A Toy Model of Electrodynamics in 1+1 Dimensions

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Abstract. A model is presented that describes a scalar field interacting with a point particle in 1+1 dimensions. The model exhibits many of the same phenomena that appear in classical electrodynamics, such as radiation and radiation damping, yet has a much simpler mathematical structure. By studying these phenomena in a highly simplified model, the physical concepts involved may be more easily understood.

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1. Introduction

In this paper, we present a field theory model that describes a point particle coupled to a scalar field in 1+1 dimensions. The system can be viewed as a toy model of classical electrodynamics†, in which the number of spatial dimensions has been reduced from three to one, the vector field has been replaced with a scalar field, and non-relativistic as opposed to relativistic dynamics are used for the particle. The model has a much simpler mathematical structure than electrodynamics, yet exhibits many of the same physical phenomena. By reducing the mathematical complexity of electrodynamics while retaining much of its conceptual structure, the toy model helps clarify the subtle physical concepts underlying the theory of coupled particles and fields. In particular, the paper could be used to supplement a discussion of radiation and radiation damping in an electrodynamics course taught at the level of Jackson's *Electrodynamics*, and should be accessible to advanced undergraduates and beginning graduate students.

The paper is organized as follows. In section 2, we write down the Hamiltonian for the toy theory, and derive the equations of motion for the system. In section 3, we discuss the scattering of radiation by the particle. After presenting a general formalism for solving scattering problems, we consider a simple example: the scattering of a nearly monochromatic wavepacket by a harmonically bound particle. In section 4, we discuss the energy and momentum of the field, and derive conservation laws for these quantities. Also, we calculate the power radiated by a moving particle. In section 5 we generalize the toy model to the case of a spatially extended particle. Finally, in section 6 we compare the toy model with ordinary electrodynamics in $3+1$ dimensions. Further discussion of the model can be found in [2].

2. Hamiltonian for the model theory

The dynamical variables for the system are the scalar field $\phi(t,x)$ and its conjugate momentum density $\pi(t,x)$, and the position z and canonical momentum p of the particle. We will take the Hamiltonian for the system to be

$$H = H_f + H_p + H_i, \tag{1}$$

where

$$H_f = \frac{1}{2}\int \left[\pi(t,x)^2 + (\partial_x\phi(t,x))^2\right] dx \tag{2}$$

is the Hamiltonian for the field,

$$H_p = \frac{p^2}{2m} + V(z) \tag{3}$$

is the Hamiltonian for the particle, and

$$H_i = 2g\phi(t,z(t)) \tag{4}$$

† The theory of charged particles in electrodynamics is quite subtle; a systematic treatment of the subject is given in [1].

describes the interaction of the particle with the field. The quantities m and g are the mass and charge of the particle, and V is an arbitrary potential. We will choose a system of units such that the speed at which waves are propagated by the field is equal to one. Although the model is mathematically well-defined for arbitrary particle velocity, we will only consider the case of a particle that moves slower than the speed of wave propagation.

Given this Hamiltonian, it follows that the equations of motion for the field variables are

$$\partial_t \phi(t,x) = \pi(t,x) \tag{5}$$

$$\partial_t \pi(t,x) = \partial_x^2 \phi(t,x) - 2g\,\delta(x - z(t)), \tag{6}$$

and the equations of motion for the particle variables are

$$\dot{z} = p/m \tag{7}$$

$$\dot{p} = -\frac{dV}{dz} - 2g\,\partial_x \phi, \tag{8}$$

where $\partial_x \phi$ is understood to be evaluated at the particle position:

$$\partial_x \phi \equiv \partial_x \phi(t,x)\,|_{x=z(t)}\ . \tag{9}$$

From the equations of motion, we find that the field equation is‡

$$\Box \phi(t,x) = -2g\,\delta(x - z(t)), \tag{10}$$

and the force on the point particle is

$$F = m\ddot{z} = -\frac{dV}{dz} - 2g\,\partial_x \phi. \tag{11}$$

It is interesting to note that if the model is generalized to describe a many-particle system, the field gives an attractive force between particles of the same charge. To see this, consider a particle with charge g at rest at z_1. From the field equation, we find that the static field generated by the charge is

$$\phi(t,x) = g|x - z_1|.$$

The force that this field exerts on a particle with charge g at z_2 is

$$F_{12} = -2g\,\partial_x \phi\,|_{x=z_2} = -2g^2\,\epsilon(z_2 - z_1),$$

so the charges attract. Note that the force is independent of the particle separation.

3. Scattering

We can think of the field ϕ as being made up of two components: a field that is bound to the particle, and a dynamical radiation field. The particle is coupled to the radiation

‡ The following notation is used in this paper: $\Box = \partial_t^2 - \partial_x^2$ is the d'Alembertian operator in $1+1$ dimensions, $\theta(x)$ is the step function, defined such that $\theta(x) = 1$ for $x > 0$, $\theta(x) = 1/2$ for $x = 0$, $\theta(x) = 0$ for $x < 0$, and $\epsilon(x)$ is the sign function, defined such that $\epsilon(x) = 1$ for $x > 0$, $\epsilon(x) = 0$ for $x = 0$, $\epsilon(x) = -1$ for $x < 0$.

field via the interaction Hamiltonian H_i, allowing it to absorb and emit radiation. This interaction between the particle and the radiation field can be formulated as a scattering problem. We will assume that in the far past ($t < t_i$, for some time t_i) and the far future ($t > t_f$, for some time t_f), the particle is stationary† and well separated from any existing radiation:

$$z(t) = \begin{cases} z_i & \text{for } t < t_i \\ z_f & \text{for } t > t_f. \end{cases} \tag{12}$$

Between t_i and t_f, incoming radiation propagates toward the particle and scatters off it. The problem we want to solve is to express the state of the system after t_f in terms of some initial state at time $t_0 < t_i$.

