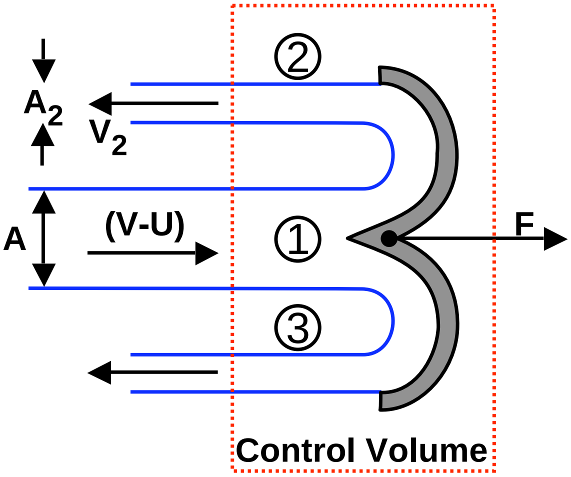
ME19b. SOLUTIONS. Jan. 4 Due Jan. 18, 2007

PROBLEM B1.

In the frame of the stationary bucket, the incident jet has a steady velocity $(V-U)$.



By neglecting gravity and viscous effects Bernoulli's equation along the streamline connecting the incident jet to one of the deflected jets yields:

$$p_1 + \frac{1}{2}\rho(V-U)^2 = p_2 + \frac{1}{2}\rho v_2^2 = p_3 + \frac{1}{2}\rho v_3^2$$

but $p_1 = p_2 = p_3 = p_{atm}$.

$$\begin{aligned} v_2^2 = v_3^2 &= (V-U)^2 \\ v_2 = v_3 &= V-U \end{aligned}$$

Additionally, by continuity and symmetry $A_2 = A_3 = A/2$. Applying momentum conservation,

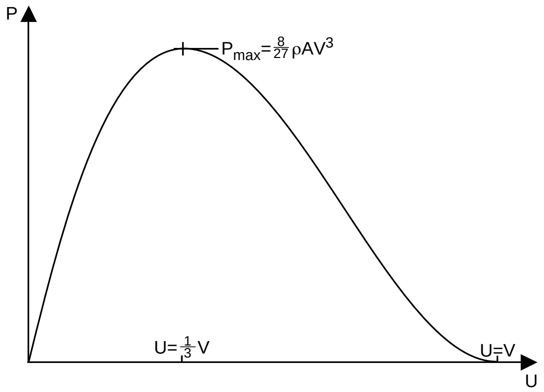
$$\begin{aligned} \Sigma F_x &= \int_{CS} u_x \rho \mathbf{u} \cdot \mathbf{n} dA \\ -F &= (V-U)\rho[-(V-U)]A + (-v_2)\rho v_2 A_2 + (-v_3)\rho v_3 A_3 \\ \rightarrow F &= 2\rho A (V-U)^2 \end{aligned}$$

Power:

$$\begin{aligned} P &= FU = 2\rho A U (V-U)^2 \\ \frac{\partial P}{\partial U} &= 2\rho A (V-U)^2 - 4\rho A U (V-U) = 0 \text{ at the maximum} \\ & (V-U) - 2U = 0 \\ \rightarrow U_{\text{max power}} &= \frac{V}{3} \end{aligned}$$

Efficiency:

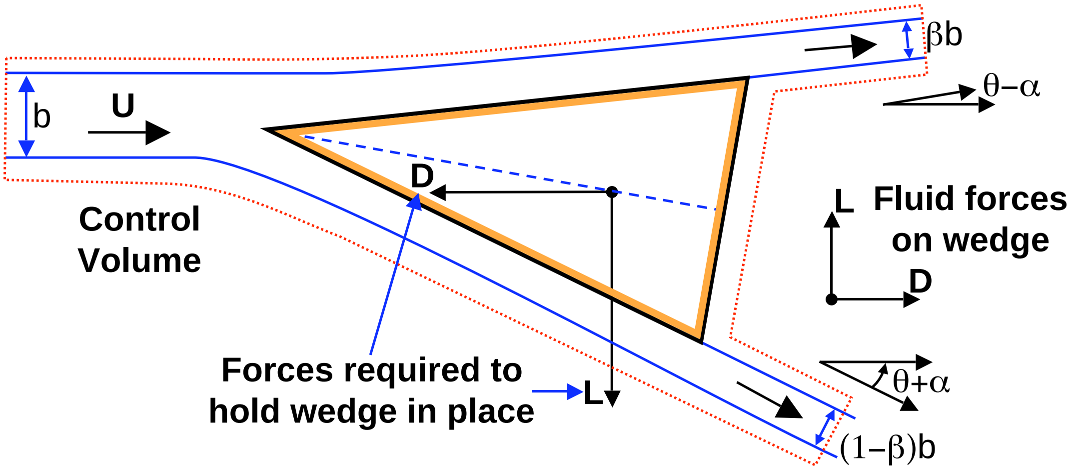
$$\begin{aligned} \eta = \frac{P}{\rho A V^3} &= \frac{2\rho A U (V-U)^2}{\rho A V^3} \\ &= 2\frac{U}{V}\left(1 - \frac{U}{V}\right)^2 \end{aligned}$$



