

# Ideological Contests with Quality Investments

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December 25, 2017

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## Abstract

We consider an asymmetric contest in which policy-motivated players make proposals with both a spatial location and a “quality” component, which must be generated at an up-front cost. Potential applications include elections, legislative policymaking, lobbying, expertise acquisition in bureaucracies, and judicial opinion writing. We show that equilibria are essentially unique and in two-dimensional mixed strategies, and provide an elegant analytical characterization of strategies, outcomes, and payoffs that yields many results. In equilibrium, more extreme proposals are both higher quality and strictly better for the decisionmaker. A more ideologically extreme *or* able player is more likely to make a proposal, and makes more extreme proposals, but is also more likely to win due to their proposals’ substantially higher quality. Simultaneously, her opponent is harmed and moderates her own proposals. When the players are highly asymmetric one almost always wins, but the decisionmaker may nevertheless strongly benefit from the potential for competition.

# 1 Introduction

In many political environments, individuals and groups often try to increase the appeal of their ideological proposals by working to improve them in other ways. In electoral contests, candidates and parties invest and advertise to make their policy positions more attractive to the median voter (Ashworth and Bueno de Mesquita (2009); Wiseman (2006)). In legislatures, lawmakers and interest groups work to draft laws that not only further their ideological aims, but are also well designed; e.g., clearly written, cost-efficient, and with predictable consequences (Hirsch and Shotts (2012); Hitt, Volden and Wiseman (2017)). Within government, bureaucrats develop expertise at implementing specific policies to incentivize their political superiors to select them and not others (Ian R Turner (2017); Ting (2011)). And on multi-member courts, justices expend effort to craft judicial opinions that are clear and persuasive to gain their colleagues' support (Lax and Cameron (2007)).

Across these dispersed literatures, a growing body of theoretical models either directly employ, or effectively reduce to, the following simple framework. First, one or more “proposers” compete by making proposals that consist of *both* of an ideology – over which players have one-dimensional spatial preference – and a level of *quality* or *valence* – which everybody values, and is costly to generate. Next, a “decisionmaker” (or decisionmakers) chooses among the available proposals. The term valence originally comes from the study of competitive elections, where it was initially used to describe exogenous candidate-specific characteristics that appeal to all voters such as charisma or competence (e.g. Ansolabehere and Snyder (2000); Groseclose (2001)).

A wide variety of permutations on this basic structure have been considered in the literature; the following is an incomplete list, and works mentioned fall in multiple categories. Some models assume that outcomes of the contest are probabilistic (e.g. Epstein and Nitzan (2004); Esteban and Ray (1999)), which with sufficient restrictions prevents mixed strategy equilibria, but makes results dependent on the form of the contest success function. Others assume that the proposers move sequentially (e.g. Hitt, Volden and Wiseman (2017); Lax and Cameron (2007); Wiseman (2006)). This also eliminates mixed strategy equilibria, but can be cumbersome to analyze, and results depend on an arbitrarily-chosen sequencing of the players. Others assume the proposers choose simultaneously, but sequentially separate their choice of policy and effort (e.g. Ashworth and Bueno de Mesquita (2009); Epstein and Nitzan (2004); Konrad (2000); Munster (2006); Serra (2010)). This choice is arguably more natural for some applications, but yields strategic incentives that depend heavily on

commitments allowed by the game sequence. For example, in both [Ashworth and Bueno de Mesquita \(2009\)](#) and [Epstein and Nitzan \(2004\)](#), the proposers effectively collude through the game sequence, selecting their ideological platforms in order to reduce the intensity of the anticipated spending contest. Moreover, in deterministic contests the structure of equilibria is complicated, and the set is incompletely characterized. Other models do not allow the proposers to propose platforms other than their ideal points (e.g. [Esteban and Ray \(1999\)](#)); with two players this reduces the game to a more-standard asymmetric all pay contest with flexible payoffs for winning and losing ([Meirowitz \(2008\)](#)). Finally, others assume that the proposers are purely motivated to win and do not care about policy directly (e.g. [Ashworth and Bueno de Mesquita \(2009\)](#); [Baye, Kovenock and Vries \(1993\)](#); [Che and Gale \(1998\)](#)).

In this paper, we solve the arguably simplest formulation of the model – that the players simultaneously choose all the characteristics of their proposals, and outcomes are deterministic. The model herein is a generalization of [Hirsch and Shotts \(2015\)](#) that allows for asymmetric proposers, different potential outside options for the decisionmaker (including no outside option, as in a two-candidate election), and varying weights on quality vs. ideology. The resulting contest has several non-standard properties that are difficult to analyze jointly, despite recent advances in literature.

First, the contest has an “all pay” component ([Siegel \(2009\)](#)) – players decide how much effort to exert before knowing what their opponent has done. Consequently, equilibria are in mixed strategies. Second, it is multidimensional – players gain support for their proposals through a mixture of ideological concessions and up-front quality investments, with more of one reducing the need for the other. Third, it is asymmetric – players may differ both in their ability at generating quality, and in their intrinsic ideological extremism. Unlike the standard all-pay contest, differences in costs and “valuations” are not isomorphic.

Fourth and most importantly, the players are purely policy-motivated – they care only about the ideology and quality of the chosen proposal, rather than whose proposal is chosen. This assumption is essential to model the behavior of policy-motivated actors, and has two implications for the contest structure. First, ideological concessions are “winner pay,” in that the cost of making a concession is only suffered if the proposal actually wins ([Che and Gale \(2003\)](#)). Second, there is a “second order rank order spillover” ([Baye, Kovenock and Vries \(2012\)](#); [Chowdhury and Sheremeta \(2010\)](#)); the strategy of the (first ranked) winner has a direct effect on the utility of the (second ranked) loser, because she must live under the winner’s policy.

We show that despite the aforementioned complexities, the equilibrium characterization

is actually very tractable – it is unique, in continuous two-dimensional mixed strategies, has an elegant analytical characterization, and yields many substantively interesting comparative statics. Our results contribute to the political science and political economy literatures by fully characterizing equilibria in a simple game with a common set of modeling assumptions. They also contribute to the contest literature by solving an applied contest model with theoretically interesting complexities that are not typically analyzed jointly, and that is closely related to existing spatial contests used to study both political and product competition.

We first derive general necessary and sufficient conditions for equilibrium. Strategies are characterized by univariate atomless probability distributions over the decisionmaker’s utility with common convex support, combined with simple functions associating each utility with a unique combination of ideology and quality. The distributions are characterized by a straightforward coupled system of differential equations. While the basic form of the conditions is unsurprising, the proofs are complicated by the second-order spillover, which rules out the use of techniques that exploit exogenously fixed payoffs from losing (Siegel (2009)). We then derive an elegant and unique analytical characterization of the solution to the system of differential equations, thereby fully characterizing equilibrium strategies and payoffs. The characterization yields a variety of substantive results and comparative statics that are detailed in the paper and conclusion, a subset of which we review here.

First, participation in the contest is generically asymmetric; one proposer always makes a positive-quality proposal, while the other sits out with strictly positive probability. Second, as in the symmetric variant in Hirsch and Shotts (2015), more ideologically extreme proposals in the support of a player’s strategy are both higher quality and strictly better for the decisionmaker. The reason is that ideological concessions are effectively “winner pay,” and therefore more expensive to make on proposals that are more likely to win. An additional implication is that the decisionmaker is always presented with, and always chooses, a proposal whose ideology diverges from his own. As applied to electoral contests, failure of median convergence thus emerges naturally from the candidates’ attempts to exploit quality to gain support for non-centrist platforms – this divergence occurs even though contest outcomes are deterministic (see Munster (2006)).

Third, more able *or* extreme proposers dominate the contest and are more likely to win by virtue of their greater motivation and ability. Thus, extreme preferences – far from a liability – are an asset in the contest. Fourth, a proposer becoming more extreme *benefits* the DM because the resulting increase in motivation is partially channelled into productive investment. This is true even though the extremist’s proposals are more extreme as a

byproduct, and the other proposer is both harmed and forced to moderate her own proposals.

Fifth, greater ability at generating quality – while common-value – has similar equilibrium effects to greater ideological extremism. It makes the decisionmaker better off, but results in more extreme proposals and harms the other proposer as a byproduct. The reason is that common value investments are exploited in equilibrium to extract ideological gains.

Finally, balance in the proposers’ extremism and ability maximizes the probability of competition, but *not* the decisionmaker’s welfare. Instead, the decisionmaker benefits if symmetry is broken by one proposer becoming more extreme or able – even though her opponent becomes less active as a byproduct. In the limiting case of one proposer becoming arbitrarily extreme or able, the probability her opponent is active approaches 0 – she thus has the appearance of being a monopoly agenda setter. But, the DM’s utility is bounded away from what it would be in a true monopoly (Hirsch and Shotts (2017)). In fact, if one proposer becomes arbitrarily extreme, the decisionmaker becomes *unboundedly better* off.

The paper proceeds as follows. Section 2 introduces the model. Section 3 develops concepts and notation, and presents some general results. Section 4 uses an intermediate result to analytically characterize the likelihood each player makes a proposal, the probabilities of victory, and the equilibrium distributions over ideologies. Section 5 fully characterizes equilibrium strategies analytically, and uses this to derive payoffs and related results. Section 6 concludes. All proofs are contained in the main text.

