ME19b. SOLUTIONS. Jan. 25 Due Feb. 1, 2007

PROBLEM B7.

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Since the flow is planar and incompressible this simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocity in the vertical direction, v, is zero at the plate and at infinity so it is zero everywhere in the flow, so conservation of mass implies that:

$$\frac{\partial u}{\partial x} = 0$$

so u is only a function of y, $u = u(y)$.

Navier-Stokes:
x-direction:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Since the flow is planar, $v = 0$, $\frac{\partial u}{\partial x} = 0$, and the pressure is constant, this becomes:

$$\rho\frac{\partial u}{\partial t} = \mu\frac{d^2 u}{dy^2} \qquad (1)$$

Using separation of variable to evalute this PDE, it is assumed:

$$u(y,t) = Y(y)T(t)$$

Plugging this into equation 1 leads to an equation with terms that are only functions of y on one side and terms of only t on the other. For this to hold both sides can only be equal to a constant, λ:

$$\frac{\dot{T}}{T} = \frac{\mu}{\rho}\frac{Y''}{Y} = \lambda$$

The equation on t is then:

$$\dot{T} = \lambda T$$

The solution to this equation is:

$$T(t) = c_1 e^{\lambda t}$$

The equation on y is:

$$Y'' - \frac{\rho}{\mu}Y = 0$$

The solution to this equation is:

$$Y(y) = c_2 e^{\sqrt{\frac{\rho}{\mu}\lambda}y} + c_3 e^{-\sqrt{\frac{\rho}{\mu}\lambda}y}$$

The boundary conditions at the plate and as $y \to \infty$ are

$$u(0,t) = U(t) = Ue^{kt}$$

$$u(y \to \infty, t) = 0$$

The second condition yields $c_2 = 0$. Combining the $T(t)$ and $Y(y)$ equations for $u(y,t)$:

$$u(y,t) = c_4 e^{-\sqrt{\frac{\rho}{\mu}\lambda}y} e^{\lambda t}$$

where $c_4 = c_1 c_3$. We now apply the first boundary condition:

$$u(0,t) = c_4 e^{\lambda t} = U e^{kt}$$

so the values of the unknown constant $c_4 = U^*$ and $\lambda = k$ and now known. This gives a velocity profile:

$$u(y,t) = U e^{kt} e^{-\sqrt{\frac{k}{\nu}}y}$$

where ν is the kinematic viscosity $\nu = \frac{\mu}{\rho}$. The vorticity $\omega(\mathbf{y},\mathbf{t}) = \nabla \times \mathbf{u} = -\frac{\partial u}{\partial y}$

$$\omega = U\sqrt{\frac{k}{\nu}}\; e^{kt} e^{-\sqrt{\frac{\rho}{\mu}k}y}$$

The boundary layer thickness, δ, is to be where the velocity is at least 10% of the plate velocity:

$$\begin{aligned} 10\%\; U e^{kt} &= U e^{kt} e^{-\sqrt{\frac{k}{\nu}}\delta} \\ 10\% &= e^{-\sqrt{\frac{k}{\nu}}\delta} \\ \delta &= \ln(10)\sqrt{\frac{\nu}{k}} \end{aligned}$$