ME19b. SOLUTIONS. Feb. 8 Due Feb. 15, 2007

PROBLEM B8.

From the Blasius solution, the drag on one side of the flat plate is

$$D = \rho U^2 w \left[(\delta_M)_{\text{trailing edge}} - (\delta_M)_{\text{leading edge}} \right] \tag{1}$$

where w is the width of the plate and δ_M is the momentum thickness of the boundary layer. At the leading edge of the plate, no boundary layer has developed, and $(\delta_M)_{\text{leading edge}} = 0$. The definition of the momentum thickness is

$$\delta_M = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

Using the Blasius boundary layer solution, the momentum thickness evaluates to

$$\delta_M = 0.664 \left(\frac{\nu x}{U}\right)^{\frac{1}{2}} \tag{2}$$

Plugging equation 2 into equation 1 gives

$$D = 0.664 \rho \sqrt{\nu} U^{\frac{3}{2}} \sqrt{L} w$$

which, after plugging in $\rho = 10^3$ kg/m^3, $\nu = 10^{-6}$ m^2/s, $L = 10$ m, and $w = 1$ m, becomes

$$D \simeq \left(2.1 U^{\frac{3}{2}}\right) \text{N}$$

The total power from the eight rowers is

$$P = \frac{1}{2}(8P_i)$$

where the factor of 1/2 comes from the fact that half of the power is uselessly dissipated. Each rower can produce $P_i = 0.1HP$, so that

$$P = 0.4HP = 298.4 \text{ W}$$

The power can be related to the force necessary to move the boat and the boat's speed speed by

$$P = Du = 2.1 U^{\frac{5}{2}} \text{ W}$$

Thus, the boat can reach a top speed of

$$U = 7.26 \text{ m/s}$$

Some students calculated the drag to be twice the above value, accounting for the drag on both sides of the flat plate. This answer was rewarded full credit as well, as the wording of the problem statement was not succifiently specific to make one of the two approaches more desirable than the other.

PROBLEM B9.

If the boundary layer is like that of a flat plate (for which $\frac{dU}{dx} = 0$) then the Blasius solution applies and

$$\begin{aligned}
\delta_D &= 1.72 \left(\frac{\nu x}{U}\right)^{\frac{1}{2}} \\
&= 1.72 \left(\frac{2.5 \times 10^{-6} \text{ m}^2/\text{s} \cdot x}{1.0 \text{ m/s}}\right)^{\frac{1}{2}} \\
&= 2.7 \times 10^{-3}\ x^{\frac{1}{2}}
\end{aligned}$$

The effect of this displacement thickness is to yield a volume flow rate in the tube which is the same as the volume flow rate of a *uniform* stream in a tube of radius $(R - \delta_D)$ where R is the actual radius of the tube.

Therefore the velocity of the flow outside the boundary layer is not U as given by 1 m/s but U_x where

$$U \pi R^2 = U_x \pi \left(R - \delta_D\right)^2$$

At $x = 200$ m, where $\delta_D = 0.038$ m this yields

$$U_x = \frac{U}{\left(1 - \frac{\delta_D}{R}\right)^2} = 1.39 \text{ m/s}$$

Then since Bernoulli's equation applies outside the boundary layer the pressure at $x = 200$ m is related to the pressure at the inlet ($x = 0$ m) by

$$\begin{aligned} p_{x=200 \text{ m}} - p_{x=0 \text{ m}} &= \frac{1}{2}\rho\left[U^2 - U_x^2\right] \\ &= -0.57 \text{ Pa} \end{aligned}$$

where $\rho = 1.2$ kg/m^3.

PROBLEM B10.

