

A toy model of quantum electrodynamics in $(1 + 1)$ dimensions

A D Boozer

Department of Physics, California Institute of Technology, Pasadena, California 91125

E-mail: boozer@caltech.edu

Abstract. We present a toy model of quantum electrodynamics (QED) in $(1 + 1)$ dimensions. The QED model is much simpler than QED in $(3 + 1)$ dimensions but exhibits many of the same physical phenomena, and serves as a pedagogical introduction to both QED and quantum field theory in general. We show how the QED model can be derived by quantizing a toy model of classical electrodynamics, and we discuss the connections between the classical and quantum models. In addition, we use the QED model to discuss the radiation of atoms and the Lamb shift.

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1. Introduction

Quantum electrodynamics (QED) is of fundamental importance in quantum field theory: it was the first quantum field theory to be constructed, and has provided a model for the construction of subsequent quantum field theories. For this reason, quantum field theory is usually introduced to students by using QED as an example. From a pedagogical standpoint, however, QED is not ideal for this purpose, because it is a rather complicated and technically demanding theory. QED is plagued by infinities of various sorts that require careful treatment; for example, a calculation of the Lamb shift yields a result that is infinite when expressed in terms of the bare electron mass; to obtain a finite result, one needs to introduce mass renormalization. In addition, the gauge invariance of electrodynamics raises a number of subtle issues when the theory is quantized. Finally, calculations in QED are often lengthy and difficult, and for the beginning student the mathematical complexity of the theory can obscure its physical meaning.

To address these issues, we present a simple toy model of QED in $(1+1)$ dimensions. The model is obtained by quantizing a toy model of classical electrodynamics that describes a scalar potential coupled to one or more charged particles in $(1+1)$ dimensions. The classical toy model is much simpler than ordinary electrodynamics, but shares many of the same physical features: there are fields that correspond to the electric and magnetic fields, these fields both mediate an interaction between the charged particles and support freely propagating radiation, and the coupling of a particle to its own field causes it to radiate and to undergo radiation damping. A full treatment of the classical model is given in [1], which provides a detailed discussion of these features[†]. In the current paper we quantize this model, and compare the classical and quantum versions of the theory.

Quantum electrodynamics is usually taught to students in two different settings: first, it is used in undergraduate quantum mechanics classes to treat the radiation of atoms, and second, it is used in graduate level quantum field theory classes to describe relativistic collisions. The toy model presented here is intended to bridge the gap between these two treatments, and could be used in either setting.

The paper is organized as follows. In section 2, we describe the classical toy model, and point out a number of features that we will later compare with corresponding features of the quantum toy model. In section 3, we quantize the classical toy model using the method of canonical quantization. In section 4, we consider a quantum field coupled to a classical source; this allows us to introduce some features of the quantized model in a simplified setting. In section 5, we investigate the full quantum theory, which describes a quantized particle coupled to a quantized field. We calculate the spontaneous decay rate and the Lamb shift for a harmonically bound particle, and discuss the connections between the QED toy model and ordinary QED.

[†] In order to make the current paper self-contained, a brief overview of the classical model is given in section 2, which summarizes all the results needed to understand the quantum model.

2. Classical theory

The classical toy model that we will be considering describes one or more point particles that obey Newtonian dynamics and are coupled to a pair of fields $E(t, x)$ and $B(t, x)$, which correspond to the electric and magnetic fields of ordinary electrodynamics. For simplicity we will assume that there is only one particle; the generalization to multiple particles is straightforward. The equation of motion for the particle is

$$m\ddot{z} = -\partial_z V(z) - 2gE(t, z), \quad (1)$$

where $z(t)$ is the position of the particle at time t , m and g are its mass and charge, and $V(z)$ is an arbitrary external potential. The equations of motion for the E and B fields are

$$\partial_t E(t, x) = \partial_x B(t, x), \quad (2)$$

$$\partial_t B(t, x) = \partial_x E(t, x) - 2\rho(t, x), \quad (3)$$

where $\rho(t, x)$, the charge density, is given by

$$\rho(t, x) = g \delta(x - z(t)). \quad (4)$$

Equation (1) can be thought of as the analog to the Lorentz force law of ordinary electrodynamics, and equations (2) and (3) can be thought of as the analogs to Maxwell's equations. Together, these equations give a complete description of the toy model.

In ordinary electrodynamics, it is often useful to express the electric and magnetic fields as derivatives of the vector potential $A^\mu(t, \mathbf{r})$. We can do something similar in the toy model: from equation (2), it follows that we can define a scalar potential $\phi(t, x)$ such that $E(t, x) = \partial_x \phi(t, x)$ and $B(t, x) = \partial_t \phi(t, x)$. By substituting these relations into equation (3), we obtain a field equation for the scalar potential[†]:

$$\square \phi(t, x) = -2\rho(t, x). \quad (5)$$

This is just the inhomogeneous wave equation, where $\rho(t, x)$ acts as the source.

As shown in [1], we can derive expressions for the total energy and momentum of the coupled particle-field system. In the toy model the E -field of the particle does not fall off with distance, so to obtain finite values for these quantities we will assume that the theory is defined on a finite region of space $[-L/2, L/2]$, where L is an arbitrary length[‡]. The total energy of the system is then given by $U_{tot} = U_p + U_f + U_i$, where

$$U_p = \frac{1}{2} m \dot{z}^2 + V(z) \quad (6)$$

is the energy of the particle,

$$U_f = \frac{1}{2} \int_{-L/2}^{L/2} (E^2(t, x) + B^2(t, x)) dx \quad (7)$$

[†] The following notation is used in this paper: $\square = \partial_t^2 - \partial_x^2$ is the d'Alembertian operator in $(1 + 1)$ dimensions, and $\epsilon(x)$ is the sign function, defined such that $\epsilon(x) = 1$ for $x > 0$, $\epsilon(x) = 0$ for $x = 0$, $\epsilon(x) = -1$ for $x < 0$. We have chosen a system of units in which the speed at which waves are propagated by the field is equal to 1.