Before t_i and after t_f, it is straightforward to decompose the field ϕ into a static bound field and a freely evolving radiation field. For $t < t_i$, the static field is

$$\psi_i(t, x) = g|x - z_i|, \tag{13}$$

while for $t > t_f$, the static field is

$$\psi_f(t, x) = g|x - z_f|. \tag{14}$$

In section 3.1 we will show that one can define retarded and advanced fields $\phi_r(t, x)$ and $\phi_a(t, x)$ that obey the inhomogeneous wave equation

$$\Box\phi_r(t, x) = \Box\phi_a(t, x) = -2g\,\delta(x - z(t)) \tag{15}$$

and satisfy the boundary conditions

$$\phi_r(t, x) = \psi_i(t, x) \text{ for } t < t_i \tag{16}$$

$$\phi_a(t, x) = \psi_f(t, x) \text{for } t > t_f. \tag{17}$$

Because $\phi(t, x)$ also satisfies the inhomogeneous wave equation, we can define fields $\phi_{in}(t, x)$ and $\phi_{out}(t, x)$ by

$$\phi_{in}(t, x) \ = \phi(t, x) - \phi_r(t, x) \tag{18}$$

$$\phi_{out}(t, x) = \phi(t, x) - \phi_a(t, x), \tag{19}$$

which satisfy the homogeneous wave equation

$$\Box\phi_{in}(t, x) = \Box\phi_{out}(t, x) = 0. \tag{20}$$

For $t < t_i$, the *in* field has a simple physical interpretation: it represents the incoming radiation that is incident on the particle. For $t > t_i$, the interpretation of the *in* field is less intuitive: it describes what the incoming radiation would look like if it did not interact with the particle and continued to evolve freely. Similar remarks can be made about the *out* field: for $t > t_f$, the *out* field represents the outgoing radiation that is the end result of the scattering process, while for $t < t_f$, the *out* field describes what the

† One might think that requiring the particle to be stationary is too restrictive, and that we should simply require that the particle move freely at a constant velocity. The reason for the restriction is that in the model theory a moving particle radiates, as we shall see in section 4.2.

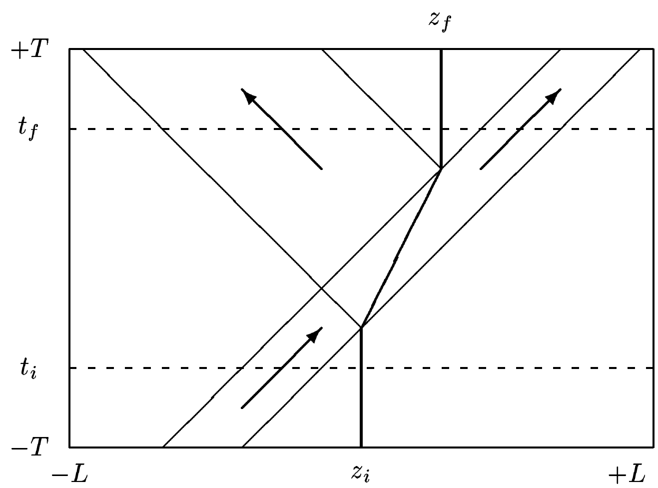

Figure 1. The region R for an example scattering process. A pulse of radiation approaches the particle from the left and scatters off it, as shown by the arrows.

outgoing radiation would look like if it was freely evolved backward in time, neglecting its interaction with the particle.

From equations (18) and (19), we see that for all times the field ϕ may be decomposed into an *in* field and a retarded field:

$$\phi(t,x) = \phi_{in}(t,x) + \phi_r(t,x), \tag{21}$$

or into an *out* field and an advanced field:

$$\phi(t,x) = \phi_{out}(t,x) + \phi_a(t,x). \tag{22}$$

These decompositions provide a solution to the scattering problem. The solution proceeds in three steps. First, as we will discuss in section 3.2, the initial data is used to solve for the *in* field at all times. Next, as we will discuss in section 3.3, the initial data together with the *in* field is used to calculate the trajectory of the particle. Finally, as we will discuss in section 3.1, the particle trajectory is used to calculate the retarded and advanced fields at all times. Using equation (21), we can combine the *in* field with the retarded field to obtain the total field at all times; and if desired, equation (22) may be used to obtain the *out* field.

In what follows, it will be convenient to restrict our attention to a finite region of spacetime R containing the entire scattering process (see Figure 1). We will define the region R to consist of events (t,x) such that $|t| < T$ and $|x| < L$, where T is chosen such that that $-T < t_i < t_f < T$, and L is chosen such that for all times $t \in [-T,T]$, the particle and the radiation fields are entirely contained in the spatial region $[-L,L]$; that is, $\phi_{in}(t,x) = \phi_{out}(t,x) = 0$ for $|x| > L$. Unless otherwise noted, events are always assumed to lie in the region R.

3.1. Retarded and advanced fields

In this section we calculate the retarded and advanced fields for a particle moving along a trajectory $z(t)$. In section 3.3 we will show how the trajectory itself may be determined, but for now let us assume it is already known.

First consider the retarded field. We can define a retarded Green's function

$$G_r(t,x) = \theta(t-|x|), \tag{23}$$

which obeys the equation

$$\Box G_r(t,x) = 2\,\delta(t)\,\delta(x) \tag{24}$$

and satisfies the boundary condition $G_r(t,x) = 0$ for $t < 0$. Also, let us define the retarded time $t_r(t,x)$ corresponding to the event (t,x) by

$$t_r = t - |x - z(t_r)|. \tag{25}$$

This definition is unique, provided that the particle always moves more slowly than the speed of wave propagation. The retarded time has a simple interpretation: if a pulse of radiation is emitted by the particle at the retarded time $t_r(t,x)$, then it will reach point x at time t.

We want to solve equation (15) for the retarded field, subject to the boundary condition given in equation (16). Let us write equation (15) as

$$\Box\phi_r(t,x) = -2\rho(t,x), \tag{26}$$

where

$$\rho(t,x) = g\,\delta(x - z(t)). \tag{27}$$

Using the retarded Green's function, we can express the solution as

$$\phi_r(t,x) = -\iint_{T_r}^{\infty} G_r(t-t', x-x')\,\rho(t',x')\,dt'\,dx' + g(t - T_r), \tag{28}$$

where T_r is an arbitrary constant, which is chosen such that $T_r < t_r(t,x)$ for all events $(t,x) \in R$. Substituting for $\rho(t,x)$, we find

$$\phi_r(t,x) = -g\int_{T_r}^{\infty} \theta(t - t' - |x - z(t')|)\,dt' + g(t - T_r). \tag{29}$$

From equation (29), it is straightforward to check that for $t < t_i$ the retarded field is just the static field of the particle, as claimed in equation (16).

