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ON THE ULTIMATE ACCURACY OF SOLAR OSCILLATION FREQUENCY MEASUREMENTS

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ABSTRACT

If one assumes that solar *p*- and *f*-mode oscillations are stochastically excited, then the measured mode properties and solar background noise can be used to calculate maximum-likelihood uncertainties in mode frequency measurements, σ_ν, for a given observation time. Such calculations agree quite well with our current data, if I use the measured background noise, which includes instrumental as well as solar contributions. Assuming negligible instrumental background, I find that it should be possible with a 3 yr continuous observation to measure individual mode frequencies $\nu_{n\ell m}$ to accuracies as high as $\sigma_\nu/\nu \approx 1.4 \times 10^{-5}$, which is a factor of $\sim$5 times better than our current best one-season measurements (the latter from 5.5 months of observation with a duty cycle of 30%). The most precise measurements should be for low-ℓ modes in the 1–2 mHz range, and the longest periods observable in a 3 yr observation (according to this model) will be approximately 20 minutes. These fundamental limitations in the eventual accuracy of *p*-mode frequency measurements are set by the solar background noise and the stochastic nature of the driving mechanism.

Subject headings: methods: data analysis — Sun: oscillations

Measurements of *p*-mode frequencies have become steadily more accurate over the past few years, and inversions of precision *p*-mode data have revealed a number of interesting properties of the solar interior, such as the precise location of the base of the convection zone, and the Sun's internal rotation profile (see Libbrecht & Woodward 1991, and references therein). Eventually the stochastic nature of the *p*-mode excitation mechanism, along with the solar background noise from convection, will severely limit further improvements in precision *p*-mode frequency measurements. Here I estimate these fundamental limits on *p*-mode frequency measurements, given the known *p*-mode amplitudes and linewidths (Libbrecht 1988; Libbrecht & Woodward 1991) along with the measured solar convective noise spectrum (Duvall & Harvey 1986; Jiménez et al. 1988). The results predict the outcome of the next generation of *p*-mode observations and can be used to examine the potential observability (via precision *p*-mode frequency measurements) of different solar phenomena, given a model for the *p*-mode frequency perturbations.

Following the discussion of Duvall & Harvey (1986) and Anderson, Duvall, & Jefferies (1990), we assume the oscillation modes can be treated as independent harmonic oscillators (Kumar & Goldreich 1989), stochastically excited by solar convection. The mean power spectrum for a single mode is written as

$$\bar{S}_{n\ell m}(\nu) = \frac{A_{n\ell m}(\Gamma_{n\ell m}/2)^2}{(\nu - \nu_{n\ell m})^2 + (\Gamma_{n\ell m}/2)^2} + B(\nu) , \qquad (1)$$

where $A_{n\ell m}$ is the mode amplitude, $\nu_{n\ell m}$ is the mode frequency, $\Gamma_{n\ell m}$ is the FWHM linewidth, and $B(\nu)$ is a broad-band background noise spectrum. If we observe for a time T, then at frequencies separated by $\Delta\nu = 1/T$ the values of a sample spectrum $S(\nu)$ are independent with mean $\bar{S}(\nu)$ and distributed as χ^2 with 2 degrees of freedom (Duvall & Harvey 1986).

If $T \gg \Gamma^{-1}$, then it is possible without loss of signal to split the time series into N pieces of length T/N (with $1 \ll N \ll T\Gamma$) and add the independent power spectra together,[1] so that by the central limit theorem the combined sample spectrum is approximately normally distributed, with mean $N\bar{S}$ and variance $\sigma_S^2 = N\bar{S}^2$. With a normally distributed power spectrum the best maximum-likelihood estimate of the mode multiplet frequency $\nu_{n\ell m}$ is accompanied by an uncertainty σ_ν (hereafter dropping the $(n\ell m)$ subscript), where

$$\sigma_\nu^{-2} = \sum \frac{1}{\sigma_S^2}\left(\frac{dS}{d\nu}\right)^2$$

and the sum is over frequency (Brandt 1970). The sum converts to an integral that can be done analytically, giving