[‡] We will assume that L is large enough that the particle is always confined to this region.

is the energy of the E and B fields, and

$$U_i = 2 \int_{-L/2}^{L/2} \phi(t, x) \rho(t, x) dx = 2g\phi(t, z) \quad (8)$$

is the energy due to the coupling of the particle to the field. The total momentum of the system is given by $P_{tot} = P_f + m\dot{z}$, where

$$P_f = - \int_{-L/2}^{L/2} E(t, x)B(t, x) dx. \quad (9)$$

is the momentum of the field. Using the equations of motion (1), (2) and (3), it is straightforward to show that $\dot{U}_{tot} = 0$ and $\dot{P}_{tot} = -\partial_z V(z)$. Thus, the total energy is conserved, and in the absence of an external potential the total momentum is also conserved.

In section 3 we will quantize the model using the method of canonical quantization, which requires that we first express the classical theory in terms of a Hamiltonian. We will take the generalized coordinates of the system to be the particle position z and the scalar potential $\phi(t, x)$, and we will define p and $\pi(t, x)$ to be the generalized momenta conjugate to these coordinates. The Hamiltonian is given by $H = H_p + H_f + H_i$, where

$$H_p = \frac{p^2}{2m} + V(z) \quad (10)$$

is the Hamiltonian for the particle,

$$H_f = \frac{1}{2} \int_{-L/2}^{L/2} ((\partial_x \phi(t, x))^2 + \pi^2(t, x)) dx \quad (11)$$

is the Hamiltonian for the field, and

$$H_i = 2 \int_{-L/2}^{L/2} \phi(t, x) \rho(t, x) dx = 2g\phi(t, z) \quad (12)$$

describes the coupling of the particle to the field. Using Hamilton's equations, we find that the equations of motion for the particle variables are

$$\dot{z} = p/m, \quad (13)$$

$$\dot{p} = -\partial_z V(z) - 2g \partial_x \phi(t, x) |_{x=z(t)}, \quad (14)$$

and the equations of motion for the field variables are

$$\partial_t \phi(t, x) = \pi(t, x), \quad (15)$$

$$\partial_t \pi(t, x) = \partial_x^2 \phi(t, x) - 2\rho(t, x). \quad (16)$$

It is straightforward to verify that these equations of motion are equivalent to equations (1), (2), and (3). Note that when the equations of motion are satisfied, $p = m\dot{z}$ is the mechanical momentum of the particle and $\pi(t, x) = B(t, x)$ is the B -field.

2.1. Solutions to the field equation

Now that we have described the model, let us consider some solutions to the field equation (5). Although we are primarily interested in the situation in which the charge distribution $\rho(t, x)$ is given by equation (4), it is more instructive to solve the field equation for the case of an arbitrary static charge distribution $\rho(x)$. We can express the solutions to the field equation as $\phi(t, x) = \phi_s(x) + \phi_r(t, x)$, where

$$\square\phi_s(x) = -\partial_x^2\phi_s(x) = -2\rho(x) \quad (17)$$

and

$$\square\phi_r(t, x) = 0. \quad (18)$$

We can think of $\phi_s(x)$ as describing the static potential generated by the charge distribution, and $\phi_r(t, x)$ as describing any radiation that might be present. From equation (17), we find that the static potential is given by†

$$\phi_s(x) = \phi_0 + \int |x - y| \rho(y) dy, \quad (19)$$

where ϕ_0 is an arbitrary constant. The corresponding E and B fields are

$$E_s(x) = \partial_x\phi_s(x) = \int \epsilon(x - y) \rho(y) dy, \quad (20)$$

$$B_s(x) = \partial_t\phi_s(x) = 0. \quad (21)$$

Let g denote the total quantity of charge:

$$g = \int \rho(x) dx. \quad (22)$$

If we substitute equations (19), (20), and (21) into equations (7) and (8), we find that the total energy is $U_{tot} = U_f + U_i = U_b + U_c$, where

$$U_b = \iint |x - y| \rho(x) \rho(y) dx dy \quad (23)$$

is the binding energy of the charge distribution, and $U_c = g^2L/2 + 2g\phi_0$ is a constant that depends only on the total quantity of charge g , not on how the charge is spatially distributed.

As an example, let us consider a charge distribution that consists of two stationary point charges:

$$\rho(x) = g_1 \delta(x - z_1) + g_2 \delta(x - z_2). \quad (24)$$

From equation (23), we find that the binding energy of the two charges is

$$U_b = 2g_1g_2 |z_1 - z_2|. \quad (25)$$

If the two charges have the same sign, then U_b describes an attractive interaction potential between the charges. Thus, in contrast to electrodynamics, charges of the

† The most general solution includes an additional term E_0x that describes a constant background E -field E_0 , but for simplicity we will assume that $E_0 = 0$. Also, we have assumed that $\rho(x)$ is nonzero only in the region $[-L/2, L/2]$.

same sign attract. This can also be shown by solving for the field produced by one of the charges, and then using equation (1) to calculate the force that this field exerts on the other charge.

2.2. Solutions to the particle equation of motion

Now that we have considered some solutions to the field equation, let us turn our attention to equation (1), the equation of motion for the particle. One can show that the field $E(t, z)$ that appears on the right hand side of this equation can be divided into two components: a component $E_r(t, z) = g\dot{z}$ that describes the retarded E -field generated by the particle itself, and a component $E_{ext}(t, z)$ that describes the E -field due to other charges and to incoming radiation[†]. By dividing the E -field in this way, we can express the equation of motion as

$$m\ddot{z} + m\gamma\dot{z} = -\partial_z V(z) - 2gE_{ext}(t, z), \quad (26)$$

where $\gamma \equiv 2g^2/m$. Thus, the coupling of the particle to its own field results in a damping term with damping constant γ . As the particle moves it radiates energy to the field, and this damping term describes the reaction force that the radiation exerts on the particle.

Let us solve equation (26) for the special case of a harmonically bound particle in the absence of an external field ($E_{ext}(t, z) = 0$). The potential is

$$V(z) = \frac{1}{2}m\omega_A^2 z^2, \quad (27)$$

where ω_A is the harmonic frequency, so the equation of motion is

$$\ddot{z} + \gamma\dot{z} + \omega_A^2 z = 0. \quad (28)$$

The solution to this equation is

$$z(t) = A e^{-\gamma t/2} \cos(\omega t + \theta), \quad (29)$$

where $\omega \equiv \omega_A(1 - \gamma^2/4\omega_A^2)^{1/2}$, and A and θ are set by the initial conditions. Note that because of the damping term, the oscillation frequency is shifted by $\delta\omega = \omega - \omega_A$; this is the classical analog of the quantum Lamb shift, which we will discuss in section 5.2. Let us assume that the damping constant is small compared to the harmonic frequency ($\gamma \ll \omega_A$). In this limit, the frequency shift is given by $\delta\omega = -\gamma^2/8\omega_A$. Also, if we substitute the particle trajectory given by equation (29) into equation (6) for the particle energy, we find that

$$U_p = \frac{1}{2}m\omega_A^2 A^2 e^{-\gamma t}. \quad (30)$$

Thus, the energy of the particle decays exponentially at rate γ .