Let us now use equation (29) to calculate the time derivative and gradient of the retarded field. We find

$$\partial_t\phi_r(t,x) = -g\int_{T_r}^{\infty} \delta(t - t' - |x - z(t')|)\,dt' + g \tag{30}$$

$$\partial_x\phi_r(t,x) = g\int_{T_r}^{\infty} \epsilon(x - z(t'))\,\delta(t - t' - |x - z(t')|)\,dt'. \tag{31}$$

There are two cases to consider: either the event (t, x) lies on the particle trajectory, or it does not. Let us first consider the case where it does not. Then we can simplify the delta functions in these integrals by using the identity‡

$$\delta(f(u)) = \sum_{k=1}^{n} |f'(u_k)|^{-1}\, \delta(u - u_k), \tag{32}$$

which holds for any function $f(u)$, where the sum is taken over the points $u_1, \cdots, u_n$ at which $f(u) = 0$. We will take

$$f(t') = t - t' - |x - z(t')|, \tag{33}$$

so

$$f'(t') = -1 + v(t')\, \epsilon(x - z(t')), \tag{34}$$

where $v(t')$ is the velocity of the particle at time t'. Since $f(t') = 0$ only when $t' = t_r$, we find

$$\delta(t - t' - |x - z(t')|) = [1 - v(t_r)\, \epsilon(x - z(t_r))]^{-1}\, \delta(t' - t_r). \tag{35}$$

We can use this result to perform the t' integrations in equations (30) and (31):

$$\partial_t \phi_r(t, x) \;=\; -\, g[1 - v(t_r)\, \epsilon(x - z(t_r))]^{-1}\, v(t_r)\, \epsilon(x - z(t_r)) \tag{36}$$

$$\partial_x \phi_r(t, x) = g[1 - v(t_r)\, \epsilon(x - z(t_r))]^{-1}\, \epsilon(x - z(t_r)). \tag{37}$$

We can also express these as

$$\partial_t \phi_r(t, x) \;=\; -\, g[1 - v(t_r)\, \epsilon(x - z(t))]^{-1}\, v(t_r)\, \epsilon(x - z(t)) \tag{38}$$

$$\partial_x \phi_r(t, x) = g[1 - v(t_r)\, \epsilon(x - z(t))]^{-1}\, \epsilon(x - z(t)). \tag{39}$$

This follows from the fact that, assuming the particle always moves slower than the speed of wave propagation,

$$\epsilon(x - z(t_r)) = \epsilon(x - z(t)). \tag{40}$$

In other words, if the particle is to the left (right) of x at time t, then it was also to the left (right) of x at time $t_r(t, x)$. To see this, note that

$$t - t_r > |z(t) - z(t_r)|, \tag{41}$$

since otherwise the average speed of the particle during the time interval $[t_r, t]$ would exceed the speed of wave propagation. Thus, from the definition of t_r,

$$|x - z(t_r)| > |z(t) - z(t_r)|, \tag{42}$$

from which (40) follows.

Now let us consider the case of events that do lie on the particle trajectory. For such events, the field gradient is

$$\partial_x \phi_r(t, x)\, \big|_{x=z(t)} \; = g \int_{-\infty}^{t-T_r} \epsilon(z(t) - z(t - \tau))\, \delta(\tau - |z(t) - z(t - \tau)|)\, d\tau.$$

‡ See [8], Appendix A.

where $\tau \equiv t - t'$. If the particle always moves slower than the speed of wave propagation then the delta function is zero for any finite value of τ, and we can expand $z(t-\tau)$ in a power series§:

$$z(t-\tau) = z(t) - v\tau + \cdots, \tag{43}$$

where, to simplify the notation, we have defined $v \equiv v(t)$. Thus,

$$\begin{aligned} \partial_x \phi_r(t,x) \mid_{x=z(t)} &= g \int_{-\infty}^{t-T_r} \epsilon(v\tau)\, \delta(\tau - |v\tau|)\, d\tau \\ &= g\,(1-v^2)^{-1}\, v. \end{aligned} \tag{44}$$

Note that this is the average of the field gradients on either side of the particle.

Everything we have said about the retarded field can be repeated for the case of the advanced field: we define an advanced Green's function by

$$G_a(t,x) = G_r(-t,x), \tag{45}$$

and an advanced field by

$$\phi_a(t,x) = -\iint_{-\infty}^{T_a} G_a(t-t', x-x')\, \rho(t',x')\, dt'\, dx' - g(t - T_a). \tag{46}$$

By proceeding in analogy with the retarded field, one can calculate the time derivative and gradient of the advanced field.

3.2. In and Out fields

Given the initial state of the system at time t_0, it is a simple matter to determine the *in* field at all times. We first obtain initial conditions for the *in* field:

$$\phi_{in}(t_0,x) = \phi(t_0,x) - g|x - z_i| \tag{47}$$

$$\pi_{in}(t_0,x) = \pi(t_0,x). \tag{48}$$

From these initial conditions, we can calculate the *in* field at an arbitrary time t by using d'Alembert's solution‖ to the wave equation:

$$\phi_{in}(t,x) = \frac{1}{2}[\phi_{in}(t_0, x+\tau) + \phi_{in}(t_0, x-\tau) + \int_{x-\tau}^{x+\tau} \pi_{in}(t_0,y)\, dy], \tag{49}$$

where $\tau = t - t_0$. It is straightforward to verify this result by checking that it satisfies the homogeneous wave equation and gives the correct initial conditions at $t = t_0$.

A similar result holds for the *out* field, although it is not very useful since we don't have initial conditions for the *out* field.

§ This method of evaluating the retarded field on the world line of the particle is based on a similar technique used to evaluate the self-force in electrodynamics, which is presented in [5], pp. 187–189.
‖ A discussion of this solution can be found in [3], pp. 344–346.

