Quantum field theory in $(0 + 1)$ dimensions

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Abstract. We show that many of the key ideas of quantum field theory can be illustrated simply and straightforwardly by using toy models in $(0 + 1)$ dimensions. Because quantum field theory in $(0+1)$ dimensions is equivalent to quantum mechanics, these models allow us to use techniques from quantum mechanics to gain insight into quantum field theory. In addition, working in $(0+1)$ dimensions considerably simplifies the mathematics, allowing the physical concepts involved to be exhibited more clearly.

PACS numbers: 03.70.+k, 11.10.-z, 11.10.Kk, 03.65.-w

1. Introduction

Quantum field theory is a difficult subject to learn, as it involves both subtle physical concepts and a complicated mathematical formalism. In addition, quantum field theory is often introduced by using it to calculate scattering cross sections for real physical processes, and the complexity of these calculations can obscure the physical principles at work. In this paper, we show that many of these principles can be illustrated simply and straightforwardly by using toy models in $(0+1)$ dimensions. These toy models provide examples that could supplement a first course in quantum field theory† or elementary particle physics, and should help clarify the conceptual structure of the theory.

Quantum field theory in $(0+1)$ dimensions is formally equivalent to quantum mechanics, as can be understood from the following considerations. A scalar field ϕ in $(n+1)$ dimensions specifies a field value for each point in spacetime, and can therefore be viewed as a mapping of spacetime into the real numbers: $(t, x_1, \cdots, x_n) \rightarrow \phi(t, x_1, \cdots, x_n)$. The trajectory of a particle in $(n+1)$ dimensions specifies a point in space for each moment of time, and can therefore be viewed as a mapping of the real numbers into space: $t \rightarrow (x_1(t), \cdots, x_n(t))$. Thus, a field in $(0+1)$ dimensions and a particle in $(1+1)$ dimensions are described in the same way: the field is described by a mapping $t \rightarrow \phi(t)$, and the particle is described by a mapping $t \rightarrow x(t)$. By using this correspondence to translate problems back and forth between quantum mechanics and quantum field theory, we can exhibit the formalism of quantum field theory in the familiar setting of quantum mechanics.

The paper is organized as follows. In section 2, we discuss free fields in $(0+1)$ dimensions, using a neutral scalar field and a charged fermion field as examples. In section 3, we consider interacting fields; after discussing the physical meaning of interacting fields in $(0+1)$ dimensions, we present the $(0+1)$-dimensional analogs of three key results: the diagrammatic expansion of correlation functions, the Källén-Lehmann spectral representation of the two-point function, and the LSZ reduction formula. Together, these results establish the Feynman diagram formalism for calculating scattering matrix elements. We discuss the significance of these results for $(3+1)$-dimensional quantum field theory, and show how they can be understood in terms of our $(0+1)$-dimensional model. Finally, in section 4, we present detailed calculations for three example theories: a scalar field with a trivial ϕ^2 coupling, a scalar field with a ϕ^4 coupling, and a scalar field coupled to a charged fermion field via a Yukawa coupling. We treat each example using both the methods of quantum mechanics and the Feynman diagram formalism of quantum field theory.

† We will use Peskin and Schroeder's *An Introduction to Quantum Field Theory* as a standard reference for quantum field theory in $(3+1)$ dimensions. Other useful textbooks at a level appropriate for this paper are [2], [3], and [4].

2. Free fields

2.1. Neutral scalar field

The natural generalization of the Klein-Gordon equation† to $(0+1)$ dimensions is

$$(\partial_t^2 + m^2)\phi = 0. \tag{1}$$

The corresponding Hamiltonian is

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 = m(a^\dagger a + 1/2), \tag{2}$$

where we have quantized the system by introducing creation and annihilation operators $a^\dagger$ and a, which obey $[a, a^\dagger] = 1$:

$$\phi = (2m)^{-1/2}(a + a^\dagger) \tag{3}$$

$$\pi = -i(m/2)^{1/2}(a - a^\dagger). \tag{4}$$

The Hamiltonian H_0 has eigenstates $|n\rangle$ and eigenvalues $E_n = m(n+1/2)$, so it describes free particles of mass m, where $|n\rangle$ represents a state in which n particles are present. Because there are no interactions among the particles, the energy of an n-particle state is just the sum of the energies of each particle, plus the vacuum energy $m/2$.

In the Schrödinger picture, the state of the quantum field is given by a state vector $|\Phi\rangle$, which evolves in time according to the Schrödinger equation

$$i\frac{d}{dt}|\Phi\rangle = H_0|\Phi\rangle. \tag{5}$$

We can expand $|\Phi\rangle$ in the basis of energy eigenstates $|n\rangle$, or in a basis of eigenstates $|\phi_c\rangle$ of the field operator ϕ:

$$|\Phi\rangle = \sum_n |n\rangle\langle n|\Phi\rangle = \int d\phi_c\,|\phi_c\rangle\langle\phi_c|\Phi\rangle. \tag{6}$$

In these expansions, $|\langle n|\Phi\rangle|^2$ gives the probability that there are n particles present, and $|\langle\phi_c|\Phi\rangle|^2$ gives the probability that the field has the value ϕ_c.

In later sections we will be interested in interacting field theories, for which the total Hamiltonian H can be expressed as the sum of the free-field Hamiltonian H_0 and an interaction Hamiltonian H_i. In describing the time evolution of such systems, we can simplify the problem by working in the interaction picture, in which the time evolution due to H_0 is already taken into account. In particular, it is useful to introduce an interaction picture field operator $\phi_I(t)$, defined by evolving the Schrödinger picture field operator ϕ under the free-field Hamiltonian H_0:

$$\phi_I(t) = e^{iH_0t}\,\phi\,e^{-iH_0t} = (2m)^{-1/2}\,(a\,e^{-imt} + a^\dagger\,e^{imt}). \tag{7}$$

We will see that useful physical information can be extracted from the theory by calculating vacuum expectation values of products of interaction picture field operators. Of particular importance is the Feynman propagator, which is defined by

$$D_F(t_a - t_b) = \langle 0|T[\phi_I(t_a)\phi_I(t_b)]|0\rangle, \tag{8}$$

† The Klein-Gordon equation in $(3+1)$ dimensions is discussed in chapter 2 of [1] and chapter 3 of [2].

where T is a time-ordering operator that shifts operators evaluated at later times to the left and operators evaluated at earlier times to the right. For example, for two fields‡

$$T[\phi_I(t_a)\phi_I(t_b)] = \theta(t_a - t_b)\phi_I(t_a)\phi_I(t_b) + \theta(t_b - t_a)\phi_I(t_b)\phi_I(t_a). \tag{9}$$

Substituting for the field operators, we find that

$$D_F(t_a - t_b) = \frac{1}{2m} e^{-i(m-i\epsilon)|t_a - t_b|}, \tag{10}$$

where we have added a small imaginary component to the mass so that time integrals over the Feynman propagator converge. We can also define a Fourier transformed propagator $\tilde{D}_F(\omega)$, related to $D_F(t)$ by

$$D_F(t) = \frac{1}{2\pi} \int \tilde{D}_F(\omega)\, e^{-i\omega t}\, d\omega = \frac{1}{2m}\, e^{-i(m-i\epsilon)|t|} \tag{11}$$

$$\tilde{D}_F(\omega) = \int D_F(t)\, e^{i\omega t}\, dt = i(\omega^2 - m^2 + i\epsilon)^{-1}. \tag{12}$$

Note that the Feynman propagator is the Green's function for the Klein-Gordon equation with a delta function source:

$$(\partial_t^2 + m^2) D_F(t) = -i\delta(t). \tag{13}$$

In what follows, we will often simplify the notation by defining $\phi_k \equiv \phi_I(t_k)$ and $D_{ab} \equiv D_F(t_a - t_b)$.