[†] This is shown in [1]. For simplicity, we have assumed that the speed of the particle is much less than the speed at which waves are propagated by the field ($|\dot{z}| \ll 1$); in general, $E_r(t, z) = g\dot{z}(1 - \dot{z}^2)^{-1}$.

3. Quantizing the toy model

We can quantize the toy model by replacing the dynamical variables z , p , $\phi(t, x)$, and $\pi(t, x)$ with quantum operators. For the particle variables z and p , this procedure is familiar from elementary quantum mechanics: in the position basis, the operators corresponding to these variables are

$$\hat{z} = z, \quad \hat{p} = -i\partial_z. \quad (31)$$

From equation (10), we find that the quantized particle Hamiltonian is

$$\hat{H}_p = \frac{\hat{p}^2}{2m} + V(\hat{z}) = -\frac{1}{2m}\partial_z^2 + V(z) = \sum_n E_n |n\rangle\langle n|, \quad (32)$$

where $|n\rangle$ and E_n , the eigenstates and eigenvalues of \hat{H}_p , are obtained by solving the time-independent Schrödinger equation. For the special case of a harmonically bound particle, it is convenient to express \hat{z} and \hat{p} in terms of creation and annihilation operators b^\dagger and b :

$$\hat{z} = (2m\omega_A)^{-1/2}(b + b^\dagger), \quad \hat{p} = -i(m\omega_A/2)^{1/2}(b - b^\dagger). \quad (33)$$

The quantized particle Hamiltonian can then be expressed as

$$\hat{H}_p = \omega_A(b^\dagger b + 1/2). \quad (34)$$

The eigenstates of \hat{H}_p are Fock states, and the eigenvalues are $E_n = \omega_A(n + 1/2)$. In what follows, it will sometimes be useful to write the position operator in the form

$$\hat{z} = (\eta/\omega_A)(b + b^\dagger), \quad (35)$$

where $\eta \equiv (2m\omega_A)^{-1/2}\omega_A$ is a dimensionless parameter, called the Lamb-Dicke parameter, that characterizes the ratio of the size of the ground state wavepacket to the wavelength of light with frequency ω_A .

To quantize the field variables $\phi(t, x)$ and $\pi(t, x)$, we first use the field Hamiltonian given in equation (11) to derive the classical equations of motion for a free field:

$$\partial_t \phi(t, x) = \pi(t, x), \quad \partial_t \pi(t, x) = \partial_x^2 \phi(t, x). \quad (36)$$

We then find the normal modes of the system[†], and quantize each mode by introducing creation and annihilation operators. The details of this procedure are described in a number of textbooks (see [2, 3, 4]), so we will omit the intermediate steps and simply quote the end result:

$$\hat{\phi}(x) = \sum_m (2\omega_m L)^{-1/2} (a_m e^{ik_m x} + a_m^\dagger e^{-ik_m x}), \quad (37)$$

$$\hat{\pi}(x) = -i \sum_m (\omega_m/2L)^{1/2} (a_m e^{ik_m x} - a_m^\dagger e^{-ik_m x}). \quad (38)$$

[†] The normal modes depend on the choice of boundary conditions. We will assume that periodic boundary conditions are imposed on the region $[-L/2, L/2]$, so $\phi(t, L/2) = \phi(t, -L/2)$ and $\pi(t, L/2) = \pi(t, -L/2)$.

Here m is the mode number, which ranges over the nonzero integers, a_m^\dagger and a_m are the creation and annihilation operators associated with mode m , and $k_m \equiv 2\pi m/L$ and $\omega_m \equiv |k_m|$ are the wavenumber and frequency of mode m . The quantized field operators are given by

$$\hat{E}(x) = \partial_x \hat{\phi}(x), \quad \hat{B}(x) = \hat{\pi}(x). \quad (39)$$

If we substitute the quantized field operators into the field Hamiltonian H_f given in equation (11), and note that

$$\int_{-L/2}^{L/2} e^{i(k_{m_1} + k_{m_2})x} dx = L \delta_{m_1 + m_2}, \quad (40)$$

we find that the quantized field Hamiltonian is

$$\hat{H}_f = \sum_m \omega_m a_m^\dagger a_m, \quad (41)$$

where for simplicity we have dropped a constant term that describes the zero-point energy. We will denote the ground state of \hat{H}_f by $|0\rangle$, and we will denote states in which one photon is present in mode m by $|1_m\rangle \equiv a_m^\dagger |0\rangle$. From equation (9), we find that the quantized field momentum is

$$\hat{P}_f = \sum_m k_m a_m^\dagger a_m. \quad (42)$$

Note that \hat{P}_f generates displacements of the field operators; for example,

$$[\hat{\phi}(x), \hat{P}_f] = -i\partial_x \hat{\phi}(x), \quad [\hat{\pi}(x), \hat{P}_f] = -i\partial_x \hat{\pi}(x). \quad (43)$$

4. Quantum field coupled to a classical charge distribution

In section 2.1, we described solutions to the classical field equation when we replaced the point particle with an arbitrary static charge distribution $\rho(x)$. Before turning to the full quantum theory, it is instructive to consider the quantum version of this problem. We will take the Hamiltonian for the system to be $\hat{H} = \hat{H}_f + \hat{H}_i$, where \hat{H}_f , the field Hamiltonian, is given by equation (41), and \hat{H}_i , the interaction Hamiltonian, is given by

$$\hat{H}_i = 2 \int \hat{\phi}(x) \rho(x) dx. \quad (44)$$

4.1. Perturbation theory

In section 2.1, we found that for the classical theory we could divide the total potential ϕ into two components: a component ϕ_s that described the static potential of the charge distribution, and a component ϕ_r that described the dynamical radiation field. We expect that the ground state of the quantum theory will correspond to the situation in which the radiation component vanishes and only the static component is present. Thus, the properties of the quantum ground state should tell us something about the static field generated by the charge distribution.