3.3. Particle trajectory

Recall that the equation of motion for the particle is given by equation (11), which we will write as

$$m\ddot{z} = -\frac{dV}{dz} + F_f, \tag{50}$$

where

$$F_f(t) = -2g\,\partial_x\phi(t,x)\,|_{x=z(t)} \tag{51}$$

is the force exerted on the particle by the field. If we decompose ϕ into an *in* field and a retarded field using equation (21), we can express this force as

$$F_f(t) = F_{in}(t) + F_r(t), \tag{52}$$

where

$$F_{in}(t) = -2g\,\partial_x\phi_{in}(t,x)\,|_{x=z(t)} \tag{53}$$

is the force that the incoming radiation field exerts on the particle, and

$$F_r(t) = -2g\,\partial_x\phi_r(t,x)\,|_{x=z(t)} \tag{54}$$

is the radiation reaction force that the particle exerts on itself. If we substitute for the gradient of the retarded field using equation (44), we find

$$F_r(t) = -2g^2\,(1 - v^2(t))^{-1}\,v(t). \tag{55}$$

Thus, the equation of motion is

$$m\ddot{z} + 2g^2\,(1 - \dot{z}^2)^{-1}\,\dot{z} = -\frac{dV}{dz} - 2g\,\partial_x\phi_{in}. \tag{56}$$

Given the initial state of the particle, together with the *in* field obtained in section 3.2, this equation of motion can be integrated to give the particle trajectory.

For small velocities, we can approximate the radiation reaction force as

$$F_r = -\gamma m\dot{z}, \tag{57}$$

where $\gamma \equiv 2g^2/m$. In this limit, the equation of motion for the particle is

$$m\ddot{z} + m\gamma\dot{z} = -\frac{dV}{dz} - 2g\,\partial_x\phi_{in}. \tag{58}$$

Thus, the radiation reaction force gives a damping term with damping constant γ.

One might be tempted to argue that the time asymmetric damping term in equation (58) is due to a time asymmetry of the theory, but this would not be correct. The time asymmetry of the damping term only reflects our time asymmetric decomposition of the field into an *in* field and a retarded field. We could equally well decompose the field into an *out* field and an advanced field, in which case the equation of motion would be

$$m\ddot{z} - m\gamma\dot{z} = -\frac{dV}{dz} - 2g\,\partial_x\phi_{out}. \tag{59}$$

Note that the sign of the damping term has flipped.

3.4. Scattering from a harmonically bound particle

Let us now consider the problem of scattering from a harmonically bound particle. Consider a wavepacket that starts to the left of the particle and propagates toward it. We will assume the wavepacket is very broad and may be approximated as monochromatic:

$$\phi_{in}(t,x) = \phi_I \, e^{-i(\omega t - kx)}, \tag{60}$$

where $\omega = k$, since the wave propagates to the right. From equation (58), we find that the equation of motion for the particle is

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = -i(\gamma\omega/g)\phi_I \, e^{-i(\omega t - kz)}, \tag{61}$$

where ω_0 is the frequency of the harmonic oscillator. Let us assume that the displacement of the particle is much less than the wavelength of the radiation ($kz \ll 1$), so we can make the dipole approximation:

$$\ddot{z} + \gamma \dot{z} + \omega_0^2 z = -i(\gamma\omega/g)\phi_I \, e^{-i\omega t}. \tag{62}$$

The solution is

$$z(t) = (i/g)\phi_I \, f(\omega) \, e^{-i\omega t}, \tag{63}$$

where

$$f(\omega) = \gamma\omega \, (\omega^2 - \omega_0^2 + i\gamma\omega)^{-1}. \tag{64}$$

The velocity of the particle is

$$v(t) = -i\omega \, z(t) = (\omega/g)\phi_I \, f(\omega) \, e^{-i\omega t}. \tag{65}$$

Thus, $kz \ll 1$ implies $v \ll 1$. In this limit, the time derivative of the retarded field is

$$\partial_t \phi_r(t,x) = -g \, v(t_r) \, \epsilon(x - z(t_r)) = -\omega\phi_I \, f(\omega) \, e^{-i\omega t_r} \, \epsilon(x - z(t_r)). \tag{66}$$

Since $t_r = t - |x - z(t_r)|$, in the dipole approximation

$$e^{-i\omega t_r} \simeq e^{-i\omega(t - |x|)}. \tag{67}$$

Also, far away from the particle ($|x| \gg |z|$),

$$\epsilon(x - z(t_r)) = \epsilon(x). \tag{68}$$

Thus, we may simplify our expression for the time derivative of the field:

$$\partial_t \phi_r(t,x) = -\omega\phi_I \, f(\omega) \, e^{-i\omega(t-|x|)} \, \epsilon(x). \tag{69}$$

The total field is given by the sum of the *in* field and the retarded field; thus, the time derivative of the total field is

$$\partial_t \phi(t,x) = \begin{cases} A_i \, e^{-i(\omega t - kx)} + A_r \, e^{-i(\omega t + kx)} & \text{for } x < 0 \\ A_t \, e^{-i(\omega t - kx)} & \text{for } x > 0, \end{cases} \tag{70}$$

where

$$A_i = -\, i\omega\phi_I \tag{71}$$

$$A_r = i f(\omega) \, A_i \tag{72}$$

$$A_t = (1 - i f(\omega)) \, A_i \tag{73}$$

are the amplitudes of the incoming, reflected, and transmitted waves. The corresponding intensities are

$$I_i = \omega^2 |\phi_I|^2 \tag{74}$$

$$I_r = |f(\omega)|^2 \, I_i \tag{75}$$

$$I_t = (1 - |f(\omega)|^2) \, I_i, \tag{76}$$

where we have used that

$$\mathrm{Im} f = -|f|^2. \tag{77}$$

Note that the intensities obey the conservation equation $I_i = I_r + I_t$.