2.2. *Charged fermion field*

The natural generalization of the Dirac equation§ to $(0 + 1)$ dimensions is

$$(i\gamma^0 \partial_t - m)\psi = 0, \tag{14}$$

where ψ is a two-component spinor, and γ^0 is given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{15}$$

Define spinors u and v by

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{16}$$

Then the quantized fermion field operator is

$$\psi = b\, u + c^\dagger\, v = \begin{pmatrix} b \\ c^\dagger \end{pmatrix}, \tag{17}$$

where b annihilates fermions and c annihilates antifermions. Because the particles are fermions, we quantize using anticommutators rather than commutators: $\{b, b^\dagger\} =$

‡ The following notation is used in this paper: $\theta(x)$ is the step function, defined such that $\theta(x) = 1$ for $x > 0$, $\theta(x) = 1/2$ for $x = 0$, $\theta(x) = 0$ for $x < 0$; $\epsilon(x)$ is the sign function, defined such that $\epsilon(x) = 1$ for $x > 0$, $\epsilon(x) = 0$ for $x = 0$, $\epsilon(x) = -1$ for $x < 0$; N is a normal-ordering operator that shifts creation operators to the left and annihilation operators to the right (for example, $N[a^\dagger a] = N[aa^\dagger] = a^\dagger a$).

§ The Dirac equation in $(3 + 1)$ dimensions is discussed in chapter 3 of [1] and chapter 4 of [2].

$\{c, c^\dagger\} = 1$, and all other anticommutators are zero. One consequence of the anticommutation relations is that $(b^\dagger)^2 = (c^\dagger)^2 = 0$. Thus, there are only four states of the system: the vacuum state $|0,0\rangle$, a state with one fermion $b^\dagger|0,0\rangle = |1,0\rangle$, a state with one antifermion $c^\dagger|0,0\rangle = |0,1\rangle$, and a state with one fermion and one antifermion $b^\dagger c^\dagger|0,0\rangle = |1,1\rangle$. We define the adjoint field operator $\bar{\psi}$ by

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 = b^\dagger\, u^\dagger - c\, v^\dagger = (b^\dagger, -c). \tag{18}$$

The Hamiltonian is

$$H_0 = m\bar{\psi}\psi = m(b^\dagger b - cc^\dagger) = m(b^\dagger b + c^\dagger c - 1). \tag{19}$$

Note that the vacuum energy is $-m$; each of the two modes contributes a vacuum energy $-m/2$, and because the modes describe fermions rather than bosons, the sign of the vacuum energy is negative rather than positive. The Feynman propagator for the Dirac equation is given by

$$S_F^{\alpha\beta}(t_a - t_b) = \langle 0|T[\psi_I^\alpha(t_a)\bar{\psi}_I^\beta(t_b)]|0\rangle, \tag{20}$$

where the time-ordering operator is defined such that there is a minus sign every time fermion field operators are exchanged‖. For two fields,

$$T[\psi_I^\alpha(t_a)\bar{\psi}_I^\beta(t_b)] = \theta(t_a - t_b)\psi_I^\alpha(t_a)\bar{\psi}_I^\beta(t_b) - \theta(t_b - t_a)\bar{\psi}_I^\beta(t_b)\psi_I^\alpha(t_a). \tag{21}$$

Substituting for the field operators, we find that the Feynman propagator $S_F(t)$ and its Fourier transform $\tilde{S}_F(\omega)$ are given by

$$S_F(t) = \frac{1}{2\pi}\int \tilde{S}_F(\omega)\, e^{-i\omega t}\, d\omega = \frac{1}{2}(1 + \epsilon(t)\,\gamma^0)\, e^{-i(m-i\epsilon)|t|} \tag{22}$$

$$\tilde{S}_F(\omega) = \int S_F(t)\, e^{i\omega t}\, dt = i(\omega\gamma^0 - m + i\epsilon)^{-1}. \tag{23}$$

3. Interactions

3.1. Physical interpretation of interacting field theories

The Hamiltonian H for a quantum field theory can often be expressed as the sum of a free Hamiltonian H_0 and an interaction Hamiltonian H_i:

$$H = H_0 + H_i. \tag{24}$$

The free Hamiltonian describes one or more free fields whose quanta are noninteracting particles, while the interaction Hamiltonian couples the fields together and causes the particles to interact. The interaction H_i modifies the free theory described by H_0 in a number of ways. First, it alters the vacuum state of the theory; in general, the vacuum state of the interacting theory is a superposition of multiparticle states of the free theory. Second, the interaction Hamiltonian not only causes particles to interact with one another, it also causes particles to interact with themselves. This means that

‖ The normal-ordering operator is also defined such that there is a minus sign every time fermion operators are exchanged (for example, $N[b^\dagger b] = -N[bb^\dagger] = b^\dagger b$).

the physical mass of the particles (also called the renormalized mass) is not the mass that appears in the free Hamiltonian; rather, it is the sum of the free mass and the mass-energy associated with the particle's self-interaction. Finally, the creation and annihilation operators that were defined for the free theory do not necessarily create and destroy particles for the interacting theory.

As an example, consider a theory describing a neutral scalar field. We discussed the free Hamiltonian H_0 for this theory in section 2.1; let us now consider what happens when we add an arbitrary interaction Hamiltonian H_i. Denote the eigenstates of the full Hamiltonian H by $|\bar{n}\rangle$ and the eigenvalues by $\bar{E}_n$. We can define creation and annihilation operators $d^\dagger$ and d for the interacting theory by

$$d \equiv \sum_{n=0}^{\infty} \sqrt{n+1}\, |\bar{n}\rangle\langle \bar{n}+1|. \tag{25}$$

From this expression, it follows that the commutation relation for the new operators is $[d, d^\dagger] = 1$, so they are related to a and $a^\dagger$ by a canonical transformation†. The operators d and $d^\dagger$ can be used to express the full Hamiltonian H in a physically revealing form. First, note that if we define constants V_k appropriately, we can write the eigenvalues of H as

$$\bar{E}_n = V_0 + V_1\, n + \frac{1}{2!} V_2\, n(n-1) + \frac{1}{3!} V_3\, n(n-1)(n-2) + \cdots + V_n \tag{26}$$

$$= \sum_{k=0}^{n} \frac{n!}{(n-k)!\, k!} V_k. \tag{27}$$

Specifically, given the eigenvalues $\bar{E}_n$, the V_n's are defined recursively by $V_0 = \bar{E}_0$, and

$$V_n = \bar{E}_n - \sum_{k=0}^{n-1} \frac{n!}{(n-k)!\, k!} V_k \tag{28}$$

for $n > 0$. For example, $V_1 = \bar{E}_1 - \bar{E}_0$, and $V_2 = \bar{E}_2 - 2\bar{E}_1 + \bar{E}_0$. Next, note that

$$(d^\dagger)^k d^k |\bar{n}\rangle = \frac{n!}{(n-k)!} |\bar{n}\rangle. \tag{29}$$