We can determine the ground state wavefunction $|\Omega\rangle$ by using perturbation theory; we will take \hat{H}_f to be the unperturbed Hamiltonian and \hat{H}_i to be the perturbation. To first order in $\rho(x)$, the ground state wavefunction is given by

$$|\Omega\rangle = |0\rangle - \sum_m (1/\omega_m) \langle 1_m | \hat{H}_i | 0 \rangle | 1_m \rangle. \quad (45)$$

From equations (37) and (44), we find that

$$\langle 1_m | \hat{H}_i | 0 \rangle = 2(2\omega_m L)^{-1/2} \int e^{-ik_m x} \rho(x) dx. \quad (46)$$

Thus, the vacuum expectation value of the scalar potential $\hat{\phi}(x)$ is

$$\langle \Omega | \hat{\phi}(x) | \Omega \rangle = -(2/L) \sum_m \int (1/\omega_m^2) e^{ik_m(x-y)} \rho(y) dy. \quad (47)$$

We show how to evaluate the sum in Appendix B; from equation (B.6), we find that

$$\langle \Omega | \hat{\phi}(x) | \Omega \rangle = \phi_0 + \int |x - y| \rho(y) dy = \phi_s(x), \quad (48)$$

where $\phi_0 = -gL/6$ and $\phi_s(x)$ is the classical solution given by equation (19). Similarly, one can show that the vacuum expectation values of the field operators $\hat{E}(x)$ and $\hat{B}(x)$ are

$$\langle \Omega | \hat{E}(x) | \Omega \rangle = E_s(x), \quad \langle \Omega | \hat{B}(x) | \Omega \rangle = B_s(x), \quad (49)$$

where $E_s(x)$ and $B_s(x)$ are the classical solutions given by equations (20) and (21). Thus, in the quantum theory the charge distribution generates a static field that agrees with the predictions of the classical theory.

We can also use perturbation theory to calculate the energy shift to the ground state; to second order in $\rho(x)$, it is given by

$$\Delta E = - \sum_m (1/\omega_m) |\langle 1_m | \hat{H}_i | 0 \rangle|^2. \quad (50)$$

If we substitute for $\langle 1_m | \hat{H}_i | 0 \rangle$ using equation (46), and evaluate the sum using equation (B.6), we find that

$$\Delta E = E_c + \iint |x - y| \rho(x) \rho(y) dx dy = E_c + U_b, \quad (51)$$

where $E_c = -g^2L/6$ and U_b is the classical binding energy given in equation (23). As we showed in section 2.1, we can take the charge distribution to consist of two point charges; the binding energy U_b can then be thought of as describing an interaction potential between the charges. Since this binding energy has the same form in both the classical and quantum theories, the interaction potential is likewise the same, and for both theories charges of the same sign feel an attractive force.

4.2. Unitary transformation

Although we only calculated the vacuum expectation values to first order and the energy shift to second order, the results we obtained are actually exact to all orders. To see this, first note that we can express the interaction Hamiltonian as

$$\hat{H}_i = \sum_m \omega_m (\lambda_m^* a_m + \lambda_m a_m^\dagger), \quad (52)$$

where we have defined

$$\lambda_m = (2/\omega_m) (2\omega_m L)^{-1/2} \int e^{-ik_m x} \rho(x) dx. \quad (53)$$

Thus, the total Hamiltonian is

$$\hat{H}_f + \hat{H}_i = \sum_m \omega_m a_m^\dagger a_m + \sum_m \omega_m (\lambda_m^* a_m + \lambda_m a_m^\dagger). \quad (54)$$

We can diagonalize this Hamiltonian by defining a unitary transformation

$$U = \exp\left(\sum_m (\lambda_m a_m^\dagger - \lambda_m^* a_m)\right), \quad (55)$$

which has the property that[†]

$$U a_m U^\dagger = a_m - \lambda_m. \quad (56)$$

From equations (54) and (56), it follows that the transformed Hamiltonian is

$$U(\hat{H}_f + \hat{H}_i)U^\dagger = \hat{H}_f - \sum_m \omega_m \lambda_m^* \lambda_m. \quad (57)$$

If we substitute for λ_m , and use the results of Appendix B to perform the sum over m , we find that

$$U(\hat{H}_f + \hat{H}_i)U^\dagger = \hat{H}_f + E_c + U_b, \quad (58)$$

where as before $E_c = -g^2 L/6$ and U_b is the classical binding energy given in equation (23).

We can understand the physical meaning of the unitary transformation U by considering its effect on the field operators. Using equation (56), it is straightforward to show that

$$U \hat{\phi}(x) U^\dagger = \hat{\phi}(x) + \phi_s(x), \quad (59)$$

where $\phi_s(x)$ is the classical static potential given by equation (19), and

$$U \hat{E}(x) U^\dagger = \hat{E}(x) + E_s(x), \quad U \hat{B}(x) U^\dagger = \hat{B}(x) + B_s(x), \quad (60)$$

where $E_s(x)$ and $B_s(x)$ are the classical static fields given by equations (20) and (21). Thus, we can think of U as removing the static component of the field, leaving only the quantized radiation field (in the classical theory, this would correspond to subtracting the static potential ϕ_s from the total potential ϕ to obtain the radiation potential ϕ_r). In equation (58) for the transformed Hamiltonian, \hat{H}_f is the Hamiltonian for the radiation

[†] For each mode m , U acts as a displacement operator $D(\lambda_m)$; see Appendix A, equation (A.6).

field and $E_c + U_b$ is the energy of the static field. Note that the ground state of the transformed Hamiltonian is $|0\rangle$; as expected, this corresponds to the situation in which there is no radiation ($\hat{H}_f|0\rangle = 0$). Also, note that the energy of the ground state is $E_c + U_b$, which agrees with the value we obtained by using second-order perturbation theory. By inverting the unitary transformation, we can obtain an exact expression for the ground state $|\Omega\rangle$ of the original Hamiltonian:

$$|\Omega\rangle = U^\dagger|0\rangle = \exp\left(-\sum_m(\lambda_m a_m^\dagger - \lambda_m^* a_m)\right)|0\rangle = \prod_m |\alpha_m\rangle, \quad (61)$$

where $|\alpha_m\rangle$ is a coherent state of mode m with amplitude $\alpha_m = -\lambda_m$ (see Appendix A). From equation (59), it follows that the vacuum expectation value of the scalar potential $\hat{\phi}(x)$ is

$$\langle\Omega|\hat{\phi}(x)|\Omega\rangle = \langle 0|U\hat{\phi}(x)U^\dagger|0\rangle = \langle 0|\hat{\phi}(x)|0\rangle + \phi_s(x) = \phi_s(x), \quad (62)$$

which agrees with the value we obtained by using first-order perturbation theory.

