

# A short-time approximation of phenotype fitness as “differential reproductive success” of normalized phenotype population coefficients

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## Abstract

We use a Taylor Expansion to investigate natural selection at short times. We examine the time-dependence of fitness in the traditional formulation of the Price Equation. We argue that the equation itself makes a fundamental statement about measurements (aka phenotypes), rather than about fitness, and stress the importance of considering the normalized population density functions underlying natural selection (and resulting unitary time evolution).

*Keywords:* Price Equation, fitness, natural selection, dynamic insufficiency, differential reproductive success, probability density functions

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## 1. Introduction

It is curious that, 42 years after the initial publication of the Price Equation, there remains debate over the fundamental parameters of the formulation. While Grafen’s p-score formulation is the most refined to date,[1] it is unclear whether gains in understanding have been made, with repeated statistical regressions of actual DNA loci unable to recover the heritability inferred from phenotype-only studies.[2]

In recent times, a curious pattern has emerged. In some studies, which infer genotypes from phenotypes, a large heritability can be found.[3, 4, 5] However, in studies which keep careful track of Single Nucleotide Polymorphisms (SNPs), this heritability is not observed.[2, 6]

The standard interpretation of this missing heritability is that it is due to genetic variation beyond the level of the SNP, the influence of gene-regulatory networks, and the possibility of inheritable methylation.[2, 7]

However, in this brief article, we examine the Price Equation under the assumption that fitness does not exist. In its place, we suggest a formal definition of differential reproductive success, and we derive this definition in the short-time limit using a Taylor Expansion in section 2. We use this to examine the Fisher equation in section 3. We discuss the analog of fitness in this pseudo-classical description and its relation to the discussion of dynamical insufficiency in section 4.

## 2. Derivation of Price Equation without fitness

In this section, we seek to recast the problem of variation in terms more familiar to physical scientists. As such, rather than talk about frequencies, we will restrict our discussion to the following system of orthogonal (completely distinguishable) observations (this model can be extended, as to continuous phenotypes in the Appendix A):

$$|\Psi(\Delta t)\rangle = \sum_i c_i(\Delta t) |\phi_i(0)\rangle \quad (1)$$

Where we can normalize the system such that:

$$1 = \sum_i c_i^2(\Delta t) \quad (2)$$

if we take  $c_i^2 = \frac{N_i}{N}$  as the probability that an animal measured at random will have phenotype  $i$  (where there are  $N$  total measurements and the  $N_i$  are the number of measurements of phenotype  $i$ ). We will refer to the  $|\Psi(\Delta t)\rangle$  as the observed phenotype vector for some population, and  $|\phi_i(0)\rangle$  as the observed phenotype  $i$ . In this case, we have the following definition for the time-dependent average phenotype:

$$P_{avg}(t) = \sum_i c_i^2(\Delta t) \phi_i \quad (3)$$

We note that this model could be extended making assumptions about causation of phenotype. We define the views of Grafen and Orr as the canonical interpretation of phenotype causation (p-scores of distinguishable genotypes), noting the goals of Grafen's Formal Darwinism Project are well designed.[1, 8]

We need not assume anything about causation of phenotype in order to understand the Price Equation, and the fundamental insufficiency of its

formulation. By assumption in our pseudo-physical model, each phenotype eigenstate corresponds to a different measurable:

$$P | \Psi(\Delta t) \rangle = \sum_i c_i(\Delta t) \phi_i | \phi_i(0) \rangle \quad (4)$$

We can then follow the derivation of Price by asking about the change in the measurable over some time. But how can we do this without fitness?

Recall that, in the classical derivation,[9] a crucial step is this substitution amounts to an assumption that fitness exists:

$$c_i^2(\Delta t) = w_i(0) c_i^2(0) \quad (5)$$

However, we do not want to make this usual assumption, given that fitness has been particularly nebulous.

