Multiple Dirichlet series and Shifted Convolutions

Jeff Hoffstein (Brown University)

April 1, 2010

Let \( f \) and \( g \) be cusp forms of even weight \( k \) for \( \Gamma_0(N_0) \), having Fourier expansions

\[
f(z) = \sum_{m \geq 1} a(m)e^{2\pi imz} \quad g(z) = \sum_{m \geq 1} b(m)e^{2\pi imz}.
\]

I’ll define a certain Dirichlet series and a related double Dirichlet series and show how the double Dirichlet series can be meromorphically continued to \( \mathbb{C}^2 \) and how it’s analytic properties can be used to discover properties of averages of shifted sums related to \( f \) and \( g \). For \( h \geq 1 \), the Dirichlet series is

\[
D(s; h) = \sum_{m_2 \geq 1} \frac{a(m_2 + h)\overline{b(m_2)}}{m_2^{s+k-1}}
\]

and the double Dirichlet series is

\[
Z(s, w) = \sum_{h \geq 1} \frac{D(s; h)}{h^w+(k-1)/2}.
\]

The related average shifted sums are of the form

\[
S(x; y) = \sum_{m_2 \sim x, h \sim y} \frac{a(m_2 + h)\overline{b(m_2)}}{m_2^{k-1}}.
\]

As an application I’ll prove the following Theorem: For \( Q \) a positive integer and \( \chi \) a character modulo \( Q \), let

\[
L(s, f, \chi) = \sum_{m \geq 1} \frac{a(m)}{m^{s+(k-1)/2}}
\]

denote the \( L \)-series of \( f \), normalized to have a functional equation as \( s \to 1 - s \). Then there exists \( \delta > 0 \) such that

\[
\frac{1}{x} \sum_{Q \sim x} \frac{1}{\phi(Q)} \sum_{\chi(Q)} |L(1/2, f, \chi)|^2 = c_f \log x + \mathcal{O}(x^{-\delta}).
\]