Thus, the full Hamiltonian H can be expressed as

$$H = \sum_{k=0}^{\infty} \frac{1}{k!} V_k\, (d^\dagger)^k d^k$$

$$= \bar{E}_0 + m_R\, d^\dagger d + \frac{1}{2!} V_2\, d^\dagger d^\dagger d d + \frac{1}{3!} V_3\, d^\dagger d^\dagger d^\dagger d d d + \cdots, \tag{30}$$

where we have substituted $\bar{E}_0$ for V_0 and defined $m_R \equiv V_1 = \bar{E}_1 - \bar{E}_0$. We can interpret this result by comparing it with the free Hamiltonian H_0:

$$H_0 = \frac{1}{2} m + m a^\dagger a. \tag{31}$$

We see that in going from H_0 to H the vacuum energy has changed from $m/2$ to $\bar{E}_0$, and the particle mass has changed from m, the free mass, to m_R, the renormalized

† We will write down this canonical transformation explicitly for the example theories given in sections 4.1 and 4.3.

mass. Also, there are n-body interactions among the individual particles, which are described by potentials V_n. In higher dimensions, the interactions lead to particle-particle scattering. In $(0+1)$ dimensions there is no scattering (there is nowhere for the particles to scatter to); rather, the interactions cause the energy of the system to depend on the number of particles in a nonlinear way. For example, consider a two-body interaction of strength V_2. If there are n particles present, then the number of particle pairs is $(1/2)n(n-1)$, and the interaction energy per pair is V_2, so the total interaction energy is $(1/2)n(n-1)V_2$. Thus, the two-body interaction gives a contribution to the total energy that is quadratic in the number of particles.

3.2. Diagrammatic expansion of correlation functions

As we shall see in the next two sections, a great deal of physical information about an interacting field theory can be extracted from its correlation functions. For the free theory, described by H_0, we define the n-point correlation function to be

$$\langle 0|T[\phi_I(t_1)\cdots\phi_I(t_n)]|0\rangle. \tag{32}$$

For the interacting theory, described by $H = H_0 + H_i$, we define the n-point correlation function to be

$$\langle\bar{0}|T[\phi(t_1)\cdots\phi(t_n)]|\bar{0}\rangle, \tag{33}$$

where $\phi(t)$ denotes the Heisenberg picture field operator, which is obtained by evolving the Schrödinger picture field operator ϕ under the full Hamiltonian H:

$$\phi(t) = e^{iHt}\,\phi\,e^{-iHt}. \tag{34}$$

One can show that the correlation functions for the interacting theory can be expressed as†

$$\langle\bar{0}|T[\phi(t_1)\cdots\phi(t_n)]|\bar{0}\rangle = \frac{\langle 0|T[\phi_I(t_1)\cdots\phi_I(t_n)\exp(-i\int H_I(t)\,dt)]|0\rangle}{\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle}, \tag{35}$$

where $H_I(t)$ is the interaction picture Hamiltonian:

$$H_I(t) = e^{iH_0t}\,H_i\,e^{-iH_0t}. \tag{36}$$

We can obtain a perturbation series by expanding equation (35) in H_I. Since H_I is built out of of interaction picture field operators, the terms of this series take the form of integrals over correlation functions of the free theory, and by using Wick's theorem (see Appendix A), the correlation functions can be expressed as products of Feynman propagators. Thus, we obtain a series expansion for the correlation functions of the interacting theory, the terms of which consist of integrals over products of Feynman propagators. Each term represents a specific spacetime process and may be depicted as a Feynman diagram: vertices in the diagram represent events, and lines connecting vertices represent the propagation of a particle from one event to another.

† This result is derived in section 4.2 of [1]; see also section 3.8 of [4]. It can be obtained by combining the Dyson expansion of the time-evolution operator with a formula due to Gell-Mann and Low [5] for the ground state of the interacting theory.

In section 4 we give several examples that illustrate this diagrammatic expansion, but before turning to these examples we will discuss two results that show how the correlation functions can be used to calculate physically useful quantities. First, in section 3.3, we show that information about the single-particle states of the interacting theory can be obtained from the two-point function. Next, in section 3.4, we show how this information can be combined with information extracted from higher-order correlation functions to calculate scattering matrix elements.

3.3. Källén-Lehmann spectral representation

Recall that for the free theory, the two-point correlation function defines the Feynman propagator:

$$\langle 0|T[\phi_I(t_a)\phi_I(t_b)]|0\rangle = D_F(t_a - t_b). \tag{37}$$

We will show that for the interacting theory, the two-point correlation function can be expressed as

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = |\langle\bar{0}|\phi|\bar{0}\rangle|^2 + \sum_{n=1}^{\infty}\rho_n\, D_F(t_a - t_b, \bar{E}_n - \bar{E}_0), \tag{38}$$

where

$$\rho_n = 2(\bar{E}_n - \bar{E}_0)|\langle\bar{n}|\phi|\bar{0}\rangle|^2, \tag{39}$$

and

$$D_F(\tau, \mu) = \frac{1}{2\mu}e^{-i(\mu - i\epsilon)|\tau|} \tag{40}$$

is the Feynman propagator for a particle of mass μ. The quantity ρ_1 is called the field strength renormalization, and is also denoted by Z:

$$Z \equiv \rho_1 = 2(\bar{E}_1 - \bar{E}_0)|\langle\bar{1}|\phi|\bar{0}\rangle|^2 = 2m_R|\langle\bar{1}|\phi|\bar{0}\rangle|^2. \tag{41}$$

The field strength renormalization can be interpreted as the probability of creating a single particle state of the interacting theory by acting on the interacting vacuum with the field operator ϕ. Since $m_R = \bar{E}_1 - \bar{E}_0$, equation (38) can also be written as

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \\ |\langle\bar{0}|\phi|\bar{0}\rangle|^2 + Z\, D_F(t_a - t_b, m_R) + \sum_{n=2}^{\infty}\rho_n\, D_F(t_a - t_b, \bar{E}_n - \bar{E}_0). \tag{42}$$

Thus, by computing the two-point function for the interacting theory and comparing it with equation (42), we can extract the renormalized mass and the field strength renormalization.

We can interpret equation (42), which is the $(0+1)$-dimensional equivalent of the Källén-Lehmann spectral representation† [6] [7], by comparing it with the analogous result for the free theory given in equation (37). For the free theory, ϕ creates single

† The Källén-Lehmann spectral representation is discussed in section 7.1 of [1]; see also section 16.4 of [3].

particle states of mass m, which propagate in time according to the Feynman propagator $D_F(t) = D_F(t, m)$. For the interacting theory, ϕ creates both single particle states of mass m_R and n-particle states of mass $\bar{E}_n - \bar{E}_0$. The probability of creating a single particle state is Z, and the probability of creating an n-particle state is ρ_n. Both single and n-particle states propagate according to the Feynman propagator, evaluated at their respective energies.

To prove (38), first note that

$$\begin{aligned}\langle\bar{0}|\phi(t_a)\phi(t_b)|\bar{0}\rangle &= \sum_{n=0}^{\infty}\langle\bar{0}|\phi(t_a)|\bar{n}\rangle\langle\bar{n}|\phi(t_b)|\bar{0}\rangle \\ &= |\langle\bar{0}|\phi|\bar{0}\rangle|^2 + \sum_{n=1}^{\infty} e^{-i(\bar{E}_n-\bar{E}_0)(t_a-t_b)}\,\langle\bar{0}|\phi|\bar{n}\rangle\langle\bar{n}|\phi|\bar{0}\rangle. \qquad (43)\end{aligned}$$

From the definition of the time-ordered product,

$$\begin{aligned}&\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = \\ &\theta(t_a - t_b)\,\langle\bar{0}|\phi(t_a)\phi(t_b)|\bar{0}\rangle + \theta(t_b - t_a)\,\langle\bar{0}|\phi(t_b)\phi(t_a)|\bar{0}\rangle. \qquad (44)\end{aligned}$$

If we substitute (43) into (44), and use the definitions (39) and (40) for ρ_n and the Feynman propagator, we obtain (38).