## 2 The Model

Two proposers (*Left* and *Right*) make competing proposals for consideration by a decisionmaker (*DM*). A proposal  $(\gamma, q)$  consists of an *ideology*  $\gamma \in \mathbb{R}$  and a level of *quality*  $q \in [0, \infty) = \mathbb{R}^+$ , which must be produced at an up-front cost. All three players care about the characteristics of the exact proposal ultimately chosen, rather than whose proposal is chosen. Utility over proposals takes the form

$$U_i(\gamma, q) = \left(\frac{1}{\lambda}\right) q - (\gamma - X_i)^2,$$

where  $X_i$  is player  $i$ ’s ideological ideal point, and  $\frac{1}{\lambda}$  is the weight all players place on quality. The proposers’ ideal points are assumed to be on either side of the decisionmaker ( $X_L < X_D < X_R$ ).

The game proceeds as follows. First, the proposers first simultaneously select their proposals  $(\gamma_i, q_i)$ . Making a proposal with quality  $q_i$  costs  $c_i(q_i) = a_i q_i$  up-front, where  $a_i > \frac{1}{\lambda}$ .

Next, the DM chooses one of the two proposals or something from a set of outside options  $\mathbb{O}$ , where  $\mathbb{O}$  may contain the DM’s ideal point with no quality  $(0, 0)$  and/or proposals that are strictly worse (and can be empty).

### 3 Preliminary Analysis

The game is a multidimensional contest (Che and Gale (2003)) in which the scoring rule applied to “bids”  $(\gamma, q)$  is simply the DM’s utility  $U_D(\gamma, q) = \left(\frac{1}{\lambda}\right)q - (X_D - \gamma)^2$ . To facilitate the analysis we thus reparameterize proposals  $(\gamma, q)$  to be expressed in terms of  $(s, y)$ , where  $y = \gamma - X_D$  denotes the (signed) distance of the proposal’s ideology from the DM’s ideal, and  $s = \left(\frac{1}{\lambda}\right)q - y^2$  denotes the DM’s utility for the proposal, or its *score*. The implied quality of a proposal  $(s, y)$  is then  $q = \lambda(s + y^2)$ . Using this we re-express the proposers’ utility and cost functions in terms of  $(s, y)$ .

**Definition 1.**

1. *Player  $i$ ’s utility for proposal  $(s, y)$  is*

$$V_i(s, y) = U_i(y + X_D, \lambda(s + y^2)) = -x_i^2 + s + 2x_i y$$

where  $x_i = X_i - X_D$  is the (signed) distance of  $i$ ’s ideal from the DM.

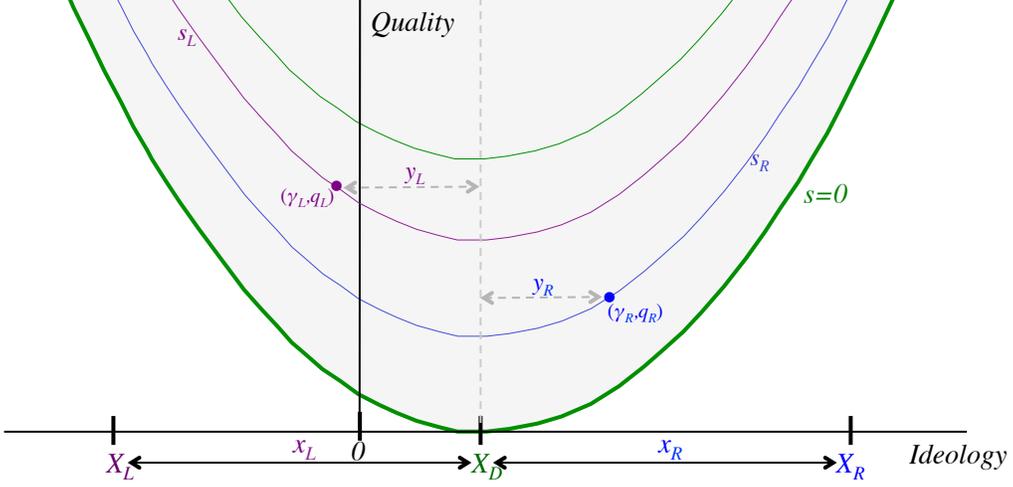
2. *Proposer  $i$ ’s cost to make proposal  $(s, y)$  is*

$$c_i(\lambda(s + y^2)) = a_i \cdot \lambda(s + y^2) = \alpha_i(s + y^2)$$

where  $\alpha_i = a_i \lambda$  is  $i$ ’s weighted marginal cost of generating quality.

Figure 1 depicts the game in ideology-quality space. The DM’s indifference curves, i.e., the policies with equal score, are depicted by green lines. Definition 1 reparameterizes proposals into score and ideological distance (henceforth just ideology)  $(s, y)$ , and the five primitives  $(X_i, a_i, \lambda)$  into four parameters  $(x_i, \alpha_i)$  capturing ideal ideological distance from the DM (henceforth just ideal ideology) and weighted marginal costs of generating quality (henceforth just costs).

Figure 1: Setup of the Policy Contest



### 3.1 Necessary and Sufficient Equilibrium Conditions

In the reparameterized game, a proposer's pure strategy  $(s_i, y_i)$  is a two-dimensional element of  $\mathbb{B} \equiv \{(s, y) \in \mathbb{R}^2 \mid s + y^2 \geq 0\}$ . A mixed strategy  $\sigma_i$  is a probability measure over the Borel subsets of  $\mathbb{B}$ , and let  $F_i(s)$  denote the CDF over scores induced by  $i$ 's mixed strategy  $\sigma_i$ .<sup>1</sup>

We now derive necessary and sufficient equilibrium conditions in a series of four lemmas. Let  $\bar{\Pi}_i(s_i, y_i; \sigma_{-i})$  denote  $i$ 's expected utility for making proposal  $(s_i, y_i)$  with  $s_i \geq 0$  if a tie would be broken in her favor. Clearly this is  $i$ 's expected utility from making a proposal at any  $s_i > 0$  where  $-i$  has no atom, and  $i$  can always achieve utility arbitrarily close to  $\bar{\Pi}_i(s_i, y_i; \sigma_{-i})$  by making  $\varepsilon$ -higher score proposals. Now  $\bar{\Pi}_i(s_i, y_i; \sigma_{-i}) =$

$$-\alpha_i (s_i + y_i^2) + F_{-i}(s_i) \cdot V_i(s_i, y_i) + \int_{s_{-i} > s_i} V_i(s_{-i}, y_{-i}) d\sigma_{-i}. \quad (1)$$

The first term is the up-front cost of generating the proposal's quality. The second term is the probability  $i$ 's proposal is selected, times her utility for it. The third term is  $i$ 's utility should she lose, which requires integrating over all her opponent's proposals with higher score than  $s_i$ .

Taking the first order condition with respect to  $y_i$  yields the first Lemma.

**Lemma 1.** *At any score  $s_i > 0$  where  $F_{-i}(\cdot)$  has no atom, the proposal  $(s_i, y_i^*(s_i))$ , where  $y_i^*(s_i) = F_{-i}(s_i) \cdot \frac{x_i}{\alpha_i}$ , is the strictly best score- $s_i$  proposal.*

<sup>1</sup>For technical convenience we restrict attention to strategies generating score CDFs that can be written as the sum of an absolutely continuous and a discrete distribution.

Lemma 1 states that for almost every score  $s_i > 0$ , proposer  $i$ 's best combination of ideology and quality to generate that score is unique. Crucially, the optimal ideology-quality combination does not depend on the specific proposals that her opponent makes; it is just a weighted average of the proposer's and DM's ideal ideologies  $\frac{x_i}{\alpha_i}$ , multiplied by the probability  $F_{-i}(s_i)$  that her opponent makes a lower-score proposal.

The second lemma establishes that at least one of the proposers is always *active*, in the sense of making a proposal with strictly positive score (all positive-score proposals are positive-quality, but the reverse is not necessarily true). Intuitively, this holds because the proposers wish to move policy in opposite directions from the DM, and can beat negative-score proposals for “free” by proposing the DM's ideal with 0-quality.

**Lemma 2.** *In equilibrium  $F_k(0) > 0$  for at most one  $k \in \{L, R\}$ .*

**Proof:** Suppose not, so  $F_i(0) > 0 \forall i$  in some equilibrium. Let  $U_i^*$  denote proposer  $i$ 's equilibrium utility, which can be achieved by mixing according to her strategy conditional on making score- $s \leq 0$  proposal. Let  $\bar{y}^0$  denote the expected ideological outcome and  $\bar{s}^0$  the expected score outcome conditional on both sides making score  $\leq 0$  proposals. Since  $x_L < 0 < x_R$ , we have  $V_k(\bar{s}^0, \bar{y}^0) \leq V_k(0, 0)$  for at least one  $k$ , which implies  $k$  has a profitable deviation since  $U_k^* \leq \bar{\Pi}_k(0, 0; \sigma_{-k}) < \bar{\Pi}_k(0, y_k^*(0); \sigma_{-k})$  (since  $F_{-k}(0) > 0$ ). QED

The third Lemma establishes that in equilibrium there is 0 probability of a tie at a strictly positive score. The absence of “score ties” is an intuitive consequence opposing ideological interests and the fact that generating quality is “all pay” – at least one proposer will find it in her interests to invest up-front in a bit more quality to break the tie, and make an ideological proposal weakly better than the expected outcome from a tie.