5. Quantum theory

Let us now consider the full quantum theory, which describes a quantized particle coupled to a quantized field. The Hamiltonian for the system is $\hat{H} = \hat{H}_p + \hat{H}_f + \hat{H}_i$, where \hat{H}_p and \hat{H}_f , the Hamiltonians for the particle and field, are given by equations (32) and (41), and \hat{H}_i , the interaction Hamiltonian, is obtained by quantizing the Hamiltonian given in equation (12):

$$\hat{H}_i = 2g \hat{\phi}(\hat{z}) = 2g \sum_m (2\omega_m L)^{-1/2} (a_m e^{ik_m \hat{z}} + a_m^\dagger e^{-ik_m \hat{z}}). \quad (63)$$

It is straightforward to check that

$$[\hat{P}_f + \hat{p}, \hat{H}] = -i\partial_z V(z), \quad (64)$$

so in the absence of an external potential the total momentum of the system is conserved, just as for the classical theory.

We can study the effects of the coupling between the particle and field by using perturbation theory: we will take $\hat{H}_0 \equiv \hat{H}_p + \hat{H}_f$ to be the unperturbed Hamiltonian and \hat{H}_i to be the perturbation. The eigenstates of \hat{H}_0 are tensor products of the particle and field eigenstates; for example, $|n, 0\rangle \equiv |n\rangle|0\rangle$ is an eigenstate of \hat{H}_0 in which the particle is in state $|n\rangle$ and the field is in state $|0\rangle$, and $|n, 1_m\rangle \equiv |n\rangle|1_m\rangle$ is an eigenstate in which the particle is in state $|n\rangle$ and the field is in state $|1_m\rangle$.

5.1. Spontaneous decay

The interaction Hamiltonian allows the particle to exchange energy with the field, and this means that excited states of the particle can spontaneously decay. A state with $n > 0$ can spontaneously decay to any state $r < n$ by emitting a photon, where the decay rate is given by Fermi's golden rule:

$$\Gamma_{n \rightarrow r} = 2\pi \sum_m |\langle r, 1_m | \hat{H}_i | n, 0 \rangle|^2 \delta(E_n - E_r - \omega_m). \quad (65)$$

From equation (63) for the interaction Hamiltonian, we find that

$$\langle r, 1_m | \hat{H}_i | n, 0 \rangle = 2g(2\omega_m L)^{-1/2} \langle r | e^{-ik_m \hat{z}} | n \rangle. \quad (66)$$

If we substitute this matrix element into equation (65), and convert the sum over m into an integral over k (see equation (B.4)), we find that

$$\Gamma_{n \rightarrow r} = 2g^2 \int (1/\omega_k) |\langle r | e^{-ik\hat{z}} | n \rangle|^2 \delta(E_n - E_r - \omega_k) dk, \quad (67)$$

where $\omega_k \equiv |k|$. The delta function in the integral tells us that the frequency of the emitted photon is equal to the energy difference between the initial and final states: $\omega_k = E_n - E_r$. We will assume that the wavelength associated with this frequency is large compared to the spatial extent of the particle wavefunction, so we can make the dipole approximation:

$$|\langle r | e^{-ik\hat{z}} | n \rangle|^2 \simeq |\langle r | 1 - ik\hat{z} | n \rangle|^2 = \omega_k^2 |\langle r | \hat{z} | n \rangle|^2. \quad (68)$$

If we substitute equation (68) into equation (67) and perform the integral over k , we find that

$$\Gamma_{n \rightarrow r} = 4g^2(E_n - E_r) |\langle r | \hat{z} | n \rangle|^2. \quad (69)$$

Note that the spatial extent of state n is given by $\langle n | \hat{z}^2 | n \rangle^{1/2}$, so the dipole approximation is justified when

$$\omega_k \langle n | \hat{z}^2 | n \rangle^{1/2} \ll 1. \quad (70)$$

Let us now specialize to the case of a harmonically bound particle. If we substitute equation (35) for \hat{z} into equation (70), and use that $\omega_k = E_n - E_r = (n - r)\omega_A$, we find that the dipole approximation is justified when $\eta(n - r)\sqrt{2n + 1} \ll 1$. Thus, we will assume that $\eta \ll 1$ and that n is small enough that this condition is met. Since the matrix element $\langle r | \hat{z} | n \rangle$ vanishes unless $r = n \pm 1$, in the dipole approximation state n only decays to state $n - 1$. From equation (69), we find that the decay rate is

$$\Gamma_{n \rightarrow n-1} = 4g^2\omega_A |\langle n - 1 | \hat{z} | n \rangle|^2 = \gamma n, \quad (71)$$

where $\gamma = 2g^2/m$ is the classical damping constant.

If we start the particle in an arbitrary initial state, it will eventually decay to the ground state by spontaneously emitting photons. To describe this process, let $p_n(t)$ denote the probability that the particle occupies Fock state n at time t . These probabilities evolve in time according to a set of rate equations:

$$\dot{p}_n(t) = \Gamma_{n+1 \rightarrow n} p_{n+1}(t) - \Gamma_{n \rightarrow n-1} p_n(t). \quad (72)$$

The first term describes the rate at which the population in n increases due to spontaneous decay from $n + 1$ to n , and the second term describes the rate at which the population in n decreases due to spontaneous decay from n to $n - 1$.