We will now show that ρ_n is normalized such that

$$\sum_{n=1}^{\infty}\rho_n = 1. \qquad (45)$$

The proof presented here is adapted from a similar proof for (3+1)-dimensional quantum field theory that is given in section 16.4 of [3]. First, note that by using equation (43) we can show that

$$\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle = -i\sum_{n=1}^{\infty}\rho_n\,G(t,\bar{E}_n - \bar{E}_0), \qquad (46)$$

where $G(t,\mu) \equiv (1/\mu)\sin\mu t$. Thus:

$$\partial_t\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = -i\sum_{n=1}^{\infty}\rho_n\,\partial_t G(t,\bar{E}_n - \bar{E}_0)\;|_{t=0} = -i\sum_{n=1}^{\infty}\rho_n. \qquad (47)$$

But we can also express the time derivative of the correlation function as

$$\partial_t\langle\bar{0}|[\phi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = \langle\bar{0}|[\pi(t),\phi(0)]|\bar{0}\rangle\;|_{t=0} = -i, \qquad (48)$$

where we have used that the Heisenberg picture operators $\phi(t)$ and $\pi(t)$ are related by $\partial_t\phi(t) = \pi(t)$ and obey the commutation relation $[\pi(t),\phi(t)] = -i$. By comparing equations (47) and (48), we obtain the normalization condition (45).

3.4. LSZ reduction formula

A key result of $(3+1)$-dimensional quantum field theory is the LSZ reduction formula† [8], which shows how scattering matrix elements can be extracted from correlation

† The LSZ reduction formula is discussed in section 7.2 of [1]; see also section 16.7 of [3].

functions. For example, consider the four-point correlation function for a theory describing a massive scalar field, in which the renormalized mass is m_R and the field strength renormalization is Z (in the previous section, we showed how these quantities can be obtained by using the Källén-Lehmann spectral representation). We will define the Fourier transform of the four-point correlation function by

$$f(p_1, p_2, q_1, q_2) \equiv \prod_{i=1}^{2} \int d^4x_i \, e^{ip_i \cdot x_i} \prod_{j=1}^{2} \int d^4y_j \, e^{-iq_j \cdot y_j} \, \langle \bar{0} | T[\phi(x_1)\phi(x_2)\phi(y_1)\phi(y_2)] | \bar{0} \rangle, \tag{49}$$

where p_i, q_j are 4-vectors. The LSZ reduction formula says that if we hold $\vec{p}_i$, $\vec{q}_j$ fixed, and consider the limit $p_i^0 \to E_{\vec{p}_i}$, $q_j^0 \to E_{\vec{q}_j}$, where $E_{\vec{k}} = (|\vec{k}|^2 + m_R^2)^{1/2}$, we find‡

$$f(p_1, p_2, q_1, q_2) \longrightarrow \left(\prod_{i=1}^{2} \frac{\sqrt{Z} i \sqrt{2E_{\vec{p}_i}}}{p_i^2 - m_R^2 + i\epsilon} \right) \left(\prod_{j=1}^{2} \frac{\sqrt{Z} i \sqrt{2E_{\vec{q}_j}}}{q_j^2 - m_R^2 + i\epsilon} \right) \langle \vec{p}_1 \vec{p}_2 | S | \vec{q}_1 \vec{q}_2 \rangle. \tag{50}$$

Here $\langle \vec{p}_1 \vec{p}_2 | S | \vec{q}_1 \vec{q}_2 \rangle$ is a scattering matrix element; it gives the probability amplitude that a pair of incoming particles with momenta $\vec{q}_1$, $\vec{q}_2$ will scatter into a pair of outgoing particles with momenta $\vec{p}_1$, $\vec{p}_2$. Similarly, scattering matrix elements for n-body to m-body scattering can be obtained from the $(n+m)$-point correlation function.

In $(0+1)$ dimensions there is no many-body scattering because there is no way to define an asymptotic limit in which multiple initial particles are spatially separated; however, we can still prove a result analogous to the LSZ reduction formula for the two-point function, corresponding to the trivial scattering process in which we start with a single particle and end up with a single particle. We define the Fourier transform of the two-point function by

$$f(\omega_a, \omega_b) \equiv \int_{-T}^{T} \int_{-T}^{T} dt_a \, dt_b \, e^{i\omega_a t_a} \, e^{i\omega_b t_b} \, \langle \bar{0} | T[\phi(t_a)\phi(t_b)] | \bar{0} \rangle, \tag{51}$$

where we have regulated the integrals by introducing a constant T, chosen such that $T \gg 1/m_R$. We will evaluate $f(\omega_a, \omega_b)$ exactly, and show that in the limit $\omega_a \to m_R$, $\omega_b \to -m_R$ it reduces to

$$f(\omega_a, \omega_b) \longrightarrow -\left(\frac{\sqrt{Z} i \sqrt{2m_R}}{\omega_a^2 - m_R^2 + i\epsilon} \right) \left(\frac{\sqrt{Z} i \sqrt{2m_R}}{\omega_b^2 - m_R^2 + i\epsilon} \right). \tag{52}$$

Because the scattering matrix element for this trivial one-body to one-body process is 1 (if we start with a single-particle state, we always end up with the same single-particle state), this result is analogous to equation (50) for $(3+1)$-dimensional two-body to two-body scattering.

To evaluate $f(\omega_a, \omega_b)$, we first substitute equation (38), the spectral representation for the two-point function, into equation (51) to obtain

$$f(\omega_a, \omega_b) =$$

‡ Equation (50) assumes that the single-particle states are normalized such that $\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q})$; this differs from the normalization condition $\langle \vec{p} | \vec{q} \rangle = (2E_{\vec{p}})(2\pi)^3 \, \delta^{(3)}(\vec{p} - \vec{q})$ used in [1].

$$|\langle\bar{0}|\phi|\bar{0}\rangle|^2\,(4/\omega_a\omega_b)\,\sin\omega_a T\,\sin\omega_b T + \sum_{n=1}^{\infty}\rho_n\,(2\mu_n)^{-1}g(\omega_a,\omega_b,\mu_n), \qquad (53)$$

where $\mu_n \equiv \bar{E}_n - \bar{E}_0$, and

$$g(\omega_a,\omega_b,\mu_n) \equiv \int_{-T}^{T}\int_{-T}^{T} dt_a\,dt_b\,e^{i\omega_a t_a}\,e^{i\omega_b t_b}\,e^{-i(\mu_n - i\epsilon)|t_a - t_b|}. \qquad (54)$$

If we define new variables $t = t_a + t_b$, $\tau = t_a - t_b$, we can express this as

$$g(\omega_a,\omega_b,\mu_n) = \frac{1}{2}\int_{-2T}^{2T} d\tau \int_{-(2T-|\tau|)}^{2T-|\tau|} dt\,e^{(i/2)(\omega_a+\omega_b)t}\,e^{(i/2)(\omega_a-\omega_b)\tau}\,e^{-i(\mu_n-i\epsilon)|\tau|}. \qquad (55)$$

We will evaluate the integrals, assuming $\epsilon T \gg 1$:

$$\begin{aligned} g(\omega_a,\omega_b,\mu_n) = &-(\omega_a+\omega_b)^{-1}[(\omega_a+\mu_n-i\epsilon)^{-1}\,e^{i(\omega_a+\omega_b)T} + \\ &(\omega_a-\mu_n+i\epsilon)^{-1}\,e^{-i(\omega_a+\omega_b)T} + \\ &(\omega_b+\mu_n-i\epsilon)^{-1}\,e^{i(\omega_a+\omega_b)T} + \\ &(\omega_b-\mu_n+i\epsilon)^{-1}\,e^{-i(\omega_a+\omega_b)T}] \end{aligned} \qquad (56)$$

Consider the limit $\omega_a \to m_R$, $\omega_b \to -m_R$. In this limit, equations (53) and (56) imply that

$$f(\omega_a,\omega_b) \to Z\,(2m_R)^{-1}\,g(\omega_a,\omega_b,m_R) \qquad (57)$$

and

$$\begin{aligned} g(\omega_a,\omega_b,m_R) \to &-(\omega_a+\omega_b)^{-1}[(\omega_a-m_R+i\epsilon)^{-1}+(\omega_b+m_R-i\epsilon)^{-1}] \\ \to &-(\omega_a-m_R+i\epsilon)^{-1}(\omega_b+m_R-i\epsilon)^{-1}. \end{aligned} \qquad (58)$$

Substituting equation (58) into equation (57), we obtain equation (52).