**Lemma 3.** *In equilibrium there is 0-probability of a tie at scores  $s > 0$ .*

**Proof:** Suppose not, so each proposer's strategy generates an atom of size  $p_i^s > 0$  at some  $s > 0$ . Proposer  $i$  can achieve her equilibrium utility  $U_i^*$  by mixing according to her strategy conditional on a score- $s$  proposal. Let  $\bar{y}^s$  denote the expected ideological outcome conditional on both sides making score- $s$  proposals. As before  $V_k(s, \bar{y}^s) \leq V_k(s, 0)$  for at least one  $k$ ; this implies  $k$  has a profitable deviation. If  $k$ 's proposal at score  $s$  is just  $(s, 0)$ , then  $U_k^* \leq \bar{\Pi}_k(s, 0; \sigma_{-k}) < \bar{\Pi}_k(s, y_k^*(s); \sigma_{-k})$  (since  $F_{-k}(s) > 0$ ). If  $k$  sometimes proposes something else, then  $U_k^* < \left(1 - \frac{p-k}{F_{-k}(s)}\right) \bar{\Pi}_k(s, E[y_k|s]; \sigma_{-k}) + \left(\frac{p-k}{F_{-k}(s)}\right) \bar{\Pi}_k(s, 0; \sigma_{-k})$ ; the r.h.s. is  $k$ 's utility if she were to instead propose  $(s, 0)$  with probability  $\frac{p-k}{F_{-k}(s)}$ , and the

expected ideology  $E[y_k|s]$  of her strategy at score  $s$  with the remaining probability (and always win ties). QED

Lemmas 1 – 3 jointly imply that in equilibrium, proposer  $i$  can compute her expected utility as if her opponent only makes proposals of the form  $(s_{-i}, y_{-i}^*(s_{-i}))$ . The utility from making *any* proposal  $(s_i, y_i)$  with  $s_i > 0$  where  $-i$  has no atom (or a tie would be broken in  $i$ 's favor) is therefore  $\bar{\Pi}_i^*(s_i, y_i; F) =$

$$-\alpha_i (s_i + y_i^2) + F_{-i}(s_i) \cdot V_i(s_i, y_i) + \int_{s_i}^{\infty} V_i(s_{-i}, y_{-i}^*(s_{-i})) dF_{-i}. \quad (2)$$

Proposer  $i$ 's utility from making the *best* proposal with score  $s_i$  is  $\bar{\Pi}_i^*(s_i, y_i^*(s_i); F)$ , which we henceforth denote  $\bar{\Pi}_i^*(s_i; F)$ .

Fourth and finally, we establish that equilibrium score CDFs must satisfy the following natural properties arising from the all pay nature of the contest.

**Lemma 4.** *The support of the equilibrium score CDFs over  $\mathbb{R}^+$  is common, convex, and includes 0.*

**Proof:** We argue  $\hat{s} > 0$  in support of  $F_i \rightarrow F_{-i}(s) < F_{-i}(\hat{s}) \forall s < \hat{s}$ , which yields the properties since absence of common support over  $\mathbb{R}^+$ , nonconvexity, or excluding 0 would all imply that one player has support at a strictly positive score immediately below which the other's score CDF is constant. Suppose not; there  $\exists s < \hat{s}$  where  $-i$  has no atom and  $F_{-i}(s) = F_{-i}(\hat{s})$ . Then  $\bar{\Pi}_i(\hat{s}, y_i; F) - \bar{\Pi}_i(s, y_i; F) = -(\alpha_i - F_{-i}(\hat{s})) \cdot (\hat{s} - s) < 0$ , implying  $i$ 's best score- $s$  proposal is strictly better than her best score- $\hat{s}$  proposal, a contradiction. QED

We conclude by combining the preceding lemmas to state a preliminary characterization of all equilibria in the form of necessary and sufficient conditions.

**Proposition 1.** *Necessary conditions for SPNE are as follows:*

1. (**Ideological Optimality**) *With probability 1, proposals are either*

(a) *negative score  $s_i \leq 0$  and 0-quality ( $s_i + y_i^2 = 0$ )*

(b) *positive score  $s_i > 0$  with ideology  $y_i = y_i^*(s_i) = \left(\frac{x_i}{\alpha_i}\right) F_{-i}(s_i)$ .*

2. (**Score Optimality**) *The profile of score CDFs  $(F_i, F_{-i})$  satisfy the following boundary conditions and differential equations.*

- **(Boundary Conditions)**  $F_k(0) > 0$  for at most one proposer  $k$ , and there  $\exists \bar{s} > 0$  such that  $\lim_{s \rightarrow \bar{s}} \{F_i(s)\} = 1 \forall i$ .
- **(Differential Equations)** For all  $i$  and  $s \in [0, \bar{s}]$ ,

$$\alpha_i - F_{-i}(s) = f_{-i}(s) \cdot 2x_i(y_i^*(s) - y_{-i}^*(s))$$

The above and  $F_i(s) = 0 \forall i, s < 0$  are sufficient for equilibrium.

**Proof:** (*Score Optimality*) A score  $\hat{s} > 0$  in the common support implies  $[0, \hat{s}]$  in the common support (by Lemma 4) implying  $\lim_{s \rightarrow \hat{s}^-} \{\bar{\Pi}_i(s; F)\} \geq U_i^*$ . Equilibrium also requires  $\bar{\Pi}_i(s; F) \leq U_i^* \forall s$  so  $\bar{\Pi}_i(s; F) = U_i^* \forall s \in [0, \bar{s}]$ , further implying the  $F$ 's are absolutely continuous over  $(0, \infty)$  (given our initial assumptions), and therefore  $\frac{\partial}{\partial s} (\Pi_i^*(s; F)) = 0$  for almost all  $s \in [0, \bar{s}]$ . This straightforwardly yields the differential equations for score optimality, with the boundary conditions implied by Lemma 4. (*Ideological Optimality*) At most one proposer ( $k$ ) makes  $\leq 0$ -score proposals with positive probability, so  $F_{-k}(0) = 0$ . Such proposals lose for sure and never influence a tie, and therefore must be 0-quality with probability 1, yielding property (a). Atomless score CDFs  $\forall s > 0$  implies  $(s, y_i^*(s))$  is the strictly best score- $s$  proposal (by Lemma 1), yielding property (b). (*Sufficiency*) Necessary conditions imply all  $(s, y_i^*(s))$  with  $s \in (0, \bar{s}]$  yield a constant  $U_i^*$ .  $F_{-k}(0) = 0$  implies  $k$ 's strictly best score-0 proposal is  $(0, y_k^*(0)) = (0, 0)$  and yields  $\bar{\Pi}_k(0; F)$ , and  $F_k(s) = 0$  for  $s < 0$  implies  $k$  has a size  $F_k(0)$  atom here. Thus both proposers' mixed strategies yield  $U_i^*$ , and neither can profitably deviate to  $s \in (0, \bar{s}]$ . To see neither can profitably deviate to  $s > \bar{s}$ , observe  $\Pi_i^*(s; F) - \Pi_i^*(\bar{s}; F) = -(\alpha_i - 1)(s - \bar{s}) < 0$ . To see  $k$  cannot profitably deviate to  $s_k \leq 0$ ,  $F_{-k}(0) = 0$  implies such proposals lose and never influence a tie, and so yield utility  $\leq U_k^*$ . To see  $-k$  cannot profitably deviate to  $s_{-k} \leq 0$ , observe all such proposals result in either  $(0, y_{-k})$  or  $(0, 0)$  when  $s_k \leq 0$  (since the DM's other choices are  $(0, 0)$  and  $\emptyset$ ), and thus yield utility  $\leq \max\{\bar{\Pi}_{-k}(0, 0; F), \bar{\Pi}_{-k}(0, y_{-k}; F)\}$  which is  $\leq U_{-k}^*$ . QED

### 3.2 Preliminary Observations about Equilibria

Proposition 1 implies that equilibria have a simple form. At least one proposer (henceforth labelled  $-k$ ) is always active – thus, competition not only strictly benefits the DM in expectation, but with probability 1. The other proposer (henceforth labelled  $k$ ) may also always be active ( $F_k(0) = 0$ ), or be inactive with strictly positive probability ( $F_k(0) > 0$ ). Inactivity

may manifest as proposing the DM’s ideal point with no quality  $(0, 0)$ , or as “position-taking” with more distant 0-quality proposals that always lose ( $s_k < 0$  and  $s_k + y_k^2 = 0$ ). However, any equilibrium exhibiting the latter is payoff-equivalent to one exhibiting the former; we thus focus on the former for comparative statics.<sup>2</sup> When either proposer  $i$  is active, she mixes smoothly over the ideologically-optimal proposals  $(s, y_i^*(s)) = \left(s, \frac{x_i}{\alpha_i} F_{-i}(s)\right)$  with scores in a common mixing interval  $[0, \bar{s}]$  according to the CDF  $F_i(s)$ .<sup>3</sup>

The differential equations characterizing the equilibrium score CDFs arise intuitively from the proposers’ indifference condition over  $[0, \bar{s}]$ . The left hand side is  $i$ ’s net marginal cost of making a higher-score proposal, given a fixed probability  $F_{-i}(s)$  of winning the contest; the proposer pays marginal cost  $\alpha_i > 1$  for sure, but with probability  $F_{-i}(s)$  her proposal is chosen and she enjoys a marginal benefit of 1 (because she values quality). The right hand side represents  $i$ ’s marginal ideological benefit of increasing her score. Doing so increases by  $f_{-i}(s)$  the probability that her proposal wins, which changes the ideological outcome from her opponent’s optimal ideology  $y_{-i}^*(s)$  at score  $s$  to her own optimal ideology  $y_i^*(s)$ .