For a wedge flow,

$$U = Cx^m$$

and

$$\theta = \frac{\pi m}{m+1}$$

hence m can be solved as:

$$m = \frac{\theta}{\pi - \theta}$$

We can determine that $m_1 = 1/9$, $m_2 = 1/3$ and $m_3 = 1$. For the laminar boundary thickness $\delta_{0.99}$, we are looking in the figure for values of $(2(m+1))^{1/2}\eta_{0.99}$ when $u/U = 0.99$. The value for $\delta_{0.99}$ can be calculated by realizing that:

$$\eta_{0.99} = \delta_{0.99}\left(\frac{U}{4\nu x}\right)^{1/2} = \frac{1}{2}\delta_{0.99}\left(\frac{c}{\nu}\right)^{1/2} x^{\frac{m-1}{2}}$$

Hence, $\delta_{0.99}$ is given by:

$$\delta_{0.99} = 2\eta_{0.99}\left(\frac{v}{c}\right)^{1/2} x^{\frac{1-m}{2}}$$

α	m	$(2(m+1))^{1/2}\eta_{0.99}$	$\eta_{0.99}$	$\delta_{0.99}$
$\pi/10$	1/9	3.2	2.15	$4.3\left(\frac{v}{c}\right)^{1/2} x^{4/9}$
$\pi/4$	1/3	2.9	1.78	$3.6\left(\frac{v}{c}\right)^{1/2} x^{1/3}$
$\pi/2$	1	2.4	1.2	$2.4\left(\frac{v}{c}\right)^{1/2}$

PROBLEM B11.

Assume the velocity profile of a laminar boundary layer on a flat plate with zero pressure gradient is of the form:

$$\frac{u(y)}{U} = sin\left(\frac{\pi y}{2\delta}\right)$$

for $0 < y < \delta$ and $\frac{u(y)}{U} = 1$ for $y > \delta$. Find the displacement thickness δ_D, the momentum thickness δ_M and the skin friction drag D on the plate with length L and breadth B.

Introduce the dimensionless parameter $\eta = y/\delta$ such that:

$$\frac{u(\eta)}{U} = sin\left(\frac{\pi\eta}{2}\right)$$

The profile parameters are:

$$\gamma = \frac{\delta_D}{\delta} = \int_0^1 \left(1 - \frac{u(\eta)}{U}\right) = 1 - \frac{2}{\pi} \approx 0.3634$$

$$\alpha = \frac{\delta_M}{\delta} = \int_0^1 \frac{u(\eta)}{U}\left(1 - \frac{u(\eta)}{U}\right) = \frac{2}{\pi} - \frac{1}{2} \approx 0.1366$$

and

$$\beta = \frac{\partial \frac{u(y)}{U}}{\partial \frac{y}{\delta}}|_{y/\delta=0} = \frac{d\frac{u(y)}{U}}{d\eta}|_{\eta=0} = \frac{\pi}{2} \approx 1.571$$

The von Karman boundary layer momentum integral can be solved such that:

$$\delta = \left(\frac{2\beta}{\alpha}\right)^{1/2}\left(\frac{\nu x}{U}\right)^{1/2} = 4.796\left(\frac{\nu x}{U}\right)^{1/2}$$

The displacement thickness is given by:

$$\delta_D = \gamma\delta = 1.743\left(\frac{\nu x}{U}\right)^{1/2}$$

For the Blasius flat plate boundary layer, we got for the displacement thickness:

$$\delta_D = 1.72\left(\frac{\nu x}{U}\right)^{1/2}$$

The momentum thickness is given by:

$$\delta_M = \alpha\delta = 0.655\left(\frac{\nu x}{U}\right)^{1/2}$$

For the Blasius flat plate boundary layer, we got for the momentum thickness:

$$\delta_M = 0.664\left(\frac{\nu x}{U}\right)^{1/2}$$

In two-dimensional flow, we have:

$$\tau_w = \mu\left.\frac{\partial u(y)}{\partial y}\right|_{\text{wall}}$$

Therefore, we get:

$$\tau_w = \frac{\rho\nu U\pi}{2\delta} = 0.3275\rho U^{3/2}\left(\frac{\nu}{x}\right)^{1/2}$$

The drag is given by:

$$D = \int_0^L \tau_w dx = 0.655\rho U^{3/2}(\nu L)^{1/2}$$

For the Blasius flat plate boundary layer, we found:

$$D = 0.664\rho U^{3/2}(\nu L)^{1/2}$$

This form of a velocity profile is a good approximation to the exact Blasius profile.