4. Field energy and momentum

Let us now consider the conservation of energy and momentum in the model field theory. The energy density $u(t,x)$ and momentum density $s(t,x)$ for the field are

$$u(t,x) = \frac{1}{2}(\pi(t,x)^2 + (\partial_x \phi(t,x))^2) \tag{78}$$

$$s(t,x) = -\pi(t,x) \, \partial_x \phi(t,x). \tag{79}$$

These expressions may be obtained using the relations†

$$u(t,x) = \partial_t \phi(t,x) \, \frac{\partial \mathcal{L}_f}{\partial(\partial_t \phi(t,x))} - \mathcal{L}_f \tag{80}$$

$$s(t,x) = -\partial_x \phi(t,x) \, \frac{\partial \mathcal{L}_f}{\partial(\partial_t \phi(t,x))}, \tag{81}$$

where

$$\mathcal{L}_f = \frac{1}{2}((\partial_t \phi(t,x))^2 - (\partial_x \phi(t,x))^2) \tag{82}$$

is the Lagrangian density corresponding to the field Hamiltonian H_f. Note also that the field Hamiltonian may be expressed as an integral over the energy density:

$$H_f = \int u(t,x) \, dx. \tag{83}$$

The energy density $u(t,x)$ can also be interpreted as a momentum flux, and the momentum density $s(t,x)$ can also be interpreted as an energy flux. The total energy and momentum of the field in the region $[-L, +L]$ are given by

$$E_f = \int_{-L}^{+L} u(t,x) \, dx \tag{84}$$

$$P_f = \int_{-L}^{+L} s(t,x) \, dx. \tag{85}$$

† See [6], Chap. 12 for a derivation of the energy and momentum density using Lagrangian methods. Expressions for these quantities can also be obtained by considering a mechanical model of the field; this approach is discussed in [7], Chap. 1.

The total energy of the particle is

$$E_p = \frac{p^2}{2m} + V(z), \tag{86}$$

and the interaction energy due to the coupling of the particle to the field is

$$E_i = 2g\phi(t, z(t)). \tag{87}$$

Using the equations of motion, we can calculate the rates of change of these energies. For the particle energy, we find

$$\dot{E}_p = -2g\, v\, \partial_x \phi \mid_{x=z} = vF_f, \tag{88}$$

where, as before, F_f is the force that the field exerts on the particle. For the interaction energy, we find

$$\dot{E}_i = 2g(\pi + v\, \partial_x \phi) \mid_{x=z} . \tag{89}$$

Using the field equation, we can write down a continuity equation for the field energy density:

$$\partial_t u + \partial_x s = -2g\pi\, \delta(x - z(t)) = -(\dot{E}_p + \dot{E}_i)\, \delta(x - z(t)). \tag{90}$$

If we integrate the continuity equation over $[-L, +L]$, we find

$$\dot{E}_f + s(t, +L) - s(t, -L) = -(\dot{E}_p + \dot{E}_i). \tag{91}$$

Because the radiation fields are entirely contained in the spatial region $[-L, +L]$, there is no outgoing energy flux ($s(t, \pm L) = 0$). Thus, we obtain an energy conservation equation:

$$\frac{d}{dt}(E_f + E_i + E_p) = 0. \tag{92}$$

We can also use the field equation to write down a continuity equation for the field momentum density:

$$\partial_t s + \partial_x u = 2g\, \delta(x - z(t))\, \partial_x \phi = -F_f\, \delta(x - z(t)). \tag{93}$$

If we integrate this equation over the region $[-L, +L]$, we find

$$\dot{P}_f + u(t, +L) - u(t, -L) = -F_f. \tag{94}$$

Again, because the radiation fields are contained within $[-L, +L]$, only the static field is present at $\pm L$, so $u(t, +L) = u(t, -L)$. Thus,

$$\dot{P}_f = -F_f, \tag{95}$$

and we obtain a momentum conservation equation:

$$\frac{d}{dt}(P_f + p) + \frac{dV}{dz} = 0, \tag{96}$$

where equation (11), the equation of motion for the particle, has been used to relate F_f and $\dot{p}$.

4.1. Energy of In and Out Fields

We have seen that ϕ can be decomposed into an *in* field and a retarded field, and that for $t < t_i$ the *in* field represents the dynamical radiation field while the retarded field represents the static field bound to the particle. Thus, for $t < t_i$ it is of interest to consider the energy density of the *in* field alone:

$$u_{in}(t,x) = \frac{1}{2}[\pi_{in}^2(t,x) + (\partial_x \phi_{in}(t,x))^2]. \tag{97}$$

For $t < t_i$ we have that

$$\phi(t,x) = \phi_{in}(t,x) + \psi_i(t,x). \tag{98}$$

If we substitute this result into equation (78) for the energy density of the total field, we find

$$u = u_{in} + g\partial_x(\phi\, \epsilon(x - z_i)) - 2g\phi\, \delta(x - z_i) - \frac{1}{2}g^2. \tag{99}$$

Thus, the total field energy can be expressed as

$$E_f = \int_{-L}^{+L} u(t,x)\, dx \tag{100}$$

$$= E_{in} + g[\phi(t,L) + \phi(t,-L)] - 2g\phi(t,z_i) - \frac{1}{2}g^2(2L), \tag{101}$$

where

$$E_{in} = \int_{-L}^{+L} u_{in}(t,x)\, dx \tag{102}$$

is the total energy of the *in* field. Since

$$\phi(t,L) + \phi(t,-L) = \psi_i(t,L) + \psi_i(t,-L) = 2gL, \tag{103}$$

we can express the total field energy as

$$E_f = E_{in} + E_{bound} - 2g\phi(t,z_i), \tag{104}$$

where

$$E_{bound} = \frac{1}{2}\int_{-L}^{+L} (\partial_x \psi_i)^2\, dx = \frac{1}{2}g^2(2L) \tag{105}$$

is the energy of the static field bound to the stationary charge. Note that because the charge is stationary, its energy is $E_p = V(z_i)$. The total energy of the system is therefore

$$E_f + E_i + E_p = E_{in} + E_{bound} + V(z_i). \tag{106}$$

Thus, for $t < t_i$, the total energy is just the sum of the energies of the radiation field and the static field, plus the potential energy of the particle.