An interesting property of equilibria (also noted in the symmetric variant in [Hirsch and Shotts \(2015\)](#)) is the positive association between a proposal’s distance from the DM and its score, implied by  $y_i^*(s) = \left(\frac{x_i}{\alpha_i}\right) F_{-i}(s)$ . That is, in equilibrium more distant proposals are not merely higher-quality than closer ones – they are also strictly better for the DM, implying that their additional quality *overcompensates* the DM for his ideological losses. Intuitively, a proposal giving the DM greater utility will be paired with a more-extreme ideology because it has a higher chance of being chosen, so the proposer is more willing to pay the sure costs of producing quality for the uncertain benefits of ideological change.

Figure 2 depicts a typical equilibrium involving symmetrically located proposers ( $-x_L = x_R$ ) but a cost advantage for the right proposer ( $\alpha_R < \alpha_L$ ). The top panel depicts proposers’ score CDFs. The right proposer is always active due to her cost advantage ( $F_R(0) = 0$ ), whereas the left proposer is sometimes inactive ( $F_L(0) > 0$ ). The right proposer’s proposals are better for the DM in a first-order stochastic sense; this is a general property of symmetric ideologies and asymmetric costs.

The bottom panel depicts the ideological locations and qualities of the equilibrium pro-

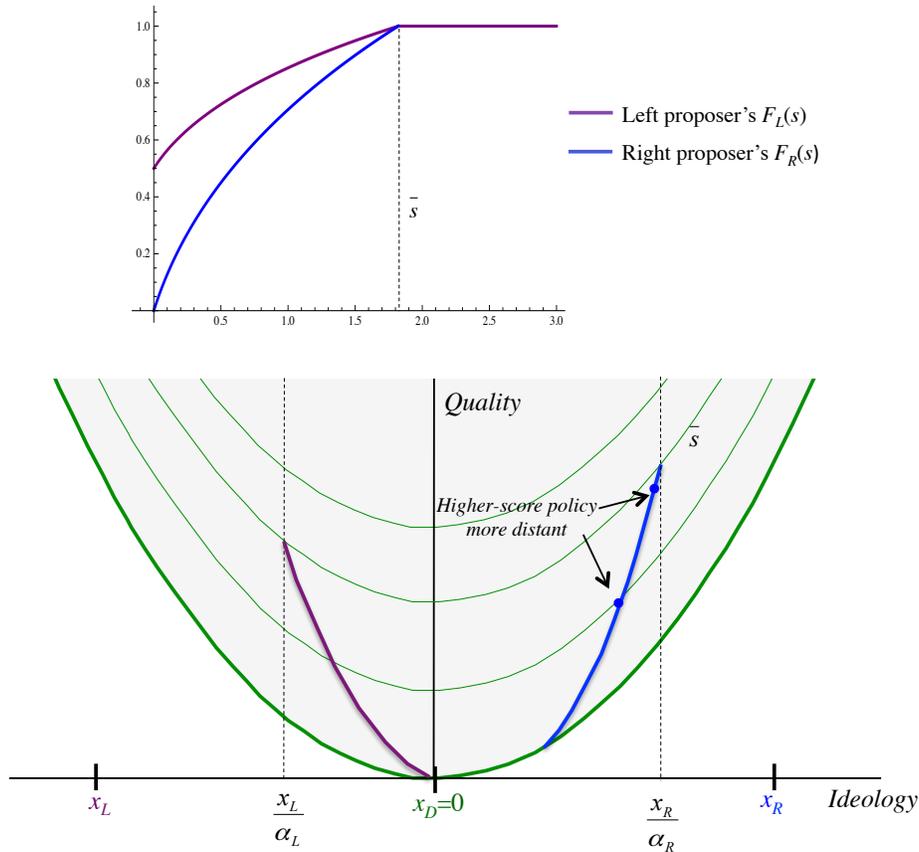
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<sup>2</sup>Profiles with “position-taking” are equilibria if the position-taking does not invite a deviation by  $-k$  to negative scores; whether this is the case depends on  $k$ ’s score-CDF below 0 and the DM’s outside options  $\mathbb{O}$ . When  $(0, 0) \in \mathbb{O}$  the necessary conditions are also sufficient.

<sup>3</sup>Technically, the proposition does not state that the support interval is also bounded ( $\bar{s} < \infty$ ), but this is later shown indirectly through the analytical equilibrium derivation.

posals – that is, a parametric plot of  $(y_i^*(s), s + (y_i^*(s))^2)$  for  $s \in [0, \bar{s}]$ . The ideology of  $i$ 's proposals extends out to  $\frac{x_i}{\alpha_i}$ , which is  $i$ 's optimal ideology to propose absent competition. The right proposer exploits her cost advantage to make more ideologically-extreme proposals at every score. Her proposals are also first-order stochastically more extreme. This is also a general feature of symmetric ideologies paired with asymmetric costs.

Figure 2: Equilibrium Score CDFs and Policies ( $x_L = -x_R, \alpha_L > \alpha_R$ )



## 4 Activity, Strength, Dominance, and Ideology

We first use the following simple result to characterize a variety of equilibrium properties that do not require knowing the equilibrium score CDFs. <sup>4</sup>

<sup>4</sup>We later constructively show existence and uniqueness of the equilibrium score CDFs.

**Lemma 5.** *In any SPNE,  $\epsilon_i(F_{-i}(s)) = \epsilon_{-i}(F_i(s)) \forall s \geq 0$ , where*

$$\epsilon_i(p) = \int_p^1 \frac{|x_i|}{\alpha_i - q} dq = |x_i| \log \left( \frac{\alpha_i - p}{\alpha_i - 1} \right)$$

**Proof:** Rearranging the differential equation in score optimality yields  $\frac{f_{-i}(s) \cdot |x_i|}{\alpha_i - F_{-i}(s)} = \frac{f_i(s) \cdot |x_{-i}|}{\alpha_{-i} - F_i(s)} \forall s \in [0, \bar{s}] \rightarrow \int_s^{\bar{s}} \frac{f_{-i}(s) \cdot |x_i|}{\alpha_i - F_{-i}(s)} ds = \int_s^{\bar{s}} \frac{f_i(s) \cdot |x_{-i}|}{\alpha_{-i} - F_i(s)} ds \forall s \in [0, \bar{s}]$ ; a change of variables and the boundary condition  $F_i(\bar{s}) = 1$  yields  $\int_s^{\bar{s}} \frac{f_{-i}(s) \cdot |x_i|}{\alpha_i - F_{-i}(s)} ds = \int_{F_{-i}(s)}^1 \frac{|x_i|}{\alpha_i - q} dq = \epsilon_i(F_{-i}(s))$ . The relationship holds trivially for  $s > \bar{s}$ . QED

We refer to the property in Lemma 5 as the *engagement equality*. To see why, observe that the decreasing function  $\epsilon_i(p)$  captures  $i$ 's relative willingness to deviate from a proposal winning with probability  $p$  to one that wins for sure (since the marginal ideological benefit of moving policy in her direction is  $|x_i|$ , and the net marginal cost of increasing score on a proposal winning the contest with probability  $q$  is  $\alpha_i - q$ ). We call this function  $i$ 's *engagement at probability  $p$* . The engagement equality  $\epsilon_i(F_{-i}(s)) = \epsilon_{-i}(F_i(s))$  states that at every score  $s \geq 0$  both proposers must be equally engaged given the resulting probabilities of winning the contest, and therefore equally willing to deviate to the maximum score  $\bar{s}$ . It is easily verified that  $\epsilon_i(1) = 0 \forall i$  and  $\epsilon_i(p)$  is strictly increasing in  $|x_i|$  and decreasing in  $\alpha_i \forall p \in [0, 1)$ .

Usefully, the engagement equality implies a simple functional relationship between the players' score CDFs that must hold in equilibrium regardless of their exact values. Letting

$$p_i(\epsilon) = \alpha_i - (\alpha_i - 1) e^{\frac{\epsilon}{|x_i|}}$$

denote the inverse of  $\epsilon_i(p)$  (which is decreasing in  $p$ , increasing in  $|x_i|$ , and decreasing in  $\alpha_i$ ) equilibrium requires that  $F_i(s) = p_{-i}(\epsilon_i(F_{-i}(s))) \forall s \in [0, \bar{s}]$ .

## 4.1 Activity and Strength

We first use the engagement equality to derive the identity of the sometimes-inactive proposer  $k$  and the probability  $F_k(0)$  that she is sometimes inactive, as well as perform comparative statics on  $F_k(0)$ .

**Proposition 2.** *In equilibrium  $k \in \arg \min_i \{\epsilon_i(0)\}$  and*

$$F_k(0) = p_{-k}(\epsilon_k(0)) = \alpha_{-k} - (\alpha_{-k} - 1) \left( \frac{\alpha_k}{\alpha_k - 1} \right)^{\left| \frac{x_k}{x_{-k}} \right|}.$$

The probability  $k$  is inactive  $F_k(0)$  is decreasing in her distance from the DM  $|x_k|$  and her opponent's quality costs  $\alpha_{-k}$ , and increasing in her opponent's distance from the DM  $|x_{-k}|$  and her own quality costs  $\alpha_k$ . In addition,  $\lim_{|x_k| \rightarrow 0} \{F_k(0)\} = \lim_{|x_{-k}| \rightarrow \infty} \{F_k(0)\} = \lim_{\alpha_k \rightarrow \infty} \{F_k(0)\} = \lim_{\alpha_{-k} \rightarrow 1} \{F_k(0)\} = 1$ .