4. Examples

4.1. ϕ^2 interaction

We will first consider a theory describing a neutral scalar field, where the free Hamiltonian H_0 is given by equation (2) and the interaction Hamiltonian H_i is given by

$$H_i = \frac{\lambda}{2}\phi^2 = \frac{\lambda}{4m}(a+a^\dagger)^2. \qquad (59)$$

The full Hamiltonian is

$$H = H_0 + H_i = \frac{1}{2}\pi^2 + \frac{1}{2}(m^2+\lambda)\phi^2 = \frac{1}{2}\pi^2 + \frac{1}{2}m_R^2\phi^2, \qquad (60)$$

where $m_R = m(1+\lambda/m^2)^{1/2}$. Thus, the entire effect of the interaction is to shift the mass of the field from m, the bare mass, to m_R, the renormalized mass. We can simplify H by introducing operators d and $d^\dagger$, defined by

$$\phi = (2m_R)^{-1/2}(d+d^\dagger) = (2m)^{-1/2}(a+a^\dagger). \qquad (61)$$

In terms of the new operators, the Hamiltonian is

$$H = m_R(d^\dagger d + 1/2). \tag{62}$$

Using equation (61), we can write down the canonical transformation from a and $a^\dagger$ to d and $d^\dagger$:

$$d = \cosh r\, a + \sinh r\, a^\dagger, \tag{63}$$

where $r = (1/4)\log(1 + \lambda/m^2)$. We can also express d as†

$$d = S a S^\dagger, \tag{64}$$

where S is a unitary transformation given by

$$S = \exp((r/2)(aa - a^\dagger a^\dagger)). \tag{65}$$

To determine the eigenstates of the interacting theory, first note that from equation (64) it follows that the vacuum state $|\bar{0}\rangle$ for the interacting theory is related to the vacuum state $|0\rangle$ for the free theory by $|\bar{0}\rangle = S|0\rangle$. By acting on the vacuum state $|\bar{0}\rangle$ with the creation operator $d^\dagger$, the full spectrum of eigenstates for the interacting theory can be constructed.

The field strength renormalization Z can be obtained by calculating the two-point correlation function for the interacting theory. Using equation (61) to express the field operator ϕ in terms d and $d^\dagger$, we see that

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle = D_F(t_a - t_b, m_R). \tag{66}$$

If we compare this result with the spectral representation of the two-point function given in equation (42), we find that $Z = 1$ and $\rho_n = 0$ for $n > 1$.

We can also obtain these results by using the diagrammatic expansion given in equation (35). For the numerator, we find

$$\begin{aligned}
&\langle 0|T[\phi_a\phi_b \exp(-i\int H_I(t)\,dt)]|0\rangle = \\
&D_{ab} + \\
&(1/1!)(-i\lambda/2)\int (D_{ab}D_{11} + 2D_{a1}D_{1b})\,dt_1 + \\
&(1/2!)(-i\lambda/2)^2\iint (D_{ab}D_{11}D_{22} + 2D_{ab}D_{12}^2 + 2D_{a1}D_{1b}D_{22} + \\
&2D_{a2}D_{2b}D_{11} + 4D_{a1}D_{12}D_{2b} + 4D_{a2}D_{21}D_{1b})\,dt_1\,dt_2 + \cdots.
\end{aligned} \tag{67}$$

For the denominator,

$$\begin{aligned}
&\langle 0|T[\exp(-i\int H_I(t)\,dt)]|0\rangle = \\
&1 + \\
&(1/1!)(-i\lambda/2)\int D_{11}\,dt_1 + \\
&(1/2!)(-i\lambda/2)^2\iint (D_{11}D_{22} + 2D_{12}^2)\,dt_1\,dt_2 + \cdots.
\end{aligned} \tag{68}$$

† This can be shown by using the relation $e^A B e^{-A} = B + [A,B] + (1/2!)[A,[A,B]] + \cdots$, which holds for arbitrary operators A, B. The unitary transformation S is known as the squeezing transformation, and has important applications in quantum optics (see section 2.7 of [9]).

Thus, the two-point function for the interacting theory is

$$\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle =$$
$$D_{ab} + (-i\lambda)\int D_{a1}D_{1b}\,dt_1 + (-i\lambda)^2\iint D_{a1}D_{12}D_{2b}\,dt_1\,dt_2 + \cdots. \qquad (69)$$

Note that the denominator cancels the terms in the numerator that correspond to disconnected diagrams‡. The series takes on a simpler form if we look at the Fourier transform of the two-point function:

$$\begin{aligned}\int\langle\bar{0}|T[\phi(t)\phi(0)]|\bar{0}\rangle\, e^{i\omega t}\,dt &= \tilde{D}_F(\omega) + (-i\lambda)\tilde{D}_F^2(\omega) + (-i\lambda)^2\tilde{D}_F^3(\omega) + \cdots \\ &= (1 + i\lambda\tilde{D}_F(\omega))^{-1}\,\tilde{D}_F(\omega) \\ &= i(\omega^2 - (m^2+\lambda) + i\epsilon)^{-1} \\ &= \tilde{D}_F(\omega, m_R). \end{aligned} \qquad (70)$$

Thus, we find that $Z = 1$ and $m_R = m(1+\lambda/m^2)^{1/2}$, which agrees with our previous results.

4.2. ϕ^4 *interaction*

Let us now consider a neutral scalar field interacting via a ϕ^4 interaction (for simplicity, in this section we will set $m = 1$; it can always be restored by using dimensional analysis):

$$H_i = \lambda\phi^4 = (\lambda/4)(a + a^\dagger)^4. \qquad (71)$$

The total Hamiltonian is

$$H = H_0 + H_i = \frac{1}{2}\pi^2 + \frac{1}{2}\phi^2 + \lambda\phi^4. \qquad (72)$$

The Hamiltonian H cannot be solved in closed form; however, we can use the time-independent perturbation theory of quantum mechanics to obtain approximate expressions for the eigenstates and eigenvalues. To second order in λ, the energy eigenvalues are

$$\bar{E}_n = E_n + \langle n|H_i|n\rangle + \sum_{m\neq n}\frac{|\langle n|H_i|m\rangle|^2}{n-m}. \qquad (73)$$

Using these eigenvalues, we can calculate the renormalized mass:

$$m_R = \bar{E}_1 - \bar{E}_0 = 1 + 3\lambda - 18\lambda^2. \qquad (74)$$

To second order in λ, the eigenstates are (see equation (5.1.44) of [10])

$$|\bar{n}\rangle = \left(1 - \frac{1}{2}\sum_{k\neq n}\frac{|\langle k|H_i|n\rangle|^2}{(n-k)^2}\right)|n\rangle +$$

‡ A diagram is said to be "connected" if for any two vertices v_α, v_β there are vertices $w_1, \cdots, w_r$ such that $w_1 = v_\alpha$, $w_r = v_\beta$, and a propagator connects w_i to w_{i+1} for $i = 1, \cdots, r-1$. A diagram is "disconnected" if it is not connected. One can show that the denominator of (35) always cancels the terms in the numerator corresponding to disconnected diagrams (see section 4.4 of [1]); we have shown this explicitly for this example.