We can obtain a similar result for the *out* field, for times $t > t_f$:

$$E_f + E_i + E_p = E_{out} + E_{bound} + V(z_f). \tag{107}$$

4.2. Radiated power

Let us calculate the power radiated by a particle moving along a trajectory $z(t)$. This can be accomplished by evaluating the energy density of the *out* field for $t > t_f$:

$$u_{out}(t,x) = \frac{1}{2}[(\partial_t\phi_{out}(t,x))^2 + (\partial_x\phi_{out}(t,x))^2]. \tag{108}$$

We will assume that there is no incoming radiation ($\phi_{in} = 0$). Then for $t > t_f$, the total field may be expressed as

$$\phi(t,x) = \phi_{out}(t,x) + \psi_f(t,x) = \phi_r(t,x), \tag{109}$$

so

$$\phi_{out}(t,x) = \phi_r(t,x) - \psi_f(t,x). \tag{110}$$

Since $z(t) = z_f$ for $t > t_f$, the time derivative and gradient of the retarded field, as given by equations (38) and (39), are

$$\partial_t\phi_r(t,x) \; = \; -\, g[1 - v\,\epsilon(x - z_f)]^{-1}\, v\,\epsilon(x - z_f) \tag{111}$$

$$\partial_x\phi_r(t,x) = g[1 - v\,\epsilon(x - z_f)]^{-1}\,\epsilon(x - z_f), \tag{112}$$

where $v \equiv v(t_r)$. The time derivative and gradient of the static field are

$$\partial_t\psi_f(t,x) \; = 0 \tag{113}$$

$$\partial_x\psi_f(t,x) = g\,\epsilon(x - z_f). \tag{114}$$

Thus, the time derivative and gradient of the *out* field are

$$\partial_t\phi_{out}(t,x) \; = \; -\, g[1 - v\,\epsilon(x - z_f)]^{-1}\, v\,\epsilon(x - z_f) \tag{115}$$

$$\partial_x\phi_{out}(t,x) = g[1 - v\,\epsilon(x - z_f)]^{-1}\, v. \tag{116}$$

If we substitute these expressions into equation (108) for the *out* field energy density, we obtain

$$u_{out}(t,x) = g^2 v^2 [1 - v\,\epsilon(x - z_f)]^{-2}. \tag{117}$$

Thus, the total energy radiated by the particle during the interval $[t_r, t_r + \Delta t]$ is (see Figure 2)

$$\Delta E = u_{out}(t, z(t_r) + (t - t_r))\,(1 - v)\,\Delta t + \\ u_{out}(t, z(t_r) - (t - t_r))\,(1 + v)\,\Delta t \tag{118}$$

$$= 2g^2 \frac{v^2}{1 - v^2}\,\Delta t. \tag{119}$$

The radiated power is therefore

$$P = \frac{\Delta E}{\Delta t} = 2g^2 \frac{v^2}{1 - v^2}. \tag{120}$$

This agrees with what we would expect based on the radiation reaction force calculated in section 3.3. Note that a particle moving at a constant velocity radiates; as we will discuss in section 6, the theory is not Galilean invariant.

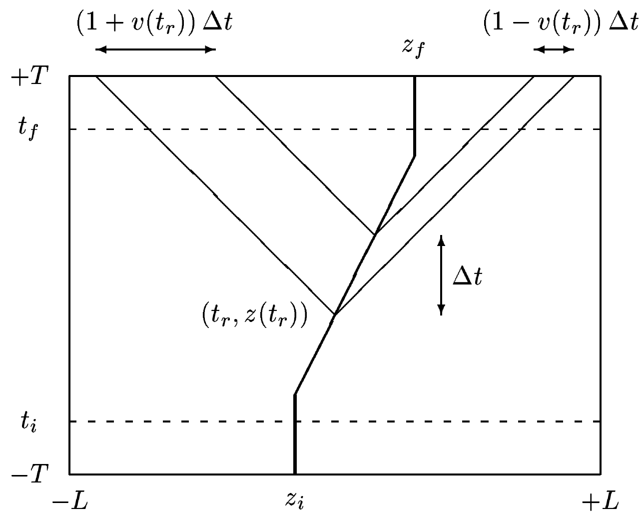

Figure 2. Energy radiated by the particle during the time interval $[t_r, t_r + \Delta t]$.

5. Extended particles

We can generalize the model theory to the case of spatially extended particles by changing the form of the interaction Hamiltonian:

$$H_i = 2\int \phi(t,x)\rho(t,x)\,dx. \tag{121}$$

Here $\rho(t,x)$ is the charge density; it is given by

$$\rho(t,x) = g\,f(x - z(t)), \tag{122}$$

where $f(x)$ is the charge density profile of the particle and $z(t)$ is the position of the particle at time t. We will assume that the charge density profile is normalized such that

$$\int f(x)\,dx = 1. \tag{123}$$

Note that the charge density satisfies the continuity equation

$$\partial_t\rho(t,x) + v(t)\,\partial_x\rho(t,x) = 0. \tag{124}$$

The new equations of motion for the field variables are

$$\partial_t\phi(t,x) = \pi(t,x) \tag{125}$$

$$\partial_t\pi(t,x) = \partial_x^2\phi(t,x) - 2\rho(t,x), \tag{126}$$

and the equations of motion for the particle variables are

$$\dot{z} = p/m \tag{127}$$

$$\begin{aligned}\dot{p} &= -\frac{dV}{dz} + 2\int \partial_x\rho(t,x)\,\phi(t,x)\,dx \\ &= -\frac{dV}{dz} - 2\int \rho(t,x)\,\partial_x\phi(t,x)\,dx,\end{aligned} \tag{128}$$

where in the equation of motion for the particle momentum we have integrated by parts and assumed that the charge density vanishes as $x \to \pm\infty$. From the equations of motion, we find that the field equation is

$$\Box\phi(t,x) = -2\rho(t,x), \tag{129}$$

and the force on the particle is

$$F = m\ddot{z} = -\frac{dV}{dz} + F_f, \tag{130}$$

where

$$F_f = -2\int \rho(t,x)\,\partial_x\phi(t,x)\,dx \tag{131}$$

is the force that the field exerts on the particle. As before, we can express the total field as the sum of an *in* field and a retarded field ($\phi = \phi_{in} + \phi_r$), so $F_f = F_{in} + F_r$, where

$$F_{in}(t) = -2\int \rho(t,x)\,\partial_x\phi_{in}(t,x)\,dx \tag{132}$$

is the force that the *in* field exerts on the particle, and

$$F_r(t) = -2\int \rho(t,x)\,\partial_x\phi_r(t,x)\,dx \tag{133}$$

is the force that the particle exerts on itself. The retarded field is still given by equation (28), only now $\rho(t,x)$ is given by equation (122). From equation (28) we find that the gradient of the retarded field is

$$\partial_x\phi_r(t,x) = \int \epsilon(x-y)\,\rho(t-|x-y|,y)\,dy. \tag{134}$$

In the following sections we present exact calculations for the total field energy of a stationary extended particle and for the radiation reaction force of an extended particle moving at a constant velocity, and then we derive approximate results for the self-energy and radiation reaction force of an extended particle in arbitrary motion.