**Proof:** Suppose  $\epsilon_k(0) < \epsilon_{-k}(0)$ ; then  $F_k(0) = 0$  and the engagement equality would imply  $F_{-k}(0) < 0$ , a contradiction. Since  $F_i(0) = 0$  for some  $i$  we must have  $F_{-k}(0) = 0$  and  $F_k(0) = p_{-k}(\epsilon_k(0)) > 0$ . Comparative statics and limit statements follow from previous observations on  $\epsilon_i(\cdot)$  and  $p_i(\cdot)$ . QED

The sometimes-inactive proposer is thus the one with the lowest engagement at probability 0 – that is, who is least willing to participate in the contest entirely. Figure 3 is a contour plot of the probability that the sometimes-inactive proposer enters the contest as a function of the ideology  $x_R$  and costs  $\alpha_R$  of the right proposer, holding parameters of the left proposer fixed. In the purple region the right proposer is sometimes inactive; increases in her ideological extremism  $|x_R|$  or decreases in her costs  $\alpha_R$  promote greater parity in the contest and an increase in the right proposer's activity. In the blue region the left proposer is sometimes inactive, and increases in the right proposer's extremism or decreases in her costs promote imbalance and a decrease in the left proposer's activity. Along the white curve both proposers are always active.

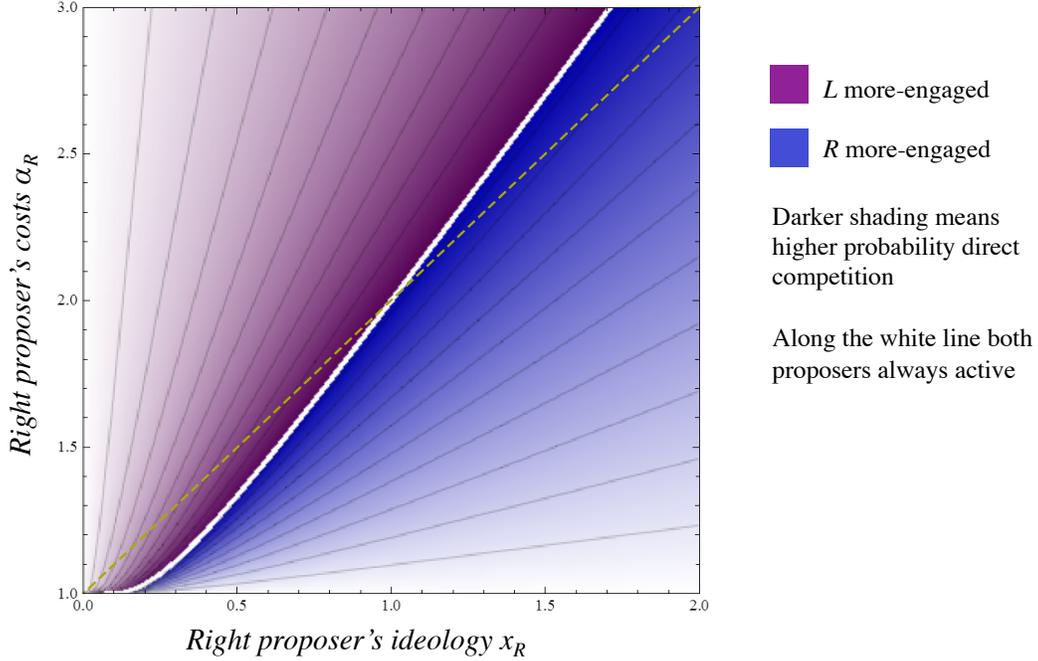
The pattern of activity thus resembles two-player all pay contests without spillovers. Observable competition is a function of *parity* in the motivations and abilities of the participants. In addition, the probability one player stays out approaches 1 as the *imbalance* between them becomes arbitrarily extreme (Baye, Kovenock and de Vries (1996); Hillman and Riley (1989)). In the present contest, this occurs if the sometimes-inactive proposer's ideal  $x_k$  approaches the DM's (so she loses her ideological motive to participate in the contest), her marginal cost of producing quality  $\alpha_k$  becomes arbitrarily high, her opponent's ideal  $x_{-k}$  moves arbitrarily far from the DM (so he becomes arbitrarily motivated), or her opponent's marginal cost of quality  $\alpha_{-k}$  approaches its intrinsic value of 1.

We next use the engagement equality to express and analyze the players' probabilities of victory in the contest.

**Proposition 3.** *In equilibrium the probability proposer  $k$  loses the contest is*

$$\int_0^1 p_{-k}(\epsilon_k(p)) dp = \int_0^1 \left( \alpha_{-k} - (\alpha_{-k} - 1) \left( \frac{\alpha_k - p}{\alpha_k - 1} \right)^{\left| \frac{x_k}{x_{-k}} \right|} \right) dp$$

Figure 3: Effect of R's Parameters on Probability of Direct Competition ( $x_L = -1, \alpha_L = 2$ )



which is decreasing in her distance from the DM  $|x_k|$  and her opponent's quality costs  $\alpha_{-k}$ , and increasing in her opponent's distance from the DM  $|x_{-k}|$  and her own quality costs  $\alpha_k$ .

**Proof:** The probability  $k$  loses the contest is  $\int_0^{\bar{s}} f_{-k}(s) F_k(s) ds$ ; applying the engagement equality this is  $\int_0^{\bar{s}} p_{-k}(\epsilon_k(F_{-k}(s))) f_{-k}(s) ds$ , and applying a change of variables of  $F_{-k}(s)$  for  $p$  (recalling  $F_{-k}(0) = 0$ ) yields the result. QED

The probability  $k$  loses thus obeys the same comparative statics as her probability of inactivity. She becomes *less* likely to win when her preferences are closer to the DM or her opponent's are more distant. Conversely, she becomes more likely to win if she is more able or her opponent less able.

## 4.2 Dominance

We may also use engagement equality to characterize when the sometimes-inactive proposer  $k$  will also be *dominated*, in the sense of making proposals that are first-order stochastically worse for the DM (and by implication less likely to win).

**Lemma 6.** *Proposer  $i$  is dominated ( $F_{-i}(s) < F_i(s) \forall s \in (0, \bar{s})$ ) i.f.f. she is less engaged at every probability  $p$  ( $\epsilon_i(p) < \epsilon_{-i}(p) \forall p \in (0, 1)$ ). Equivalently, she is dominated i.f.f. she is both less engaged at probability 0 ( $i = k \iff \int_0^1 \frac{|x_k|}{\alpha_k - q} dq \leq \int_0^1 \frac{|x_{-k}|}{\alpha_{-k} - q} dq$ ) and  $\frac{|x_k|}{\alpha_k - 1} \leq \frac{|x_{-k}|}{\alpha_{-k} - 1}$ .*

**Proof:** Lemma 5 and  $\epsilon_i(p)$  strictly decreasing when  $p \in [0, 1)$  immediately implies that  $\text{sign}(\epsilon_{-k}(F_{-k}(s)) - \epsilon_k(F_{-k}(s))) = \text{sign}(F_k(s) - F_{-k}(s)) \forall s \in [0, \bar{s})$ , which straightforwardly yields the first statement. Now let  $\delta(p) = \epsilon_{-k}(p) - \epsilon_k(p)$ , so  $\delta(0) \geq 0 = \delta(1)$ . We argue  $\delta'(1) \leq 0$  is necessary and sufficient. For necessity,  $\delta'(1) > 0 = \delta(1) \rightarrow \delta(p) < 0$  in a neighborhood below 1. For sufficiency, it is easily verified that  $\delta'(p) = \frac{|x_k|}{\alpha_k - p} - \frac{|x_{-k}|}{\alpha_{-k} - p}$  crosses 0 at most once when the proposers are asymmetric; thus  $\delta(0) \geq 0 = \delta(1) \geq \delta'(0)$  implies  $\delta(p)$  strictly quasi-concave over  $[0, 1]$  and  $\delta(p) > \min\{\delta(0), \delta(1)\} \geq 0$  for  $p \in (0, 1)$ . QED

The sometimes-inactive proposer (who by definition is less engaged at probability 0, i.e.  $\int_0^1 \frac{|x_k|}{\alpha_k - q} dq \leq \int_0^1 \frac{|x_{-k}|}{\alpha_{-k} - q} dq$ ) is therefore not necessarily dominated. Rather, domination requires the stronger condition that  $k$  be less engaged at *every* probability  $p \in [0, 1)$ ; the lemma shows that lower engagement at 0 and  $\frac{|x_k|}{\alpha_k - 1} \leq \frac{|x_{-k}|}{\alpha_{-k} - 1}$  are jointly necessary and sufficient for this property. Thus, unlike the standard two-player all-pay contest, there may not be an unambiguously “stronger” player.

It is transparently the case that a proposer  $k$  who is *both* less extreme ( $|x_k| \leq |x_{-k}|$ ) and less able ( $\alpha_k \geq \alpha_{-k}$ ) (with one strict) satisfies both conditions and is therefore dominated. This implies somewhat counterintuitively that a proposer who is more ideologically distant from the DM is actually dominant and strictly more likely to have her proposal selected (provided she is no less able at generating quality). Less obvious is which condition binds when the one proposer is more extreme while the other is more able; analyzing this case yields the following.