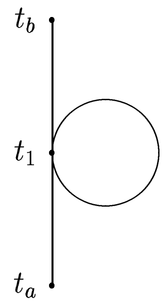

Figure 1. First order diagram for the two-point correlation function for ϕ^4 theory. Each vertex indicates a time t_j; a line connecting vertices for times t_j and t_k indicates a Feynman propagator D_{jk}.

$$\sum_{k\neq n}\left(\frac{\langle k|H_i|n\rangle}{n-k}-\frac{\langle n|H_i|n\rangle\langle k|H_i|n\rangle}{(n-k)^2}+\sum_{m\neq n}\frac{\langle k|H_i|m\rangle\langle m|H_i|n\rangle}{(n-k)(n-m)}\right)|k\rangle.$$

Using these eigenstates, together with our expression for the renormalized mass, we find that the field strength renormalization is

$$Z = 2m_R|\langle\bar{1}|\phi|\bar{0}\rangle|^2 = 1-\frac{9}{8}\lambda^2. \tag{75}$$

We can also obtain these results by using the diagrammatic expansion given in equation (35). As we saw in section 4.1, the denominator cancels the terms in the numerator corresponding to disconnected diagrams, so we can obtain the correlation function by expanding the numerator and retaining only the connected diagrams. In evaluating these diagrams, the following results will be useful (we have defined $\tau \equiv t_a - t_b$):

$$\int D_{a1}D_{1b}\,dt_1 = -\frac{i}{2}(1+i|\tau|)D_F(\tau) \tag{76}$$

$$\int D_{a1}D_{1b}^3\,dt_1 = \frac{i}{8}D_F^3(\tau)-\frac{3i}{32}D_F(\tau) \tag{77}$$

$$\int |t|\,D_F(t-\tau)\,D_F(t)\,dt = \frac{1}{4}(\tau^2-i|\tau|-1)D_F(\tau) \tag{78}$$

$$\int D_{12}^2\,dt_2 = -i/4. \tag{79}$$

The lowest order diagram is given by

$$\langle 0|T[\phi_a\phi_b]|0\rangle = D_{ab}. \tag{80}$$

To find the first order diagrams, note that

$$\langle 0|T[\phi_a\phi_b\phi_1^4]|0\rangle = 3\,D_{ab}\,D_{11}^2 + (4\cdot 3)\,D_{a1}D_{11}D_{1b}. \tag{81}$$

We see that there is one connected diagram and one disconnected diagram. The connected diagram, shown in figure 1, is given by

$$(4\cdot 3)\,(1/1!)\,(-i\lambda)\int D_{a1}D_{11}D_{1b}\,dt_1 = -3\lambda(1+i|\tau|)D_F(\tau). \tag{82}$$

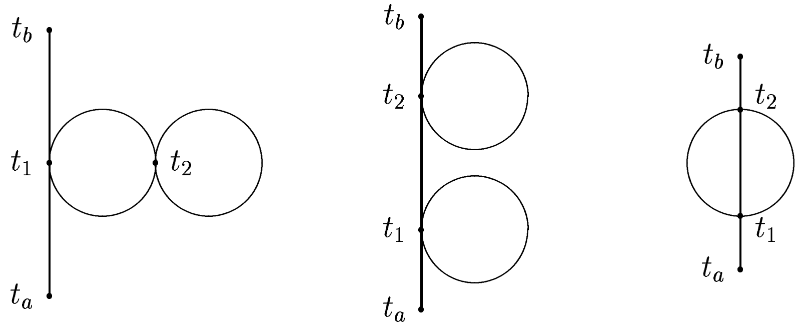

Figure 2. Second order diagrams for the two-point correlation function for ϕ^4 theory. Each vertex indicates a time t_j; a line connecting vertices for times t_j and t_k indicates a Feynman propagator D_{jk}.

To find the second order diagrams, note that

$$\begin{aligned}
&\langle 0|T[\phi_a\phi_b\phi_1^4\phi_2^4]|0\rangle = \\
&D_{ab}\,\langle 0|T[\phi_1^4\phi_2^4]|0\rangle + \\
&(4\cdot 3\cdot 3)\,(D_{a1}D_{11}D_{1b}D_{22}^2 + D_{a2}D_{22}D_{2b}D_{11}^2) + \\
&(4\cdot 3)^2\,(D_{a1}D_{b1}D_{12}^2D_{22} + D_{a2}D_{b2}D_{21}^2D_{11}) + \\
&(4\cdot 3)^2\,(D_{a1}D_{11}D_{12}D_{22}D_{2b} + D_{a2}D_{22}D_{21}D_{11}D_{1b}) + \\
&(4^2\cdot 3\cdot 2)\,(D_{a1}D_{12}^3D_{2b} + D_{a2}D_{21}^3D_{1b}).
\end{aligned}$$

The first two lines correspond to disconnected diagrams, while the third, fourth, and fifth lines correspond to connected diagrams (these are shown in figure 2). The first diagram is

$$\begin{aligned}
&2\,(4\cdot 3)^2\,(1/2!)\,(-i\lambda)^2 \iint D_{a1}D_{b1}D_{12}^2D_{22}\,dt_1\,dt_2 \\
&= 9\lambda^2(1+i|\tau|)D_F(\tau).
\end{aligned} \tag{83}$$

The second diagram is

$$\begin{aligned}
&2\,(4\cdot 3)^2\,(1/2!)\,(-i\lambda)^2 \iint D_{a1}D_{11}D_{12}D_{22}D_{2b}\,dt_1\,dt_2 \\
&= \frac{9}{2}\lambda^2(3+3i|\tau|-\tau^2)D_F(\tau).
\end{aligned} \tag{84}$$

The third diagram is

$$\begin{aligned}
&2\,(4^2\cdot 3\cdot 2)\,(1/2!)\,(-i\lambda)^2 \iint D_{a1}D_{12}^3D_{2b}\,dt_1\,dt_2 \\
&= \frac{9}{8}\lambda^2 D_F(\tau,3) + \frac{1}{8}\lambda^2(27+36i|\tau|)D_F(\tau).
\end{aligned} \tag{85}$$

Collecting all these results, we find that the two-point correlation function to second order is

$$\begin{aligned}
&\langle\bar{0}|T[\phi(t_a)\phi(t_b)]|\bar{0}\rangle \\
&= [1 - 3\lambda(1+i|\tau|) + \frac{9}{8}\lambda^2(23+24i|\tau|-4\tau^2)]D_F(\tau) + \frac{9}{8}\lambda^2 D_F(\tau,3) \\
&= Z\,D_F(\tau,m_R) + \rho_3\,D_F(\tau,3) + O(\lambda^3),
\end{aligned}$$