5.1. *Extended particle at rest*

We will begin by calculating the field energy for an extended particle at rest. Recall that the corresponding quantity for a point charge, which we called E_{bound}, was given by equation (105). From equation (28), we find that the retarded field is

$$\phi_r(x) = \int |x-y|\,\rho(y)\,dy. \tag{135}$$

The field energy is

$$\begin{aligned} E_f &= \frac{1}{2}\int_{-L}^{+L} (\partial_x\phi_r(x))^2\,dx \\ &= \frac{1}{2}\phi_r(x)\,(\partial_x\phi_r(x))|_{x=-L}^{x=+L} - \int \phi_r(x)\rho(x)\,dx \\ &= E_{bound} - \int \phi_r(x)\rho(x)\,dx, \end{aligned} \tag{136}$$

and the interaction energy is

$$E_i = 2 \int \phi_r(x)\rho(x)\, dx, \tag{137}$$

so the total energy is

$$E_f + E_i = E_{bound} + \int \phi_r(x)\rho(x)\, dx = E_{bound} + \iint |x-y|\, \rho(x)\rho(y)\, dx\, dy.$$

Note that the total energy increases if the particle expands; this is because like charges attract.

5.2. Extended particle moving at constant velocity

Let us now calculate the radiation reaction force for an extended particle moving at a constant velocity v. From equation (134), we find that the gradient of retarded field is

$$\partial_x \phi_r(t,x) = gv(1-v^2)^{-1} + g(1-v^2)^{-1} \int \epsilon(x - vt - u)\, f(u)\, du. \tag{138}$$

If we substitute this result into equation (133) for the retarded force, we obtain

$$F_r = \, - 2g^2 v(1-v^2)^{-1}, \tag{139}$$

which is the same as the retarded force for a point particle.

5.3. Extended particle in arbitrary motion

Now let us generalize to the case of arbitrary motion†. Substituting equation (134) for the gradient of the retarded field into equation (133) for the retarded force, we obtain

$$F_r(t) = -2 \iint \rho(t,x)\, \rho(t - |x-y|, y)\, \epsilon(x-y)\, dx\, dy.$$

We can expand the charge density in a power series in $|x-y|$:

$$\rho(t - |x-y|, y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} |x-y|^n\, \partial_t^n \rho(t,y). \tag{140}$$

Using equation (122) for the charge density, we see that if we keep only terms of linear order, then for $n > 0$

$$\partial_t^n \rho(t,y) = -\frac{d^n z(t)}{dt^n}\, \partial_y \rho(t,y). \tag{141}$$

So

$$\rho(t - |x-y|, y) = \rho(t,y) - \partial_y \rho(t,y) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} |x-y|^n \frac{d^n z(t)}{dt^n}.$$

If we substitute this into equation (140), we obtain

$$F_r(t) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{d^n z(t)}{dt^n} \iint \rho(t,x)\, (\partial_y \rho(t,y))\, |x-y|^n\, \epsilon(x-y)\, dx\, dy.$$

† The calculations in this section are patterned after the derivation of the Abraham-Lorentz self-force presented in [4], Chap. 17.3.

Using that

$$\partial_y[|x-y|^n\,\epsilon(x-y)] = -n|x-y|^{n-1}, \tag{142}$$

we can integrate by parts to obtain

$$F_r(t) = -2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\,\frac{d^{n+1}z(t)}{dt^{n+1}}\iint \rho(t,x)\,\rho(t,y)\,|x-y|^n\,dx\,dy. \tag{143}$$

The first few terms in the series are

$$F_r = -2g^2\,\dot{z} - m_S\,\ddot{z} + \cdots, \tag{144}$$

where

$$m_S \equiv -2g^2\iint |u-v|\,f(u)f(v)\,du\,dv \tag{145}$$

is the self-energy of the particle.

If the velocities and accelerations are low enough that a linear approximation is justified, and if the particle is small enough that we can truncate the series expansion of the retarded force at the second order, then the equation of motion for the particle is

$$m\,\ddot{z} = F_{in} + F_r = -2g\partial_x\phi_{in} - 2g^2\,\dot{z} - m_S\,\ddot{z}. \tag{146}$$

Let us define a renormalized mass $m_R = m + m_S$ and a damping constant $\gamma = 2g^2/m_R$. We can then express the equation of motion as

$$m_R\,\ddot{z} + m_R\gamma\dot{z} = -2g\partial_x\phi_{in}. \tag{147}$$

6. Comparison of the toy model and electrodynamics

In this section we compare the toy model in $1+1$ dimensions with electrodynamics in $3+1$ dimensions.

6.1. Scattering

Let us first consider the scattering of radiation by a charged particle in electrodynamics. Just as for the toy model, we can define times t_i (t_f) before which (after which) the particle and radiation do not interact, and we can decompose the total field $F^{\mu\nu}$ into an *in* field and a retarded field, or into an *out* field and an advanced field:

$$F^{\mu\nu} = F^{\mu\nu}_{in} + F^{\mu\nu}_{r} = F^{\mu\nu}_{out} + F^{\mu\nu}_{a}. \tag{148}$$

For electrodynamics we can decompose the total field even further by splitting† the retarded (advanced) field into a velocity field $F^{\mu\nu}_{vel,r}$ ($F^{\mu\nu}_{vel,a}$) that falls off like $1/r^2$, and an acceleration field $F^{\mu\nu}_{acc,r}$ ($F^{\mu\nu}_{acc,a}$) that falls off like $1/r$:

$$F^{\mu\nu}_{r} = F^{\mu\nu}_{vel,r} + F^{\mu\nu}_{acc,r} \tag{149}$$

$$F^{\mu\nu}_{a} = F^{\mu\nu}_{vel,a} + F^{\mu\nu}_{acc,a}. \tag{150}$$

† For a discussion of this splitting into velocity and acceleration fields, see [1], Chap. 4.8, [4], p. 657, [3], p. 424, or [5], p. 169.