$$\sigma_\nu^2 = f(\beta)\,\frac{\Gamma}{4\pi T} , \qquad (2)$$

where $\beta = B/A$ is the inverse signal-to-noise ratio and

$$f(\beta) = (1 + \beta)^{1/2}[(1 + \beta)^{1/2} + \beta^{1/2}]^3 .$$

This expression is a generalization of the more familiar expression for the limit of no background noise (for which $\beta = 0$) (Duval 1990). Note that this simple analytical result is only accurate in the limit $T \gg \Gamma^{-1}$ as stated above. For the lowest measurable *p*-mode frequencies, $\nu \sim 1$ mHz, this limit is not satisfied for current measurements, and the expression loses accuracy. For these circumstances a more precise maximum-likelihood technique can be applied to better estimate the measurement uncertainty (Anderson et al. 1990). For intermediate frequencies, $\nu \sim 3$ mHz, β is small, and the limiting expression $\sigma_\nu \approx \sqrt{\Gamma/4\pi T}$ applies. In spite of its shortcomings, equation (2) does adequately describe the expected frequency measurement uncertainties over most of the *p*-mode range; numerical simulations using parameters like those encountered in solar oscillation measurements give results in fairly good agreement with this expression (Duvall & Harvey 1986; Anderson et al. 1990).

It is interesting to note that if a day/night observing window is used, and if the sidelobes do not interfere with the fit to the central Lorentzian peak (which is the case for many of the low-ℓ *p*-modes), then the only effect on σ_ν is to change β. If β is already small, which is the case for modes near the 3 mHz

[1] Combining power spectra from the different m in a $(n\ell)$ multiplet would have a similar effect, if the frequency splitting is known to sufficient accuracy.

power peak, then going from a single observing site to 24 hr coverage results in only a modest improvement in σ_ν. Of course the gain is much greater at low frequencies where $\beta \gtrsim 1$, or where sidelobe interference is a problem.

It is illuminating to compare our current measurements with this expression, and to calculate the best expected σ_ν for a reasonable observing time (here we take this to be 3 yr, which is the proposed lifetime of both the GONG and SOHO measurements (Harvey et al. 1988; Scherrer, Hoeksema, & Bogart 1988)). Since the mode properties are fairly well known, at least for $\nu > 1$ mHz (Libbrecht 1988; Libbrecht & Woodard 1991), the only additional input needed is the solar broad-band velocity background noise. For this we take $B(\nu) = 100(\nu/\nu_0)^{-1.5}$ (m s^{-1})^2 Hz^{-1} for $\nu \approx \nu_0 = 10^{-3}$ Hz, which is based on both models and measurements of the continuous spectrum (Duvall & Harvey 1986; Jiménez et al. 1988). This noise spectrum is expected to be roughly independent of ℓ for $\ell \lesssim 100$. Note that there is still considerable uncertainty in the measurements and models of $B(\nu)$; the above representation could easily be off by a factor of 2 or more, and it will depend somewhat on the spectral line used in the Doppler observations.

Figure 1 shows a comparison of the measured β from Big Bear Solar Observatory (BBSO) data, along with a calculated β using the above $B(\nu)$ and the measured $A(\nu)$ (Libbrecht & Woodard 1991). The data points in this figure are averages using data from 1986, 1988, and 1989, with $5 \leq \ell \leq 60$; the observed β show negligible time dependence and only a small ℓ dependence in this ℓ range. The calculated curve uses $A_{n\ell}$ data averaged over $5 \leq \ell \leq 60$, but with a small residual ℓ dependence removed in such a way to give $A(\nu)$ extrapolated to $\ell = 0$. The calculated curve for $\nu < 1.4$ mHz assumes a power-law $A(\nu) \sim \nu^{4.6}$ consistent with the observations at slightly higher frequencies. This figure shows that the current BBSO measurements are fairly close to being limited by the solar background noise, not instrumental noise, at least at the lower frequencies. The large discrepancy between the measured and calculated $\beta(\nu)$ at intermediate frequencies is due mostly (and perhaps entirely) to leakage of many neighboring modes and sidelobes into the various $S_{n\ell m}(\nu)$ power spectra, plus the fact