Equation (72) can be thought of as a quantum analog[†] to equation (28), the classical equation of motion for a harmonically bound particle undergoing radiation damping, and

[†] The probabilities p_n are the diagonal elements of the density matrix for the quantum particle, and one can derive an equation of motion that describes the evolution of the entire density matrix. This equation, called a master equation, is an even closer analog to the classical equation of motion, but a discussion of these matters is beyond the scope of this paper (see [5], chapter 8).

it is instructive to compare the classical and quantum theories. The closest analog in the quantum theory to a classical state of the oscillator is a coherent state (see Appendix A). Suppose that at $t = 0$ we start the particle in a coherent state $|\alpha_0\rangle$. Using equation (A.2) to expand the coherent state in a basis of Fock states, we find that the initial probabilities are

$$p_n(0) = |\langle n|\alpha_0\rangle|^2 = (1/n!) e^{-\bar{n}_0} \bar{n}_0^n, \quad (73)$$

where $\bar{n}_0 = |\alpha_0|^2$ is the mean vibrational quantum number. Using equations (71) and (72), it is straightforward to check that the probabilities at time t are

$$p_n(t) = (1/n!) e^{-\bar{n}(t)} \bar{n}^n(t), \quad (74)$$

where $\bar{n}(t)$, the mean vibrational quantum number at time t , is given by $\bar{n}(t) = \bar{n}_0 e^{-\gamma t}$. Since the mean energy of the particle is $\bar{E}(t) = \omega_A \bar{n}(t)$, we see that the mean energy of the particle decays exponentially at rate γ , just as we found for the classical theory.

5.2. Lamb shift

In addition to causing spontaneous decay, the interaction Hamiltonian shifts the energy levels of the particle from E_n to $E_n + \delta_n$; for ordinary QED, this effect is known as the Lamb shift[†]. We can calculate the shift to state n by using second-order perturbation theory:

$$\delta_n = \sum_r \sum_m |\langle r, 1_m | \hat{H}_i | n, 0 \rangle|^2 (E_n - E_r - \omega_m)^{-1} = \sum_r \Delta_{n,r}, \quad (75)$$

where

$$\Delta_{n,r} = (2g)^2 \sum_m (2\omega_m L)^{-1} |\langle r | e^{-ik_m \hat{z}} | n \rangle|^2 (E_n - E_r - \omega_m)^{-1}. \quad (76)$$

We can use the results of Appendix B to evaluate the $\Delta_{n,n}$ term explicitly:

$$\Delta_{n,n} = -(2g^2/L) \sum_m (1/\omega_m^2) |\langle n | e^{-ik_m \hat{z}} | n \rangle|^2 \quad (77)$$

$$= E_c + g^2 \iint |x - y| |\psi_n(x)|^2 |\psi_n(y)|^2 dx dy, \quad (78)$$

where $E_c = -g^2 L/6$. Here we have substituted

$$\langle n | e^{-ik_m \hat{z}} | n \rangle = \int \langle n | z \rangle e^{-ik_m z} \langle z | n \rangle dz = \int e^{-ik_m z} |\psi_n(z)|^2 dz, \quad (79)$$

where $|z\rangle$ is a position eigenstate with eigenvalue z , and $\psi_n(z) = \langle z | n \rangle$ is the position-space wavefunction for the state $|n\rangle$. If we use equation (78) to substitute for $\Delta_{n,n}$ in equation (75), we find that

$$\delta_n = E_c + g^2 \iint |x - y| |\psi_n(x)|^2 |\psi_n(y)|^2 dx dy + \sum_{r \neq n} \Delta_{n,r}. \quad (80)$$

We can interpret this result as follows. The first term is a constant that is the same for all energy levels, and simply redefines the zero of energy. Since it is only the

[†] Discussions of the Lamb shift in hydrogen that are accessible to undergraduates are given in [6, 7, 8].

energy differences between levels that are physically meaningful, we can neglect this constant term and consider only the n -dependent portion of the level shift that is given by $\delta'_n \equiv \delta_n - E_c$. The second term is just the binding energy for a classical charge distribution with charge density $\rho(x) = g|\psi_n(x)|^2$; we will call this the semiclassical binding energy. The third term describes corrections to the semiclassical binding energy due to the contributions of other levels. Note that if the energy spacings $|E_n - E_r|$ are large, then these corrections are suppressed and δ'_n is dominated by the semiclassical binding energy.

Let us now specialize to the case of a harmonic oscillator potential. From equation (76), we find that

$$\sum_{r \neq n} \Delta_{n,r} = -(\gamma/2\pi\eta)I_n(\eta), \quad (81)$$

where we have defined a dimensionless quantity

$$I_n(\eta) = \sum_{r \neq n} \int_0^\infty |\langle r | e^{-iu(b+b^\dagger)} | n \rangle|^2 (u^2 + (r-n)\eta u)^{-1} du. \quad (82)$$

Here we have substituted for \hat{z} using equation (35), converted the sum on m to an integral over k using equation (B.4), and defined a dimensionless variable $u = (2m\omega_A)^{-1/2} k$. Note that when $r < n$ there are poles in the the integral; these are to be evaluated by taking the principal part. If we substitute the completeness relation

$$|\langle n | e^{-ik\hat{z}} | n \rangle|^2 = 1 - \sum_{r \neq n} |\langle r | e^{-ik\hat{z}} | n \rangle|^2 \quad (83)$$

into equation (77), we find that

$$\Delta_{n,n} = E_c + (\gamma/2\pi\eta)I_n(0). \quad (84)$$

Thus, the level shift is given by

$$\delta'_n = \delta_n - E_c = (\gamma/2\pi\eta)(I_n(0) - I_n(\eta)). \quad (85)$$

The $I_n(0)$ term describes the semiclassical binding energy, and the $I_n(\eta)$ term describes the corrections due to the contributions of other levels.

As an example, let us calculate the Lamb shift to the ground state. Using the displacement operator $D(\alpha)$ that is defined in Appendix A, we find that

$$|\langle r | e^{-iu(b+b^\dagger)} | 0 \rangle|^2 = |\langle r | D(\alpha) | 0 \rangle|^2 = |\langle r | \alpha \rangle|^2 = \frac{1}{r!} e^{-u^2} u^{2r}, \quad (86)$$

where $|\alpha\rangle$ is a coherent state with amplitude $\alpha = -iu$. Thus,

$$I_0(\eta) = \sum_{r=1}^{\infty} \frac{1}{r!} \int_0^\infty e^{-u^2} u^{2r-1} (u + r\eta)^{-1} du. \quad (87)$$

Note that

$$I_0(0) = \sum_{r=1}^{\infty} \frac{1}{r!} \int_0^\infty e^{-u^2} u^{2r-2} du = \int_0^\infty (1/u^2)(1 - e^{-u^2}) du = \sqrt{\pi}. \quad (88)$$