Because of the way these fields fall off with distance, we see that the acceleration fields transport energy out to (or in from) infinity, but the velocity fields do not. Physically, we interpret this by saying that the acceleration fields correspond to radiation, while the velocity fields describe the field that is permanently bound to the charge. Indeed, the velocity fields are just the Coulomb field of the charge boosted into a moving frame, and are sometimes referred to as generalized Coulomb fields. Thus, at all times we can split the total field $F^{\mu\nu}$ into a part that corresponds to radiation and a part that is bound to the charge:

$$F^{\mu\nu} = F^{\mu\nu}_{rad,r} + F^{\mu\nu}_{vel,r} = F^{\mu\nu}_{rad,a} + F^{\mu\nu}_{vel,a}, \tag{151}$$

where

$$F^{\mu\nu}_{rad,r} = F^{\mu\nu}_{in} + F^{\mu\nu}_{acc,r} \tag{152}$$

$$F^{\mu\nu}_{rad,a} = F^{\mu\nu}_{out} + F^{\mu\nu}_{acc,a} \tag{153}$$

represent the part of the field that corresponds to radiation.

The decomposition into velocity fields and acceleration fields does not carry over to the toy model, and therefore at an arbitrary time t we have no way of splitting the total field ϕ into a radiation field and a bound field, as we did for electrodynamics. As we have seen, however, such a splitting can be performed in the toy model for times $t < t_i$ or times $t > t_f$. To understand this, let us restrict our attention to times $t > t_f$ and compare the situation in the toy model with the situation in electrodynamics. For the toy model, at times $t > t_f$ the advanced field is just the static field ψ_f, so the total field is

$$\phi = \phi_{out} + \psi_f. \tag{154}$$

In electrodynamics, the field $F^{\mu\nu}_{acc,a}$ vanishes for $t > t_f$, so the total field is

$$F^{\mu\nu} = F^{\mu\nu}_{out} + F^{\mu\nu}_{vel,a}. \tag{155}$$

In both systems, energy that has been lost by the particle due to radiation damping shows up in the *out* field at times $t > t_f$, and the *out* field transports this energy out to infinity (we showed this for the toy model in section 4.2). Note that it is the *out* field alone that transports energy to infinity. For electrodynamics, the field $F^{\mu\nu}_{vel,a}$ does not transport energy to infinity because it falls off like $1/r^2$. For the toy model, the field ψ_f gives a uniform energy density that extends to infinity, but it does not transport energy because it is static‡. Thus, if we take the property of transporting energy out to infinity to be our criterion for radiation§, then in both systems we can split the total field into a radiation field and a bound field. The radiation field is just the *out* field, and the bound field is $F^{\mu\nu}_{vel,a}$ for electrodynamics and ψ_f for the toy model. In both systems, splitting

‡ Note that by using equation 154, we can express the energy flux as $s = -\pi\,\partial_x\phi = s_{out} - \pi_{out}\,\partial_x\psi_f$. Thus, if the *out* field vanishes and only the static field is present, there is no energy flux.

§ Indeed, for electrodynamics radiation is often defined in this very way (see [3], p. 414). There are, however, other criteria for radiation that are sometimes used in electrodynamics, which do not generalize to the toy model. For example, there is the null field criterion, in which one considers a field $F^{\mu\nu}$ to be a radiation field if it satisfies $F^{\mu\nu}F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} = 0$ (see [5], Chap. V.2).

the total field into a radiation field and a bound field gives a corresponding splitting of the field energy, so the total energy is $E = E_{out} + E_{bound} + V$, where E_{out} is the field energy associated with the *out* field, E_{bound} is the field energy associated with the bound field, and V is the potential energy of the particle (we showed this for the toy model in section 4.1).

6.2. Spacetime symmetry

An important difference between electrodynamics and the toy model is that electrodynamics is Lorentz invariant, whereas the toy model is neither Lorentz invariant nor Galilean invariant. The toy model is not Lorentz invariant because non-relativistic dynamics are used for the particle‖, and it is not Galilean invariant because the field equation for the model, the inhomogeneous wave equation, is not invariant under Galilean transformations. Because of this lack of spacetime symmetry, there is a preferred reference frame for the toy model: namely, the frame in which the radiation reaction force on the particle vanishes.

Thus, the situation in the toy model is similar to the situation in electrodynamics before the development of special relativity. At that time, it seemed that the fact that Maxwell's equations were not Galilean invariant implied that there must be a preferred reference frame for electrodynamics. This preferred frame was taken to be the rest frame of the ether, a hypothetical medium which was thought to support the propagation of electromagnetic waves. It was later realized that if one modified the laws of mechanics appropriately, the coupled particle-field system would be invariant under Lorentz transformations, and this invariance would rule out the existence of a preferred frame.

6.3. Radiation damping and self-energy

In section 3.3, we showed that in the toy model the equation of motion for the particle involves a radiation damping term that is proportional to velocity in the low-velocity limit. The analogous equation of motion for electrodynamics involves the time derivative of the acceleration, which leads either to "runaway" solutions or to solutions that "preaccelerate" when an external force is applied (these solutions are discussed in [4], Chap. 17). Because the equation of motion for the toy model only involves the velocity, it is free from these difficulties.

In section 5, we calculated the self-energy for an extended particle in the toy model. We found that the self-energy scales linearly with the size of the particle, and vanishes in the limit of a point particle. This is in contrast to electrodynamics, for which the self-energy of a particle of size a scales like $1/a$, and diverges in the limit of a point particle. The difference is due to the very different behavior of static fields in the two systems. In general, electrodynamics is well behaved at long distances, but poorly behaved at short

‖ Note also that the source term in the field equation (equation 10) is not a Lorentz scalar.

distances, since the Coulomb field of a point particle goes to zero as we move away from the particle but diverges as we approach the particle. The opposite is true for the toy model: the static field gradient $\partial_x \psi$ of a point particle remains finite as we approach the particle, but does not fall off as we move away from the particle. This is why in the toy model we needed to regulate integrals over space by introducing a length L, whereas in electrodynamics we can usually just take integrals out to infinity without having to worry about about convergence issues.

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