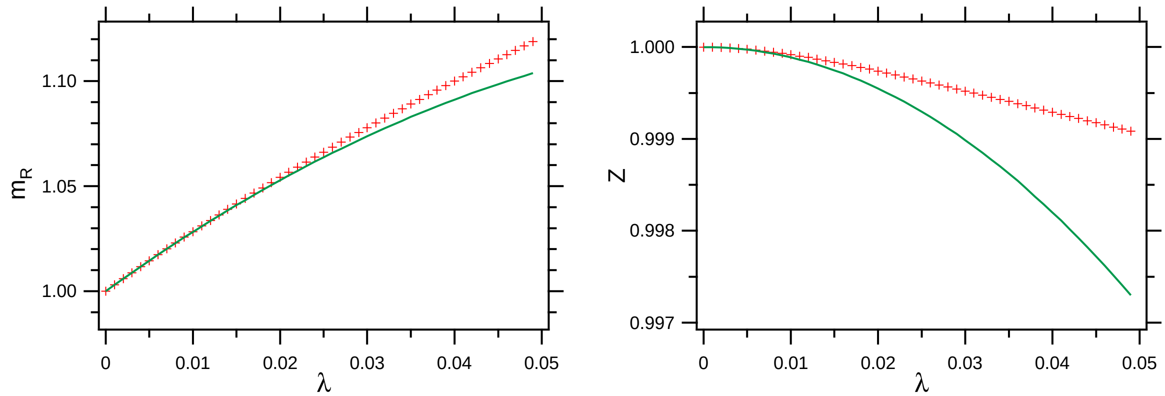

Figure 3. Renormalized mass m_R and field strength renormalization Z for ϕ^4 theory, plotted as a function of the coupling strength λ. The crosses (red) are the results of a numerical calculation; the curves (green) are the theoretical predictions to second order in λ.

where Z and m_R are as before, and $\rho_3 = (9/8)\lambda^2$.

As a final check on these results, we can calculate the mass renormalization and the field strength renormalization by numerically solving the Schrödinger equation

$$(-\frac{1}{2}\frac{d^2}{d\phi^2} + \frac{1}{2}\phi^2 + \lambda\phi^4)\,\Phi_n(\phi) = \bar{E}_n\Phi_n(\phi). \tag{86}$$

Once the eigenvalues $\bar{E}_n$ and eigenstates $\Phi_n(\phi)$ have been determined, the mass renormalization is given by $m_R = \bar{E}_1 - \bar{E}_0$ and the field strength renormalization is given by

$$Z = 2m_R \left| \int \Phi_1^*(\phi)\,\phi\,\Phi_0(\phi)\,d\phi \right|^2 . \tag{87}$$

In figure 3 we plot the numerically determined values for m_R and Z as a function of λ, along with our second order theoretical predictions for these quantities.

4.3. Yukawa interaction

Let us now consider a theory that describes an interaction between bosons and fermions. The Hamiltonian for the system is $H = H_0 + H_i$, where

$$H_0 = \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 + \mu\bar{\psi}\psi = E_0 + m a^\dagger a + \mu(b^\dagger b + c^\dagger c) \tag{88}$$

is the free Hamiltonian, describing neutral bosons of mass m and charged fermions of mass μ, and

$$H_i = N[\lambda\phi\bar{\psi}\psi] = \eta m(a + a^\dagger)(b^\dagger b + c^\dagger c), \tag{89}$$

is the interaction Hamiltonian, describing a Yukawa coupling between the bosons and fermions. We have defined a vacuum energy $E_0 = m/2 - \mu$ and a dimensionless coupling constant $\eta = (2m)^{-1/2}\,(\lambda/m)$. The eigenstates and eigenvalues of the free Hamiltonian H_0 are given by

$$H_0|n_a, n_b, n_c\rangle = (E_0 + m n_a + \mu(n_b + n_c))|n_a, n_b, n_c\rangle, \tag{90}$$

where $|n_a, n_b, n_c\rangle$ is an eigenstate of $a^\dagger a$, $b^\dagger b$, and $c^\dagger c$ with eigenvalues n_a, n_b, and n_c. We can find the eigenstates and eigenvalues of the interacting Hamiltonian H by introducing an operator d, defined by

$$d = a + \eta(b^\dagger b + c^\dagger c). \tag{91}$$

We can also express d as†

$$d = DaD^\dagger, \tag{92}$$

where D is a unitary transformation given by

$$D = \exp(-\eta(b^\dagger b + c^\dagger c)(a^\dagger - a)). \tag{93}$$

Note that

$$d^\dagger d = a^\dagger a + \eta(a + a^\dagger)(b^\dagger b + c^\dagger c) + \eta^2(b^\dagger b + 2\, b^\dagger b\, c^\dagger c + c^\dagger c), \tag{94}$$

so we can express the total Hamiltonian as

$$H = E_0 + m_R\, d^\dagger d + \mu_R\, (b^\dagger b + c^\dagger c) + V_2\, b^\dagger b\, c^\dagger c, \tag{95}$$

where we have defined $m_R = m$, $\mu_R = \mu - \eta^2 m$, and $V_2 = -2\eta^2 m$. Thus, the eigenstates and eigenvalues of the interacting Hamiltonian are given by

$$H|\overline{n_d, n_b, n_c}\rangle = (E_0 + m_R n_d + \mu_R(n_b + n_c) + V_2\, n_b n_c)|\overline{n_d, n_b, n_c}\rangle, \tag{96}$$

where $|\overline{n_d, n_b, n_c}\rangle$ is an eigenstate of $d^\dagger d$, $b^\dagger b$, and $c^\dagger c$ with eigenvalues n_d, n_b, and n_c. We see that the interacting theory can be interpreted as a system of noninteracting bosons of mass m_R, together with a system of fermions of mass μ_R, where fermions couple to antifermions with strength V_2. If we introduce new boson field operators

$$\Phi = (2m)^{-1/2}(d + d^\dagger) \tag{97}$$

$$\Pi = -\, i(m/2)^{1/2}(d - d^\dagger), \tag{98}$$

then we can also express the interacting Hamiltonian as

$$H = \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\Phi^2 + \mu\bar{\psi}\psi - \eta^2 m N[\bar{\psi}\psi]^2. \tag{99}$$

This describes a free boson field and a self-interacting fermion field.

We can understand the sign of the fermion-antifermion coupling by the following argument. In higher dimensions, a Yukawa coupling gives an attractive force between fermions and antifermions‡. The analogous statement for $(0+1)$ dimensions is that if we take two systems, one consisting of an isolated fermion and one consisting of an isolated antifermion, and add their energies, then the result is greater than the energy of a single system consisting of both a fermion and an antifermion.

† This can be shown by using the relation $e^A\, B\, e^{-A} = B + [A, B] + (1/2!)[A, [A, B]] + \cdots$, which holds for arbitrary operators A, B. The unitary transformation D is closely related to the displacement operator of quantum mechanics (see section 2.2 of [9]).

‡ See the discussion following equation (4.128) in [1].

To relate the eigenstates of H to the eigenstates of H_0, first note that equation (92) implies that the eigenstates of H with $n_d = 0$ can be expressed as $|\overline{0, n_b, n_c}\rangle = D|0, n_b, n_c\rangle$. If we substitute for D, and use the relation§

$$e^{\alpha a^\dagger - \alpha^* a} = e^{-|\alpha|^2/2} \, e^{\alpha a^\dagger} \, e^{-\alpha^* a}, \tag{100}$$

we find

$$|\overline{0, n_b, n_c}\rangle = e^{-\eta^2 (n_b + n_c)^2/2} \sum_{n_a=0}^{\infty} \frac{(-1)^{n_a}}{\sqrt{n_a!}} \, \eta^{n_a} (n_b + n_c)^{n_a} \, |n_a, n_b, n_c\rangle. \tag{101}$$

Eigenstates of the interacting theory with $n_d > 0$ can be constructed from the $n_d = 0$ eigenstates by applying the creation operator $d^\dagger$. Note that the vacuum states of the free and interacting theories are the same: $|\overline{0,0,0}\rangle = |0,0,0\rangle$.

