned}$$

so the net additional normal stress, the Reynolds stress, is

$$\sigma_{xx} = -\rho\overline{u'^2}$$