$$\begin{aligned}\eta\big|_{\frac{U}{V}=\frac{1}{3}} &= \frac{2}{3}\left(\frac{2}{3}\right)^2 \\ \rightarrow \eta_{\max} &= \frac{8}{27} \approx .296\end{aligned}$$

PROBLEM B2.

Using a control volume which includes the wedge and denoting the drag force on the wedge in the U direction by D:



Applying the momentum theorem in the x direction yields

$$\begin{aligned}F_x = -D &= \rho U^2 \beta b \cos(\theta-\alpha) + \rho U^2 (1-\beta) b \cos(\theta+\alpha) - \rho b U^2 \\ D &= \rho U^2 b \left[1 - \beta \cos(\theta-\alpha) - (1-\beta)\cos(\theta+\alpha)\right]\end{aligned}$$

Similarly,

$$L = \rho U^2 b \left[(1-\beta)\sin(\theta+\alpha) - \beta \sin(\theta-\alpha)\right]$$

Therefore the angle of attack for which L is zero is

$$\alpha = \tan^{-1}\left[(2\beta-1)\tan(\theta)\right]$$

Also the β for zero lift is

$$\beta = \frac{1}{2}\left[1 + \tan(\alpha)\cot(\theta)\right]$$

Finally to determine whether this position is stable with respect to β, it is required that

$$\begin{aligned}\left.\frac{\partial \text{ lift}}{\partial \beta}\right|_{\text{at zero lift}} &= 0\\ &= \rho b U^2\left[-\sin(\theta+\alpha)-\sin(\theta-\alpha)\right]\\ &= \text{a } \textbf{negative} \text{ quantity for } (\theta+\alpha)<\pi \text{ and } \theta>\alpha>0\end{aligned}$$

Thus we have a **unstable equilibrium**. If β is increased (body is moved down) the lift becomes negative and further pushes the body down.

PROBLEM B3.

Continuity gives exhaust jet velocity, $U_J = \frac{UA_1}{A_J}$ or $Q = UA_1 = U_J A_J$. Therefore, the total head rise accross the propeller

$$\begin{aligned}\frac{U_J^2}{2g}-\frac{U_1^2}{2g} &= \Delta H\\ &= \;= \frac{K_1 - K_2 Q}{g}\end{aligned}$$

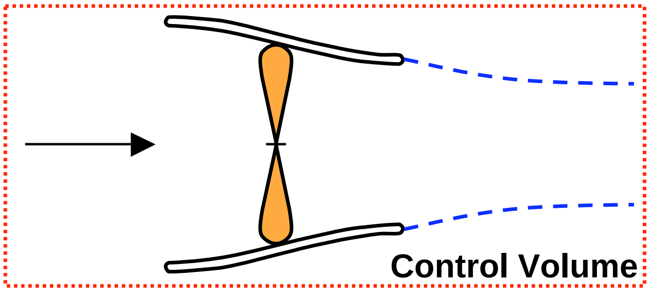
which yields the following quadratic

$$\frac{Q^2}{2gA_J^2}-\frac{Q^2}{2gA_1^2} = \frac{K_1-K_2Q}{g}$$

and therefore

$$A_J = \frac{UA_1}{\left[U^2+2(K_1-K_2UA_1)\right]^{\frac{1}{2}}}$$

Now take a large control volume around the jet. The net flux of momentum in the x direction out of the control volume is given as



$$\begin{aligned}F_x &= \rho A_J U_J^2 - \rho A_1 U^2\\ &= \text{Thrust of ducted propeller}\end{aligned}$$

because there are no pressure forces. Substituting for A_J,

$$\text{Thrust} = \rho A_1 U^2\left[\left\{1+\frac{1(K_1-K_2UA_1)}{U^2}\right\}^{\frac{1}{2}}-1\right]$$