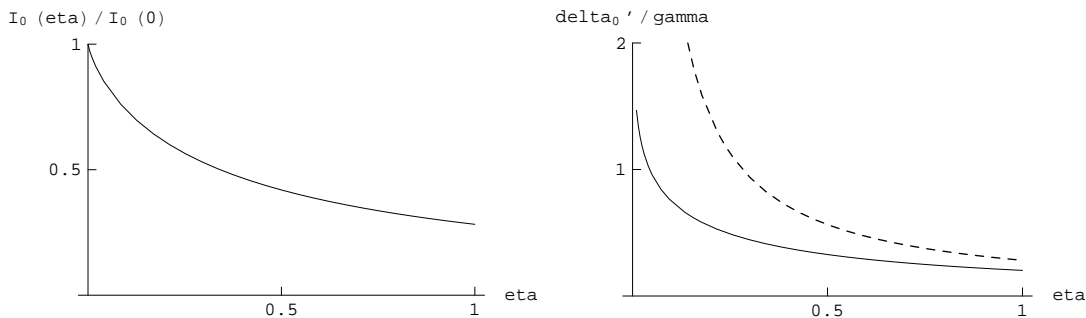


Figure 1. Lamb shift to the ground state. (a) $I_0(\eta)/I_0(0)$ versus η . (b) δ'_0/γ versus η . For comparison, the dashed curve is U_b/γ , where U_b is the semiclassical binding energy.

This result could also be obtained by substituting the ground state wavefunction $\psi_0(x)$ into equation (78) for $\Delta_{n,n}$. From equation (87) we see that $0 \leq I_0(\eta) \leq I_0(0)$, so δ'_0 is always positive.

In Figure 1, we numerically evaluate[‡] $I_0(\eta)$, and plot $I_0(\eta)/I_0(0)$ and δ'_0/γ as a function of η . Note that as $\eta \rightarrow \infty$, $I_0(\eta)/I_0(0) \rightarrow 0$, so in the limit of large η the level shift is dominated by the semiclassical binding energy. It is interesting that although the quantum and classical versions of the theory predict the same spontaneous decay rate γ , the quantum level shift δ'_n is completely different from the classical frequency shift $\delta\omega$ that we calculated in section 2.2. The situation is the same in ordinary QED: the classical and quantum theories agree regarding radiative decay rates, but disagree regarding radiative frequency shifts (this is discussed in [9], chapter 17).

5.3. Unitary transformation

In section 4.2, we considered a quantum field coupled to a classical charge distribution, and showed that one could remove the static field of the distribution by defining a suitable unitary transformation. For the full quantum theory, we can define a similar unitary transformation that removes the static field generated by the particle. Following the procedure described in section 4.2, we write the interaction Hamiltonian in the form

$$\hat{H}_i = 2g\hat{\phi}(\hat{z}) = \sum_m \omega_m (\lambda_m^\dagger a_m + \lambda_m a_m^\dagger), \quad (89)$$

where

$$\lambda_m = (2g/\omega_m)(2\omega_m L)^{-1/2} e^{-ik_m \hat{z}}, \quad (90)$$

and we define a unitary transformation

$$U = \exp\left(\sum_m (\lambda_m a_m^\dagger - \lambda_m^\dagger a_m)\right). \quad (91)$$

[‡] As a consistency check, we numerically evaluate $I_0(0)$ and compare the result with the exact value $I_0(0) = \sqrt{\pi}$; we find that that the numerical and exact values differ by 1.24×10^{-4} .

Since $U a_m U^\dagger = a_m - \lambda_m$, we find that

$$U(\hat{H}_f + \hat{H}_i)U^\dagger = \hat{H}_f - \sum_m \omega_m \lambda_m^\dagger \lambda_m = \hat{H}_f + E_c, \quad (92)$$

where $E_c = -g^2 L/6$. We can also transform the field operators; for example, the transformed scalar potential is

$$U \hat{\phi}(x) U^\dagger = \hat{\phi}(x) + \hat{\phi}_s(x), \quad (93)$$

where

$$\hat{\phi}_s(x) = \phi_0 + g \int |x - z| |z\rangle \langle z| dz \quad (94)$$

and $\phi_0 = -gL/6$. The operator $\hat{\phi}_s(x)$ describes the instantaneous static potential generated by the particle: note that if the state of the particle is $|\psi\rangle$, then the expectation value of $\hat{\phi}_s(x)$ is

$$\langle \psi | \hat{\phi}_s(x) | \psi \rangle = \phi_0 + g \int |x - z| |\psi(z)|^2 dz. \quad (95)$$

This is just the static potential for a classical charge distribution with charge density $\rho(x) = g|\psi(x)|^2$, where $\psi(z) = \langle z | \psi \rangle$ is the position-space wavefunction of the particle.

Equation (92) describes how the field and interaction Hamiltonians transform under U ; let us now consider the transformation of the particle Hamiltonian \hat{H}_p . It is convenient to work in the position basis, in which $\hat{z} = z$, and to express the transformation as $U = \exp(-2ig \hat{S}(z))$, where we have defined a new quantum field

$$\hat{S}(x) = -i \sum_m (1/\omega_m) (2\omega_m L)^{-1/2} (a_m e^{ik_m x} - a_m^\dagger e^{-ik_m x}). \quad (96)$$

Note that

$$U \hat{p} U^\dagger = U(U^\dagger \hat{p} + [\hat{p}, U^\dagger]) = \hat{p} - iU \partial_z U^\dagger = \hat{p} + 2g \partial_z \hat{S}(z), \quad (97)$$

so the transformed particle Hamiltonian is

$$U \hat{H}_p U^\dagger = (1/2m)(\hat{p} + 2g \partial_z \hat{S}(z))^2 + V(z). \quad (98)$$

Combining these results, we find that the transformed total Hamiltonian is

$$U \hat{H} U^\dagger = U(\hat{H}_p + \hat{H}_f + \hat{H}_i)U^\dagger = \hat{H}_{pi} + \hat{H}_f + E_c, \quad (99)$$

where we have defined $\hat{H}_{pi} \equiv U \hat{H}_p U^\dagger$. In the transformed representation, the static field of the particle has been completely eliminated from our description of the system: \hat{H}_f describes the pure radiation field, and \hat{H}_{pi} describes the particle and its coupling to this radiation field. The transformed Hamiltonian is directly analogous to the Hamiltonian for the nonrelativistic limit of ordinary QED in the Coulomb gauge[†]:

$$\hat{H} = (1/2m)(\hat{\mathbf{p}} + q \hat{\mathbf{A}}(\mathbf{r}))^2 + \hat{H}_{rad}. \quad (100)$$

[†] This is the form of the QED Hamiltonian that is usually presented in undergraduate quantum mechanics courses, and should be familiar to students; see [6]. Our unitary transformation U is analogous to a similar transformation used in ordinary QED; see [10], pages 425-9.