**Proposition 4.** *Suppose the proposers’ extremism and costs are both distinct. Then proposer  $k$  is dominated i.f.f. her opponent is lower cost ( $\alpha_{-k} < \alpha_k$ ) or higher cost ( $\alpha_{-k} > \alpha_k$ ) but also substantially more extreme ( $|x_{-k}| \geq \left(\frac{\alpha_{-k} - 1}{\alpha_k - 1}\right) |x_k|$ ).*

**Proof:** We argue  $\delta(0) \geq 0$  and  $\alpha_k > \alpha_{-k} \rightarrow \delta'(1) < 0$ . Observe that  $\alpha_k < \alpha_{-k}$  and  $\delta'(0) = \frac{x_k}{\alpha_k} - \frac{x_{-k}}{\alpha_{-k}} \leq 0 \rightarrow \delta'(1) = \frac{|x_k|}{\alpha_k} \left(\frac{1}{1 - 1/\alpha_k}\right) - \frac{|x_{-k}|}{\alpha_{-k}} \left(\frac{1}{1 - 1/\alpha_{-k}}\right) < 0$ . If  $\delta'(0) \leq 0$  we are done; if  $\delta'(0) > 0$  then  $\delta'(1) \geq 0 \rightarrow \delta'(p) > 0 \forall p \in [0, 1) \rightarrow \delta(1) > 0$ , a contradiction. QED

If proposer  $k$  also has higher costs this is sufficient for her to be dominated (that is, the constraint of lower engagement at 0 binds). If instead proposer  $k$  is cost-advantaged,

then the addition constraint  $\frac{|x_k|}{\alpha_k-1} \leq \frac{|x_{-k}|}{\alpha_{-k}-1}$  binds, implying that  $-k$  must be substantially more ideologically extreme. This is because the proposers place some intrinsic value on quality; this decreases the *net* cost of investing quality in higher score proposals, since such proposals are more likely to win and the intrinsic benefits of quality are more likely to be enjoyed. Consequently, if the less active proposer  $k$  happens to be cost-advantaged, this cost advantage will become magnified at higher scores, potentially preventing score-domination even if she is more moderate.

To see this concretely, consider two proposers ( $L, R$ ) who are equally active ( $\epsilon_L(0) = \epsilon_R(0)$ ) because  $L$  is a less able “extremist” while  $R$  is a more able “moderate” ( $|x_L| > |x_R|$  and  $\alpha_L > \alpha_R$ ). The two panels of Figure 4 depict such an equilibrium. The moderate strictly dominates the contest and is strictly more likely to win even though both proposers are equally likely to participate. If the example is then perturbed slightly to make  $R$  slightly less able, then she will sometimes be inactive (and therefore no longer dominant), but remain strictly more likely to win the contest (by continuity) and therefore not dominated either.

### 4.3 Ideology

Surprisingly, the engagement equality also yields simple analytic expressions for the probability distribution over the *ideology* of each player’s proposals.

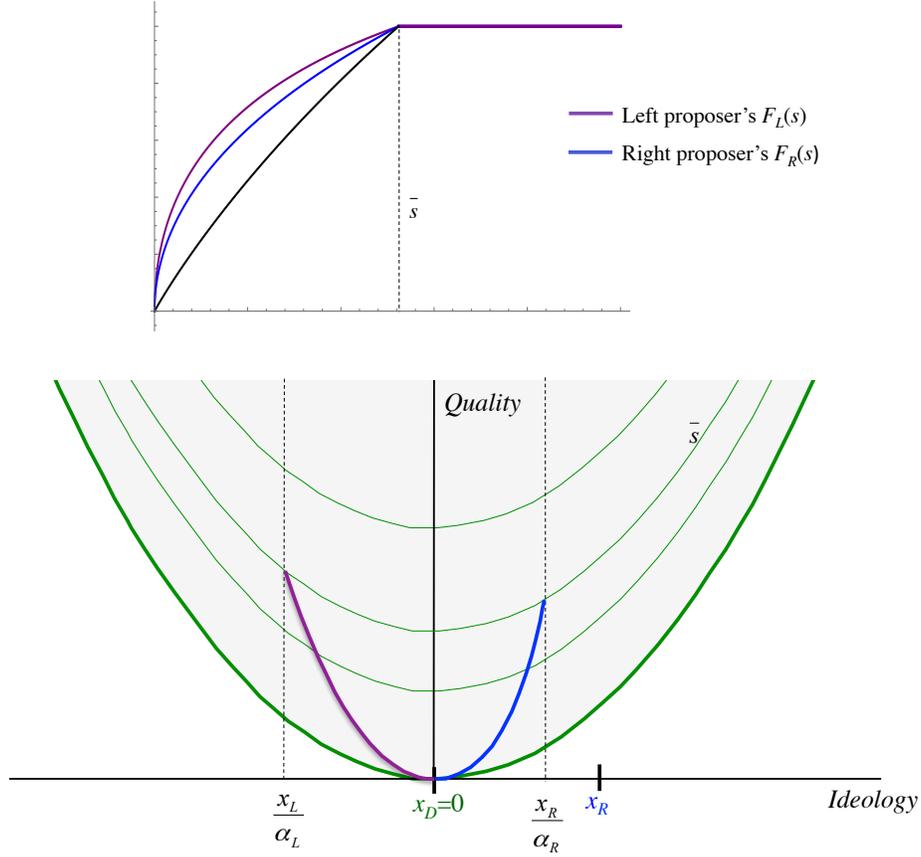
**Proposition 5.** *Let  $G_i(y) = \Pr(|y_i| \leq |y|)$  denote the probability that  $i$ ’s proposal is closer to the DM than  $y$ . Then*

$$G_i(y) = p_{-i} \left( \epsilon_i \left( \frac{y}{x_i/\alpha_i} \right) \right) = \alpha_{-i} - (\alpha_{-i} - 1) \left( \frac{x_i - y}{x_i - x_i/\alpha_i} \right)^{\left| \frac{x_i}{x_{-i}} \right|},$$

*which is first-order stochastically increasing in  $i$ ’s distance from the DM  $|x_i|$ , decreasing in her costs  $\alpha_i$ , decreasing in her opponent’s distance from the DM  $|x_{-i}|$ , and increasing in her opponent’s costs  $\alpha_i$ .*

**Proof:** Proposer  $i$ ’s ideology at score  $s$  is  $y_i^*(s) = \frac{x_i}{\alpha_i} F_{-i}(s)$  (from ideological optimality), implying  $F_{-i}(s_i^*(y)) = \frac{y}{x_i/\alpha_i}$  where  $s_i^*(y)$  is the inverse of  $y_i^*(s)$ . That is, the probability  $-i$  makes a proposal with score  $\leq s_i^*(y)$  is  $\frac{y}{x_i/\alpha_i}$ . In addition, the probability  $G(|y|)$  that  $i$  makes a proposal closer to the DM than  $y$  is just  $F_i(s_i^*(y))$ , which is  $= p_{-i}(\epsilon_i(F_{-i}(s_i^*(y)))) = p_{-i} \left( \epsilon_i \left( \frac{y}{x_i/\alpha_i} \right) \right)$  from the engagement equality. Comparative statics are straightforward. QED

Figure 4: Equilibrium with Equal Engagement ( $x_R < -x_L$ ,  $\alpha_L > \alpha_R$ )



When a proposer  $i$  becomes more ideologically extreme (higher  $|x_i|$ ) or able (lower  $\alpha_i$ ), she reacts by making first-order stochastically more extreme proposals. In the former case she is more motivated to exploit quality to realize ideological gains, and in the latter case she is better able to do so. Combined with the previous results, this means that a more extreme (but no less able) proposer *or* a more able (but no more extreme) proposer maintains her dominance in the contest *despite* the greater ideological extremism of her proposals.

How does proposer  $i$ 's opponent  $-i$  react to  $i$  becoming more extreme or more able? By *moderating* the ideological extremism of her own proposals (first-order stochastically) to improve her chances of winning the contest. A more extreme or able  $i$  endogenously increases the ideological stakes of the contest with her more-extreme proposals, thereby increasing the importance of victory to  $-i$  and the value of moderation.

## 5 Final Equilibrium Characterization and Payoffs

We complete the analysis by calculating payoffs, which requires fully characterizing the unique score CDFs  $F_i(s)$  satisfying Proposition 1.

**Proposition 6.** *The unique score CDFs over  $s \geq 0$  satisfying Proposition 1 are  $F_i(s) = p_{-i}(\epsilon(s)) \forall i$ , where  $\epsilon(s)$  is the inverse of*

$$s(\epsilon) = 2 \int_{\epsilon}^{\epsilon_k(0)} \sum_j \frac{|x_j|}{\alpha_j} p_j(\hat{\epsilon}) d\hat{\epsilon}.$$

*Alternatively, the inverse score CDFs are  $s_i(F_i) = s(\epsilon_{-i}(F_i)) \forall i$ . The function  $s(\epsilon)$  is strictly increasing in  $x_i$  and strictly decreasing in  $\alpha_i \forall \epsilon \in [0, \epsilon_k(0))$ , and the maximum score is  $\bar{s} = s(0)$ .*

**Proof:** From the engagement equality  $\epsilon_{-i}(F_i(s)) = \epsilon(s) \forall i$  for some  $\epsilon(s)$ ; we characterize the unique  $\epsilon(s)$  generating score CDFs  $F_i(s) = p_{-i}(\epsilon(s))$  satisfying score optimality. The differential equations may be rewritten as  $-\frac{f_{-i}(s) \cdot |x_i|}{\alpha_i - F_{-i}(s)} = -\left(2 \sum_j \frac{|x_j|}{\alpha_j} F_{-j}(s)\right)^{-1} \forall i$ . Substituting in  $\epsilon'(s) = \frac{\partial}{\partial s}(\epsilon_i(F_{-i}(s))) = -\frac{f_{-i}(s) \cdot |x_i|}{\alpha_i - F_{-i}(s)}$  yields  $\epsilon'(s) = -\left(2 \sum_j \frac{|x_j|}{\alpha_j} F_{-j}(s)\right)^{-1}$ . Rewriting with the inverse  $s(\epsilon)$  and using  $F_i(s(\epsilon)) = p_{-i}(\epsilon)$  yields  $s'(\epsilon) = -2 \sum_j \frac{|x_j|}{\alpha_j} p_j(\epsilon)$ . Lastly  $\epsilon_k(F_{-k}(s)) = \epsilon(s)$  and  $F_{-k}(0) = 0$  imply the boundary condition  $s(\epsilon_k(0)) = 0$  so  $s(\epsilon) = -\int_{\epsilon}^{\epsilon_k(0)} s'(\hat{\epsilon}) d\hat{\epsilon} = 2 \int_{\epsilon}^{\epsilon_k(0)} \sum_j \frac{|x_j|}{\alpha_j} p_j(\hat{\epsilon}) d\hat{\epsilon}$ . Now  $s(\epsilon)$  is straightforwardly increasing in  $|x_i|$  and decreasing in  $\alpha_i$  given previous observations about  $p_j(\hat{\epsilon})$ . QED

An interesting implication of the equilibrium characterization is that the maximum score  $\bar{s}$  changes continuously with the parameters of both proposers, even when one is dominant. This contrasts with the standard 2-player all pay contest, where the mixing interval is unaffected by the parameters of the stronger player.