Using the eigenstates of the interacting theory, we can calculate the boson and fermion field strength renormalizations. The boson field strength renormalization is

$$Z_b = 2m_R |\langle \overline{1,0,0} | \phi | \overline{0,0,0} \rangle|^2 = 1, \tag{102}$$

and the fermion field strength renormalization is

$$Z_f = |\langle \overline{0,1,0} | b^\dagger | \overline{0,0,0} \rangle|^2 = e^{-\eta^2}. \tag{103}$$

We can also obtain results for the renormalized fermion mass and the fermion field strength renormalization by using the diagrammatic expansion given in equation (35); we will work to second order in λ. The numerator is

$$\begin{aligned}
&\langle 0 | T[\psi_I^\alpha(t_a) \bar{\psi}_I^\beta(t_b) \exp(-i\lambda \int N[\phi_I(t) \bar{\psi}_I(t) \psi_I(t)] \, dt)] | 0 \rangle \\
&= S_{ab}^{\alpha\beta} - (\lambda^2/2) \iint \langle 0 | T[\psi_a^\alpha \bar{\psi}_b^\beta \, N[\phi_1 \bar{\psi}_1^\gamma \psi_1^\gamma] \, N[\phi_2 \bar{\psi}_2^\delta \psi_2^\delta]] | 0 \rangle \, dt_1 \, dt_2 \\
&= S_{ab}^{\alpha\beta} - (\lambda^2/2) \iint (D_{12} \, S_{a1}^{\alpha\gamma} \, S_{12}^{\gamma\delta} \, S_{2b}^{\delta\beta} + D_{21} \, S_{a2}^{\alpha\delta} \, S_{21}^{\delta\gamma} \, S_{1b}^{\gamma\beta}) \, dt_1 \, dt_2 + \\
&\quad (\lambda^2/2) \, S_{ab}^{\alpha\beta} \iint D_{12} \, S_{12}^{\gamma\delta} \, S_{21}^{\delta\gamma} \, dt_1 \, dt_2,
\end{aligned} \tag{104}$$

where repeated indices are to be summed over‖. The third term corresponds to a disconnected diagram and can be neglected (also, one can show that this diagram vanishes); thus, the fermion two-point correlation function is (see figure 4)

$$\langle \bar{0} | T[\psi_a^\alpha \bar{\psi}_b^\beta] | \bar{0} \rangle = [S_{ab} - i \iint S_{a1} \, \Sigma_{12} \, S_{2b} \, dt_1 \, dt_2]^{\alpha\beta}, \tag{105}$$

where we have defined

$$\Sigma(t) = -i\lambda^2 S_F(t) \, D_F(t) = -i\eta^2 m^2 \, S_F(t, \mu + m), \tag{106}$$

and

$$S_F(t, M) = \frac{1}{2}(1 + \epsilon(t) \, \gamma^0) \, e^{-i(M - i\epsilon)|t|} \tag{107}$$

is the Feynman propagator for a fermion of mass M. If we Fourier transform the

§ This follows from the Baker-Hausdorff formula (see section 2.2 of [9]).

‖ Note that the second line contains a time-ordered product in which some of the factors are normal-ordered products. Wick's theorem can still be applied in this situation, but with the restriction that contractions of field operators taken from the same normal-ordered product are not to be included in the sum over all possible contractions (see section 6.3 of [2]).

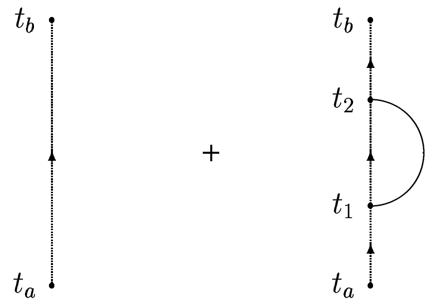

Figure 4. Fermion two-point function for Yukawa theory, to second order. Solid lines indicate boson propagators D_{ij} and dotted lines indicate fermion propagators S_{ij}.

correlation function, we find

$$\int \langle\bar{0}|T[\psi^\alpha(t)\bar{\psi}^\beta(0)]|\bar{0}\rangle\, e^{i\omega t}\, dt = [\tilde{S}_F(\omega) - i\tilde{S}_F(\omega)\,\tilde{\Sigma}(\omega)\,\tilde{S}_F(\omega)]^{\alpha\beta}$$
$$= [i(\omega\gamma^0 - \mu - \tilde{\Sigma}(\omega) + i\epsilon)^{-1}]^{\alpha\beta} + O(\eta^3),$$

where

$$\tilde{\Sigma}(\omega) = \int \Sigma(t)\, e^{i\omega t}\, dt = \eta^2 m^2 (\omega\gamma^0 - \mu - m + i\epsilon)^{-1}. \tag{108}$$

Substituting for $\tilde{S}_F(\omega)$ and $\tilde{\Sigma}(\omega)$, we find

$$\int \langle\bar{0}|T[\psi^\alpha(t)\bar{\psi}^\beta(0)]|\bar{0}\rangle\, e^{i\omega t}\, dt$$
$$= [(1-\eta^2)\,\tilde{S}_F(\omega, \mu - \eta^2 m) + \eta^2\,\tilde{S}_F(\omega, \mu + m)]^{\alpha\beta} + O(\eta^3). \tag{109}$$

From this result, we can read off the renormalized fermion mass $\mu_R = \mu - \eta^2 m$ and the fermion field strength renormalization $Z_f = 1 - \eta^2$; these quantities agree to second order with our previous results.

Appendix A. Wick's theorem

Given a product of n interaction picture field operators $\phi_1\phi_2\cdots\phi_n$, we define a contraction operation on pairs of field operators such that the result of contracting operators ϕ_i and ϕ_j is

$$\phi_1\phi_2\cdots\hat{\phi}_i\cdots\hat{\phi}_j\cdots\phi_n\, D_{ij}, \tag{A.1}$$

where the hats indicate that ϕ_i and ϕ_j are to be omitted from the product. Wick's theorem† [11] states that a time-ordered product of field operators is equal to a normal-ordered product of field operators, summed over all possible contractions. For two fields, this means that

$$T[\phi_1\phi_2] = N[\phi_1\phi_2 + D_{12}] = N[\phi_1\phi_2] + D_{12}. \tag{A.2}$$

For three fields,

$$T[\phi_1\phi_2\phi_3] = N[\phi_1\phi_2\phi_3 + \phi_1 D_{23} + \phi_2 D_{31} + \phi_3 D_{12}]. \tag{A.3}$$

† Wick's theorem is discussed in section 4.3 of [1]; see also section 17.4 of [3] and section 3.8 of [4].

Correlation functions are defined by taking vacuum expectation values of time-ordered products of fields. Because the vacuum expectation value of a normal-ordered product of fields always vanishes, Wick's theorem allows us to express correlation functions as sums over products of Feynman propagators. For example, the four-point function can be expressed as

$$\langle 0|T[\phi_1\phi_2\phi_3\phi_4]|0\rangle = D_{12}D_{34} + D_{13}D_{24} + D_{14}D_{23}. \tag{A.4}$$

Wick's theorem can also be generalized to fermion fields (see section 4.7 of [1]).

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