Here $\hat{\mathbf{p}}$ is the momentum of the particle, m and q are its mass and charge, and $\hat{\mathbf{A}}(\mathbf{r})$ and \hat{H}_{rad} are the vector potential and Hamiltonian that describe the quantized radiation field.

In performing calculations, it is useful to express the transformed Hamiltonian as

$$U\hat{H}U^\dagger = \hat{H}_p + \hat{H}_f + \hat{H}'_i, \quad (101)$$

where $\hat{H}'_i \equiv \hat{H}_{pi} - \hat{H}_p + E_c$. This has the same form as the original Hamiltonian, only the interaction Hamiltonian is now given by \hat{H}'_i instead of \hat{H}_i . Note that[‡]

$$\hat{H}_{pi} = U\hat{H}_pU^\dagger = \hat{H}_p - i[A, \hat{H}_p] - (1/2)[A, [A, \hat{H}_p]], \quad (102)$$

where $A \equiv 2g\hat{S}(\hat{z})$. Thus, we can express the new interaction Hamiltonian as

$$\hat{H}'_i = \hat{H}_{pi} - \hat{H}_p + E_c = E_c - i[A, \hat{H}_p] - (1/2)[A, [A, \hat{H}_p]]. \quad (103)$$

It is instructive to use the new form of the transformed Hamiltonian to repeat our previous calculations of the spontaneous decay rate and the Lamb shift. We will take $\hat{H}_p + \hat{H}_f$ to be the unperturbed Hamiltonian and \hat{H}'_i to be the perturbation. To second order in g , we find that the Lamb shift to state n is given by

$$\begin{aligned} \delta_n = & E_c - (1/2)\langle n, 0|[A, [A, \hat{H}_p]]|n, 0\rangle + \\ & \sum_r \sum_m |\langle r, 1_m|[A, \hat{H}_p]|n, 0\rangle|^2 (E_n - E_r - \omega_m)^{-1}. \end{aligned} \quad (104)$$

Note that

$$\langle r, 1_m|[A, \hat{H}_p]|n, 0\rangle = (E_n - E_r)\langle r, 1_m|A|n, 0\rangle \quad (105)$$

and

$$\langle r, 1_m|A|n, 0\rangle = (i/\omega_m)\langle r, 1_m|\hat{H}_i|n, 0\rangle. \quad (106)$$

Also, by inserting a complete set of states, we find that

$$(1/2)\langle n, 0|[A, [A, \hat{H}_p]]|n, 0\rangle = \sum_r \sum_m (E_n - E_r)|\langle r, 1_m|A|n, 0\rangle|^2. \quad (107)$$

If we substitute these results into equation (104), it is straightforward to check that it agrees with the expression for the Lamb shift given in equation (75). We can also calculate the spontaneous decay rate; to lowest order in g , it is given by

$$\Gamma_{n \rightarrow r} = 2\pi \sum_m |\langle r, 1_m|[A, \hat{H}_p]|n, 0\rangle|^2 \delta(E_n - E_r - \omega_m). \quad (108)$$

Using equations (105) and (106), it is straightforward to check that this agrees with the expression for the spontaneous decay rate given in equation (65).

[‡] For arbitrary operators X and Y , one can show that $e^X Y e^{-X} = Y + [X, Y] + (1/2)[X, [X, Y]] + \dots$. If we substitute $X = -iA$ and $Y = \hat{H}_p$, we find that $[X, [X, Y]]$ is a c-number, so the series truncates after the third term.

Appendix A. Coherent states

Consider a quantum harmonic oscillator that is described in terms of creation and annihilation operators a^\dagger and a . For any complex number α , one can define a state $|\alpha\rangle$ that is an eigenstate of a with eigenvalue α :

$$a|\alpha\rangle = \alpha|\alpha\rangle. \quad (\text{A.1})$$

Such states are called coherent states, and have a number of applications in quantum field theory. Here we summarize some important properties of coherent states that we make use of in the paper; a more complete treatment can be found in [5], chapter 2. A coherent state $|\alpha\rangle$ can be expanded in a basis of Fock states as follows:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_n (1/n!)^{1/2} \alpha^n |n\rangle. \quad (\text{A.2})$$

The mean vibrational quantum number of the coherent state $|\alpha\rangle$ is

$$\bar{n} = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \quad (\text{A.3})$$

One can define a unitary displacement operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \quad (\text{A.4})$$

which acts on the vacuum to produce a coherent state $|\alpha\rangle$:

$$D(\alpha)|0\rangle = |\alpha\rangle. \quad (\text{A.5})$$

The displacement operator also has the property that

$$D(\alpha) a D^\dagger(\alpha) = a - \alpha. \quad (\text{A.6})$$

Appendix B. Sum over modes

We will often want to evaluate the sum

$$S(x) = (2/L) \sum_m (1/\omega_m^2) e^{ik_m x} = (2/L) \sum_m (1/\omega_m^2) \cos k_m x. \quad (\text{B.1})$$

Note that

$$S(0) = (2/L) \sum_m 1/\omega_m^2 = (L/2\pi^2) \sum_m 1/m^2 = L/6 \quad (\text{B.2})$$

and

$$S(0) - S(x) = (2/L) \sum_m (1/\omega_m^2) (1 - \cos k_m x). \quad (\text{B.3})$$

Recall that $k_m = 2\pi m/L$, so as L increases the spacing between adjacent modes decreases. Thus, for large L we can approximate sums over m by integrals over k :

$$\sum_m \rightarrow (L/2\pi) \int dk. \quad (\text{B.4})$$

If we approximate equation (B.3) in this way, we find that

$$S(0) - S(x) = (1/\pi) \int (1/k^2) (1 - \cos kx) dk = |x|. \quad (\text{B.5})$$

Combining equations (B.2) and (B.5), we find that the sum is given by

$$(2/L) \sum_m (1/\omega_m^2) e^{ik_mx} = L/6 - |x|. \quad (\text{B.6})$$

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