The preceding characterization of the unique equilibrium score CDFs transparently yields the following comparative statics.

**Corollary 1.** *Increasing a proposer's extremism  $|x_i|$  or decreasing her costs  $\alpha_i$  first-order stochastically increases her own score CDF, but has ambiguous effects on her opponent's score CDF.*

To see that the effect of a proposer's parameters on her opponent's score CDF is necessarily ambiguous, suppose that the always-active proposer  $-k$  becomes even more extreme or able.

Then her opponent  $k$  becomes less likely to be active, but also the range of scores  $[0, \bar{s}]$  over which she mixes *when* she is active increases. She thus has a higher probability of making very high-score proposals, even while she is simultaneously less likely to enter the contest.

## 5.1 Proposer Payoffs

Using Proposition 6, the proposers' equilibrium payoffs are as follows.

**Proposition 7.** *Proposer  $i$ 's utility is  $\Pi_i^*(\bar{s}; F^*) = -\left(1 - \frac{1}{\alpha_i}\right) x_i^2 - (\alpha_i - 1) \bar{s}$ , which is decreasing in her own costs  $\alpha_i$  as well as either players' extremism  $|x_j| \forall j$ , and increasing in her opponent's costs  $\alpha_{-i}$ .*

**Proof:** A proposer's equilibrium utility is straightforward since  $(\bar{s}, y_i^*(\bar{s}))$  is in the support of their strategy and wins for sure. Comparative statics of a proposer's  $i$ 's parameters on her opponent  $-i$ 's utility, as well as of  $x_i$  on her own utility, follow immediately from previously-shown statics on  $\bar{s} = s(\epsilon)$ . Taking the derivative with respect to  $\alpha_i$ , substituting in  $\frac{\partial}{\partial \alpha_i} \left( \frac{p_i(\epsilon)}{\alpha_i} \right) = \frac{x_i p_i'(\epsilon)}{(\alpha_i - 1) \alpha_i^2}$ ,  $\frac{\partial \epsilon_k(0)}{\partial \alpha_k} = -\frac{|x_k|}{\alpha_k(\alpha_k - 1)}$ ,  $-\frac{p_i'(\epsilon) x_i}{\alpha_i - p_i(\epsilon)} = 1$ , performing a change of variables, and rearranging the expression yields  $-\mathbf{1}_{i=k} \cdot 2 \int_0^{\epsilon_k(0)} \frac{|x_{-k}|}{\alpha_{-k}} \left( p_{-k}(\epsilon) - \left( \alpha_k \log \left( \frac{\alpha_k}{\alpha_k - 1} \right) \right)^{-1} p_{-k}(\epsilon_k(0)) \right) d\epsilon - \left( \frac{|x_i|}{\alpha_i} \right)^2 \left( 1 + 2 \int_{p_i(\epsilon_k(0))}^1 \left( \frac{\alpha_i p}{\alpha_i - p} - 1 \right) dp \right)$ . The first term is negative since  $p_{-k}(\epsilon) > p_{-k}(\epsilon_k(0))$  for  $\epsilon < \epsilon_k(0)$  and  $\frac{1}{\alpha_k} < \int_0^1 \frac{1}{\alpha_k - p} dp = \log \left( \frac{\alpha_k}{\alpha_k - 1} \right)$ . The second term is also negative since  $1 + 2 \int_{p_i(\epsilon_k(0))}^1 \left( \frac{\alpha_i p}{\alpha_i - p} - 1 \right) > (1 - p_i(\epsilon_k(0))) + 2 \int_{p_i(\epsilon_k(0))}^1 \left( \frac{\alpha_i p}{\alpha_i} - 1 \right) = \int_{p_i(\epsilon_k(0))}^1 (2p - 1) dp \geq 0$ . QED

A proposer's equilibrium utility has two components. The first  $-\left(1 - \frac{1}{\alpha_i}\right) x_i^2$  is her utility if she could make proposals as a "monopolist" (and the DM's outside option included  $(0, 0)$ ). The second  $-(\alpha_i - 1) \bar{s}$  is the cost generated by competition, which forces her to make proposals that leave the DM strictly better off than the best "free" proposal  $(0, 0)$  in order to maintain influence. This competition cost is increasing in  $i$ 's marginal cost  $\alpha_i$  of generating quality (holding  $\bar{s}$  fixed) as well as the maximum score  $\bar{s}$ , which in turn is increasing in both proposers' ideological extremism and decreasing in their costs everywhere in the parameter space.

A proposer is thus strictly harmed when her competitor becomes more extreme. She is also harmed when her competitor becomes more able, despite the fact that ability is channeled into common value investments. In the model these investments are exploited to

gain ideological influence. Both effects are distinct from all pay contests without spillovers (Siegel (2009)), where the equilibrium utility of the “sometimes inactive” player is pinned at her fixed value for losing, and is unaffected by changes in her opponent’s parameters unless they influence who is more active.

A proposer also worse off when her *own* preference become more distant from the decisionmaker for two reasons; it becomes more expensive to make equally-competitive proposals that are the same distance from her ideal, and the intensity of competition increases. Finally, she is worse off when her costs of producing quality increase – even though there is a countervailing effect of reducing the intensity of competition (and indeed, the competition cost  $(\alpha_i - 1)\bar{s}$  alone is not generically monotonic in  $\alpha_i$ ).

## 5.2 DM Payoffs

Again using Proposition 6, the DM’s equilibrium utility and the proposers’ average scores (which bound the DM’s utility from below) are as follows.

**Proposition 8.** *The DM’s utility is  $U_{DM}^* = \int_{\epsilon_k(0)}^0 s(\epsilon) \cdot \frac{\partial}{\partial \epsilon} \left( \prod_j p_j(\epsilon) \right) d\epsilon =$*

$$2 \int_0^{\epsilon_k(0)} \left( 1 - \prod_j p_j(\epsilon) \right) \cdot \left( \sum_j \frac{|x_j|}{\alpha_j} p_j(\epsilon) \right) d\epsilon$$

*Proposer  $i$ ’s average score is  $E[s_i] = \int_{\epsilon_k(0)}^0 s(\epsilon) \cdot \frac{\partial}{\partial \epsilon} (p_{-i}(\epsilon)) d\epsilon =$*

$$2 \int_0^{\epsilon_k(0)} (1 - p_{-i}(\epsilon)) \cdot \left( \sum_j \frac{|x_j|}{\alpha_j} p_j(\epsilon) \right) d\epsilon$$

**Proof:**  $F_i(s) F_{-i}(s)$  is the CDF of  $\max\{s_i, s_{-i}\}$  so the DM’s utility is  $\int_0^{\bar{s}} s \cdot \frac{\partial}{\partial s} \left( \prod_j F_j(s) \right) ds = \int_0^{\bar{s}} s \cdot \frac{\partial}{\partial s} \left( \prod_j p_j(\epsilon(s)) \right) ds$ . A change of variables from  $s$  to  $\epsilon$  yields the first expression and integration by parts and rearranging yields the second. A nearly identical series of steps yields  $i$ ’s average score. QED

Direct comparative statics on the DM’s utility  $U_{DM}^*$  are not straightforward because changing a proposer’s parameters has mixed effects on her opponent’s score CDF. However,

the fact that unilaterally greater extremism and ability always strictly increases the maximum score  $\bar{s}$  suggests that *imbalance* between the participants may be beneficial to the DM if it results from one proposer's greater motivation or ability. This is quite different from the standard complete-information all pay contest, where imbalance reduces the participation of the weaker player and makes the "auctioneer" worse off ((Baye, Kovenock and de Vries (1996); Hillman and Riley (1989))). The following local comparative statics result verifies this intuition.

