ME19b. SOLUTIONS. Jan. 18 Due Jan. 25, 2007

PROBLEM B4.

Find the velocity distribution, $u(y)$, in the flow as a function of $y, U, H, dp/dx$ and the viscosity of the fluid, μ.

Continuity:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Since the flow is steady, planar, and incompressible this simplifies to:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The velocity in the vertical direction, v, is zero at both boundaries and thus everywhere in the flow, so conservation of mass implies that:

$$\frac{\partial u}{\partial x} = 0$$

so u is only a function of y, $u = u(y)$.

Navier-Stokes:
x-direction:

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

Since the flow is steady, planar, $v = 0$, and $\frac{\partial u}{\partial x} = 0$, this becomes:

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

Integrating twice with respect to y:

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + c_1 y + c_2$$

We now use the boundary conditions to evaluate the constants c_1, c_2 :

$$u(0) = c_2 = 0$$

$$u(H) = \frac{1}{2\mu}\frac{\partial p}{\partial x}H^2 + c_1 H = U$$

$$\Rightarrow c_1 = \frac{U}{H} - \frac{H}{2\mu}\frac{\partial p}{\partial x}$$

Inserting these values for the constants of integration, the velocity distribution is:

$$\frac{u(y)}{U} = \frac{y}{H} - \frac{H^2}{2\mu U}\frac{\partial p}{\partial x}\frac{y}{H}\left(1 - \frac{y}{H}\right)$$

2.) Find the magnitude and direction of the particular pressure gradient for which there would be zero net volume flow in the x direction.

$$\begin{aligned} Q &= \int_0^H u(y)dy \\ &= \int_0^H \left\{U\frac{y}{H} + \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(y^2 - Hy\right)\right\}dy \\ &= \frac{1}{2}UH - \frac{1}{12\mu}\frac{\partial p}{\partial x}H^3 \end{aligned}$$

The particular pressure gradient, $\frac{\hat{\partial p}}{\partial x}$, for which there will be no net volume flow ($Q = 0$) will be:

$$\frac{\hat{\partial p}}{\partial x} = \frac{6\mu U}{H^2}$$

The pressure gradient is positive, so the pressure will need to increase in the positive x-direction to offset the effect of the moving upper plate.

PROBLEM B5.

In a steady flow in which $u_z = u_r = 0$ continuity reduces to

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} &= 0 \\ \frac{\partial u_\theta}{\partial \theta} &= 0 \\ \rightarrow u_\theta = u_\theta(r) & \end{aligned}$$

The Navier-Stokes equation in the r-direction reduces to

$$\frac{\partial p}{\partial r} = \rho \frac{u_\theta^2}{r} \tag{1}$$

and in the θ-direction

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} = 0 \tag{2}$$

The solution to equation 2 is

$$u_\theta = \frac{A}{r} + Br$$

where A and B are integration constants. Applying the no-slip boundary conditions at (1) $r = a$, $u_\theta = \Omega a$ and (2) $r = b$, $u_\theta = 0$ yields

$$A = -\frac{\Omega a^2 b^2}{(a^2 - b^2)}; \quad B = \frac{\Omega a^2}{(a^2 - b^2)}$$

and therefore

$$u_\theta = \frac{\Omega a^2}{(a^2 - b^2)}\left[r - \frac{b^2}{r}\right]$$

Finally, applying equation 1

$$\begin{aligned} \frac{\partial p}{\partial r} &= \frac{\rho}{r}\left[\frac{A^2}{r^2} + 2AB + B^2 r^2\right] \\ &= \rho B^2 \left[\frac{b^4}{r^3} - \frac{2b^2}{r} + r\right] \\ \therefore \; p_{r=b} - p_{r=a} &= \rho B^2 \left[-\frac{b^4}{2r^2} - 2b^2 \ln r + \frac{r^2}{2}\right]_a^b \\ &= \rho B^2 \left[\frac{(b^2 - a^2)(b^2 + a^2)}{2a^2} - 2b^2 \ln \frac{b}{a}\right] \\ \rightarrow \; p_{r=b} - p_{r=a} &= \frac{\rho \Omega^2 a^4}{(a^2 - b^2)^2}\left[\frac{(b^4 - a^4)}{2a^2} - 2b^2 \ln \frac{b}{a}\right] \end{aligned}$$

PROBLEM B6.

Consider the volume flow rate for a state with n tubes:

$$Q = nA_n \overline{u}_n$$

It has been shown that average velocity for Poiseuille flow is given by:

$$\overline{u} = \frac{R^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

Therefore,

$$A_1\frac{R_1^2}{8\mu}\left(\frac{\partial p}{\partial x}\right) = nA_n\frac{R_n^2}{8\mu}\left(\frac{\partial p}{\partial x}\right)$$

Since,

$$A_n = \pi R_n^2$$

So that,

$$A_1^2 = nA_n^2$$

$$A_n = \frac{A_1}{\sqrt{n}}$$

From continuity,

$$A_1\overline{u}_1 = nA_n\overline{u}_n$$

$$\overline{u}_n = \frac{\overline{u}_1}{\sqrt{n}}$$

For the given numbers:

$$\pi(0.015)^2 = \frac{\pi(4\cdot 10^{-6})^2}{\sqrt{n}}$$

$$n = 1.98\cdot 10^{14}$$

The actual number is much smaller, which implies that the velocity (and therefore, the pressure drop) is greater in the microcirculation stages.