Rather than discuss fitness, we consider the following short-time Taylor approximation of the normallized population coefficients (assuming now, instead of fitness, that the instantaneous time derivative of each phenotype eigenstate,  $\frac{dc_i^2}{dt}$  is known):

$$c_i^2(\Delta t) \approx c_i^2(0) + \Delta t \frac{dc_i^2}{dt} + (O((\Delta t)^2)) \quad (6)$$

In this case, we can recover the following equivalent of fitness, now with the usually implicit time dependence included, which is the first result of this paper:

$$\frac{c_i^2(\Delta t)}{c_i^2(0)} \approx 1 + \Delta t \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt} \approx w_i(0) \quad (7)$$

However, we can also use 6 to rewrite the Price Equation given the usual procedure of approximating the time dependent value of the measurable (as in equation 6):

$$\Delta P_{avg} = P_{avg}(t) - P_{avg}(0) = \sum_i \left( \frac{c_i^2(\Delta t)}{c_i^2(0)} - 1 \right) (c_i^2(0)) \phi_i \quad (8)$$

which can then be re-written in the more familiar form (recognizing that, due to the unitary nature of the transformation,  $E[\frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}] = 0$ ):

$$\Delta P_{avg} = \sum_i \left( \Delta t \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt} \right) c_i^2(0) \phi_i = (\Delta t) \text{cov}(\phi_i, \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}) \quad (9)$$

Where we suggest the similarity of the  $\frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}$  term in equation 9 to the fitness term in the original Price Equation gives a good reason to take  $\frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}$  as a formal definition of “differential reproductive success” for phenotype groups at short times, noting that “success” (or “being more fit”) is a triviality of this differential reproductive success description ( $\frac{dc_i^2}{dt} > 0$ ) on short time scales. If your phenotype is gaining normalized population share, your phenotype is succeeding.

### 3. The Fisher Equation and Differential Reproductive Success

The Price Equation can be used to examine the Fisher Equation.[10] This derivation follows from assuming the fitness is the characteristic whose time-dependence is explained, although whether or not this is the meaning as Fisher originally intended it has been debated. [1] In this case, in our new terminology, we have average fitness defined as:

$$w_{avg}(0) = \sum_i c_i^2(0)w_i(0) \approx \sum_i c_i^2(0)(1 + \Delta t \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}) \quad (10)$$

$$w_{avg}(\Delta t) = \sum_i c_i^2(t)w_i(t) \approx \sum_i (c_i^2(0) + \Delta t \frac{dc_i^2}{dt})(1 + \Delta t \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}) \quad (11)$$

where we take  $w_i(t) \approx w_i(0)$  as in the usual interpretation of fitness.

We can then find the differences in the average fitness to generate the Fisher Equation analog in our notation:

$$\Delta w_{avg} \approx \sum_i (\Delta t \frac{dc_i^2}{dt})(1 + \Delta t \frac{1}{c_i^2(0)} \frac{dc_i^2}{dt}) = \sum_i ((\Delta t)^2 \frac{1}{c_i^2(0)} (\frac{dc_i^2}{dt})^2) \quad (12)$$

which reduces to the usual statement of Fisher:

$$\Delta w_{avg} \approx (\Delta t)^2 \sigma^2 (\frac{1}{c_i^2} (\frac{dc_i^2}{dt})) \quad (13)$$

Noting that the change in average fitness goes as the variance in the differential reproductive success ( $\frac{1}{c_i^2} \frac{dc_i^2}{dt}$ ). Additionally note that, at very short times,  $w_{avg}$  is smooth, as  $\Delta w_{avg} \approx 0$ .

We note that our equations 9 and 13 use only defined quantities (population coefficient and its instantaneous derivative) and still recover the interesting behavior of the population.

#### 4. Discussion and Conclusions

There has been past discussion of the Dynamic Insufficiency of the Price Equation: the inability of the equation to tell us anything about the future.[11, 1, 12] It is hard to defend as useful a definition of fitness that does not lead to testable predictions, but only airtight explanations.

We think people may have more luck with our reformulation, and its appeal to differential reproductive success. However, we note that, at times short enough to contain a single birth or death, even our newly defined phenotype differential reproductive success function is not well defined (change proceeds in multiples of individual deaths or births). Research on this analog of the quantum problem is ongoing.

In any case, can one be any more sure of the instantaneous derivative of the population coefficient than of fitness? Perhaps not, but we can take the opportunity to make our assumptions and definitions precise at short times.

We conclude that the dynamic insufficiency of the Price Equation derives from it being an explanatory statement of phenotype observation. We suggest replacing fitness with differential reproductive success of distinct phenotype measurement groups to better understand natural selection, emphasizing the time-dependence of the process at short-times.