**Proposition 9.** *When the proposers are symmetric ( $|x_i| = |x_{-i}|$  and  $\alpha_i = \alpha_{-i}$ ), the DM's utility is locally increasing in either's extremism or ability.*

**Proof:** First differentiating the DM's utility  $U_{DM}^*$  w.r.t.  $|x_{-k}|$  and applying symmetry yields  $\frac{2}{\alpha} \int_0^{\epsilon^{(0)}} \left( (1 - 3(p(\epsilon))^2) \cdot x \frac{\partial p(\epsilon)}{\partial x} + (1 - (p(\epsilon))^2) p(\epsilon) \right) d\epsilon$  which is transparently  $\geq 2 \frac{x}{\alpha} \int_0^{\epsilon^{(0)}} (1 - 3(p(\epsilon))^2) \frac{\partial p(\epsilon)}{\partial x} d\epsilon$ . Now substituting  $\frac{\partial p(\epsilon)}{\partial x} = -\log\left(\frac{\alpha-p(\epsilon)}{\alpha-1}\right) p'(\epsilon)$  and a change of variables yields  $2 \frac{x}{\alpha} \int_0^1 (1 - 3p^2) \log\left(\frac{\alpha-p}{\alpha-1}\right) dp = 2 \frac{x}{\alpha} \int_0^1 \left(\frac{p-p^3}{\alpha-p}\right) dp > 0$ . Next differentiating  $U_{DM}^*$  w.r.t.  $\alpha_{-k}$  and applying symmetry yields  $2x \int_0^{\epsilon^{(0)}} \left( (1 - (p(\epsilon))^2) \frac{\partial}{\partial \alpha} \left(\frac{p(\epsilon)}{\alpha}\right) - \frac{2}{\alpha} (p(\epsilon))^2 \frac{\partial p(\epsilon)}{\partial \alpha} \right) d\epsilon$ . Substituting  $\frac{\partial}{\partial \alpha} \left(\frac{p(\epsilon)}{\alpha}\right) = \frac{x}{(\alpha-1)\alpha^2} p'(\epsilon)$ ,  $\frac{\partial p(\epsilon)}{\partial \alpha} = -\left(\frac{1-p(\epsilon)}{\alpha-1}\right)$ ,  $-\frac{p'(\epsilon)x}{\alpha-p(\epsilon)} = 1$ , rearranging, and a change of variables yields  $\frac{2x^2}{(\alpha-1)\alpha^2} \int_0^1 \left( 2p^2 \left(\frac{\alpha-\alpha p}{\alpha-p}\right) - (1 - p^2) \right) dp < 0$ . QED

The DM thus strictly benefits if symmetry between the players is broken by one becoming more extreme or able – even though the other also becomes less active.

Lastly, the effect of asymmetry on the DM's utility may be examined globally by considering his utility in the limit as the contest becomes very imbalanced.

**Proposition 10.** *The DM's utility exhibits the following limiting behavior*

$$0 = \lim_{\alpha_i \rightarrow \infty} U_{DM}^* = \lim_{x_i \rightarrow 0} U_{DM}^* < \lim_{\alpha_i \rightarrow 1} U_{DM}^* < \lim_{x_i \rightarrow \infty} U_{DM}^* = \infty$$

**Proof:** Observe that  $E[s_{-k}] \leq U_{DM}^* \leq \bar{s}$ . For the first two limiting statements it is easily verified that  $\bar{s} \rightarrow 0$  as  $\alpha_k \rightarrow \infty$  or  $x_k \rightarrow 0$ . For the third limiting statement it is easily verified that  $\lim_{\alpha_{-k} \rightarrow 1} E[s_{-k}]$  and  $\lim_{\alpha_{-k} \rightarrow 1} \bar{s}$  are strictly positive and finite. For the fourth limiting statement observe that  $E[s_{-k}] \geq \frac{|x_{-k}|}{\alpha_{-k}} p_{-k}(\epsilon_k(0)) \cdot 2 \int_0^{\epsilon_k^{(0)}} (1 - p_k(\epsilon)) d\epsilon$  which  $\rightarrow \infty$  as  $|x_{-k}| \rightarrow \infty$  since the first term  $\rightarrow \infty$  and the remaining terms are non-decreasing. QED

If an extreme imbalance is the result of one proposer's incompetence *or* ideological moderation, the DM's utility approaches 0, her utility if  $-i$  were a "monopolist" (and the DM's

outside options included  $(0, 0)$ ). (Proposer  $-i$ 's utility also approaches her utility if she were a monopolist). The DM thus never benefits from having one proposer be an ideological “ally” vs. someone someone extreme – an ally is unmotivated to invest in quality because her ideological interests are already represented by the DM (see also [Hirsch and Shotts \(2015\)](#)).

However, if extreme imbalance is the result of one proposer’s greater ability to produce quality (specifically, if her marginal cost of producing quality approaches its intrinsic value), then the DM’s utility is bounded away from 0. In this case the DM strictly benefits from the potential for competition, even though actual competition is almost never observed (since  $F_{-i}(0) = F_k(0)$  approaches 1).

Finally and most strikingly, unilateral ideological extremism benefits the decisionmaker in a strong sense; the DM can achieve arbitrarily high utility with a proposer whose preferences are sufficiently *distant* from her own. As one proposer becomes increasingly unilaterally extreme, her proposals also become increasingly extreme, but the DM becomes increasingly better off due to the proposals’ greater quality. At the same time, the extremist’s opponent becomes increasingly unlikely to enter the contest, moderate in her proposals, and (unboundedly) worse off.

## 6 Discussion and Conclusion

This paper considers a contest among ideologically-motivated players. Two proposers have differing ideological objectives from a decisionmaker and each other. They may make costly up-front investments to increase the value of their proposals to all players, termed *quality* (valence in the political science literature), and may have differential ability at doing so. In the resulting equilibrium, the proposers exploit a mixture of quality investments and ideological concessions to gain the support of the decisionmaker. The model is a generalization of [Hirsch and Shotts \(2015\)](#) that allows for asymmetric proposers, differential weight between quality and ideology, and different potential outside options for the decisionmaker (including no outside option as in a two-candidate election).

Applications of the model in political science and political economy are widespread and include elections, lawmaking, lobbying, bureaucratic policy implementation, and judicial opinion writing. More generally, the model may be applied to organizational choice when there are a mixture of competing and common interests, freedom among several individuals or groups to make proposals, and a unitary decisionmaker lacking the ability to write ex-ante contracts and facing a deadline.

Although the model is exceedingly simple to formulate, it results in a contest with complexities that are difficult to analyze jointly. It is complete information, multidimensional, and asymmetric. Strategies have both “all-pay” and “winner-pay” components. Most importantly, it features a “second-order rank over spillover” – the strategy of the (first ranked) winner exerts a direct effect on the utility of the (second ranked) loser, because she must live under the winner’s policy. We show that the game yields a unique equilibrium in continuous mixed strategies with an elegant analytical characterization. The solution methods employed can be used to solve similar models in which players compete to have their preferred ideological policies enacted by exerting costly up-front effort (for example, if effort represented campaign advertising in an election that only influenced the voters).<sup>5</sup>

Result from the model are counterintuitive. Key themes are two-fold: (i) unilaterally greater extremism benefits the DM because the resulting motivation is partially channelled into productive investment; even though the resulting policies are also more extreme as a byproduct, and the other proposer is harmed; (ii) greater ability at common-value investments has similar equilibrium effects to greater ideological extremism, because such investments are exploited for ideological gains.

Formal results are numerous and spread throughout the paper; we provide a self-contained review here. Unless otherwise noted, comparative statics statements are in a first-order stochastic sense. First, the (essentially) unique equilibrium is in continuous mixed strategies; each proposer mixes over a curve of proposals with different ideologies and qualities. Second, in equilibrium more ideologically-extreme proposals in the support are strictly better for the DM than more moderate ones. Third, one proposer is always active in the sense of making a positive-quality proposal – she is the one with the highest value of  $\int_0^1 \frac{|x_i|}{\alpha_i - q} dq$ . Fourth, a proposer is dominant in the sense of making first-order stochastically better proposals for the DM i.f.f. she has the higher value of both  $\int_0^1 \frac{|x_i|}{\alpha_i - q} dq$  and  $\frac{|x_i|}{\alpha_i - 1}$ . Fifth, comparative statics are as follows.

**Comparative Statics.** *Increasing proposer  $i$ ’s extremism  $|x_i|$ :*

- *either increases her own probability of being active (if she is sometimes-inactive), or decreases her opponent’s probability of being active*
- *increases her probability of winning*
- *maintains her dominance if she was already dominant*

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<sup>5</sup>Solving similar models computationally is straightforward; analytic solutions are elusive.

- *makes her proposals more extreme but also better for the DM*
- *makes her opponent's proposals more moderate, and neither better nor worse unambiguously for the DM*
- *decreases both her own and her opponent's utility*

*Decreasing proposer  $i$ 's costs  $\alpha_i$  yields all the same aforementioned effects, except that it increases rather than decreases proposer  $i$ 's utility*

Sixth, the DM benefits when symmetry between the proposers is broken because one becomes unilaterally more extreme or able. Lastly, if a proposer becomes arbitrarily extreme or able, the probability of observing direct competition approaches zero, but the DM's utility is bounded away from what it would be absent competition. In fact, as one proposer becomes unboundedly extreme the DM's utility becomes unboundedly large.

Our analysis suggests several avenues for future work. One is to expand the number of possible participants in the contest. This is natural in settings where proposals are made by individuals rather than teams, or when proposals may come from varied sources (e.g., interest groups as well as legislators developing potential legislation). It is straightforward to show that when costs are common, there always exists an equilibrium in which the two most ideologically-extreme potential proposers play their equilibrium strategies in the present model, while the other potential proposers are inactive. (Hirsch and Shotts (2015) contains the weaker result for the symmetric model).

A second is to allow the proposers to also gain the decisionmaker's support using targeted benefits (pork, as in vote buying models) in addition to collective benefits (as in our model), and analyze when they use targeted vs. common benefits.

A third is to study institutional design (again without the ability to write ex-ante contracts). Primitives of the model potentially under control of an institutional designer include the marginal costs of generating quality (using subsidies), the ideology of the proposers (by choosing organizational membership), the ideology of the decisionmaker (through an appointment or collective choice procedure), and other constrained aspects of the "scoring rule" (e.g. providing a "head start" to one proposer).

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