As a result of this probability density function formulation, we are investigating the possible use of the variational principle and density function approximation in place of a fitness optimization programme approach [1] to the formalization of natural selection.

#### Appendix A. Extension to Continuous Phenotype Measurements

In this appendix we extend the discrete phenotype formulation of the text to a continuously observable phenotype. In this case, we assume a phenotype probability distribution function  $\Phi^2(z)$  which is distributed across an arbitrary range of possible phenotypes  $z$ . We take the definition of the average phenotype such that:

$$z_{avg}(t) = \int z\Phi^2(z, t) dz \tag{A.1}$$

We then ask about the change in the average measured phenotype, given a slight change in the phenotype probability distribution function. That is to say, the continuous extension of equation 6 is:

$$\Phi^2(z, \Delta t) \approx \Phi^2(z, 0) + \Delta t \frac{d\Phi^2}{dt} + (O((\Delta t)^2)) \quad (\text{A.2})$$

We can then ask, as in the Price Equation, what happens to  $z_{avg}$  after this  $\Delta t$  has passed. We initially have:

$$\Delta z_{avg}(\Delta t) = \int z \Phi^2(z, \Delta t) dz - \int z \Phi^2(z, 0) dz \quad (\text{A.3})$$

Using the equation A.2 we are able to derive the following continuous form of the Price Equation:

$$\Delta z_{avg}(\Delta t) = (\Delta t) \int \Phi^2(z, 0) z \left( \frac{1}{\Phi^2(z, 0)} \frac{d\Phi^2}{dt} \right) dz \quad (\text{A.4})$$

which, upon recognizing that  $E[\frac{1}{\Phi^2(z, 0)} \frac{d\Phi^2}{dt}] = 0$  due to normalization, can be re-written as the more familiar:

$$\Delta z_{avg} = (\Delta t) \text{cov}(z, (\frac{1}{\Phi^2} \frac{d\Phi^2}{dt})) \quad (\text{A.5})$$

Where we note we have recovered the Price Equation and the continuous phenotype analog of fitness, for all possible observables  $z$  of the phenotype probability distribution function  $\Phi^2(z, t)$ , the function  $\frac{1}{\Phi^2} \frac{d\Phi^2}{dt}$  defines differential reproductive success at short times.

- [1] A. Grafen, *Journal of Genetics*, **87**, 421 (2008).
- [2] T. A. Manolio, F. S. Collins, N. J. Cox, D. B. Goldstein, L. A. Hindorff, D. J. Hunter, M. I. McCarthy, E. M. Ramos, L. R. Cardon, A. Chakravarti, J. H. Cho, A. E. Guttmacher, A. Kong, L. Kruglyak, E. Mardis, C. N. Rotimi, M. Slatkin, D. Valle, A. S. Whittemore, M. Boehnke, A. G. Clark, E. E. Eichler, G. Gibson, J. L. Haines, T. F. C. Mackay, S. A. McCarroll, and P. M. Visscher, *Nature*, **461**, 747 (2009).
- [3] K. C. Hans-Peter Kohler, Joseph L. Rodgers, *Population and Development Review*, **25**, 253 (1999).

- [4] K. M. Kirk, S. P. Blomberg, D. L. Duffy, A. C. Heath, I. P. F. Owens, and N. G. Martin, *Evolution*, **55**, 423 (2001).
- [5] D. R. G. Sean G. Byars, Douglas Ewbank and S. C. Stearns, *Proceedings of the National Academy of Sciences*, **107**, 1787 (2009).
- [6] L. Almasy, J. E. Hixson, D. L. Rainwater, S. Cole, J. T. Williams, M. C. Mahaney, J. L. VandeBerg, M. P. Stern, J. W. MacCluer, and J. Blangero, *American Journal of Human Genetics*, **64**, 1686 (1999).
- [7] A. E. Handel and S. V. Ramagopalan, *BMC Medical Genetics*, **11** (2010).
- [8] H. A. Orr, *Nature Reviews Genetics*, **10**, 531 (2009).
- [9] G. R. Price, *Nature*, **227**, 520 (1970).
- [10] R. A. Fisher, *Genetical Theory of Natural Selection* (1930).
- [11] A. Grafen, *Journal of Evolutionary Biology*, **20**, 1243 (2007).
- [12] G. P. Karev, *Journal of Mathematical Biology*, **60**, 107 (2010).