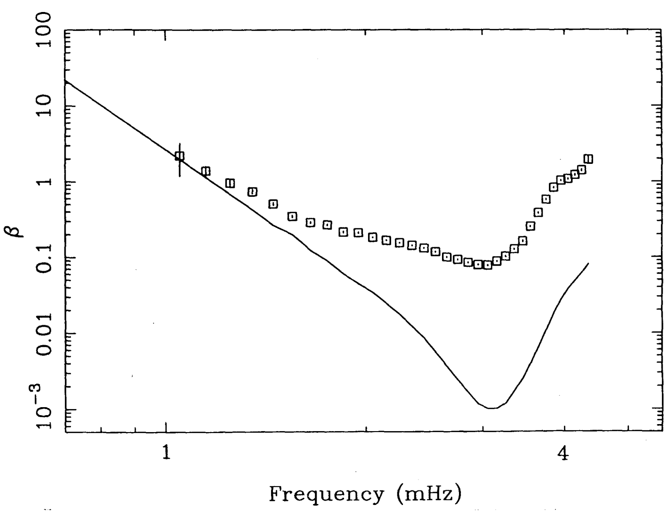

FIG. 1.—A comparison of measured and modeled $\beta = B/A$ (*see text*). The points are from B/A values measured at BBSO and extrapolated to $\ell = 0$; the calculated curve assumes a model background spectrum $B(\nu)$, and the measured mode amplitudes $A(\nu)$, extrapolated to low frequencies as described in the text.

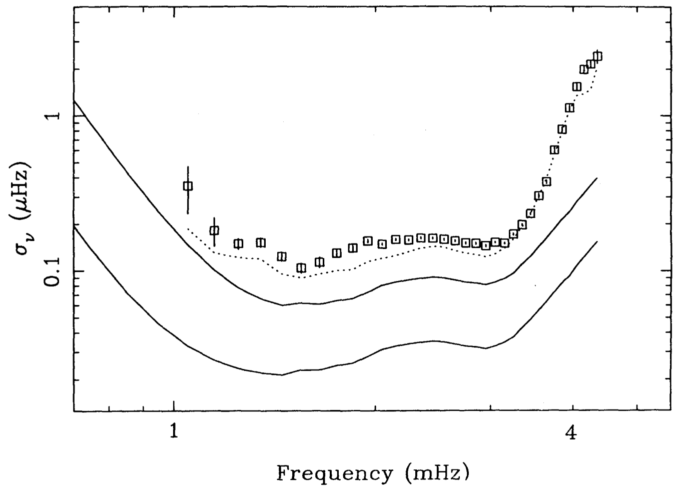

FIG. 2.—A comparison of measured and modeled frequency uncertainties for individual mode frequencies $\nu_{n\ell m}$. The points are from BBSO data, again extrapolated to $\ell = 0$. The model curves come from eq. (2), using the measured mode line widths (extrapolated at the lowest frequencies). The dotted curve uses the measured $\beta(\nu)$ from Fig. 1, with a 5.5 month observing time. The solid curves use the modeled $\beta(\nu)$ from Fig. 1, with continuous observing times of 5.5 months and 3 yr.

that the fits are done over rather small ν ranges (Libbrecht, Woodard, & Kaufman 1990). Looking between the different n ridges at $\ell = 100$, where the spacing is fairly large, the BBSO observations show the background at 3 mHz is only a factor of ~ 3 larger than that at 1 mHz, and this factor too may be due to mode leakage. Proper modeling of the data (and using continuous data to eliminate sidelobes) should help to reduce the observed β.

Figure 2 compares the observed and calculated σ_ν. The former are averages over BBSO data with $5 \leq \ell \leq 60$ from 1986, 1988, and 1989, representing a single 5.5 month observing run with an overall duty cycle of about 30%. The following corrections were also applied to the measured σ_ν: (1) the $\sigma_{\nu_{n\ell}}$ obtained from fits to combined power spectra (Libbrecht et al. 1990) were multiplied by $\sqrt{2\ell + 1}$ to simulate $\sigma_{\nu_{n\ell m}}$ from a single mode; and (2) the measured σ_ν were multiplied by $\sqrt{\Gamma(\ell = 0)/\Gamma(\ell)}$ (Libbrecht & Woodard 1991) at each ℓ to approximate σ_ν at $\ell = 0$. The computed curves assume either the measured or computed $\beta(\nu)$ from Figure 1, along with measured $\Gamma(\nu)$, again extrapolated to $\Gamma(\nu)$ at $\ell = 0$; for $\nu < 1.4$ mHz a power-law $\Gamma \sim \nu^5$ was assumed.

The good agreement in Figure 2 between the measured $\sigma_\nu(\nu)$ and that calculated using the measured $\Gamma(\nu)$ and $\beta(\nu)$ suggests that the above model provides a reasonable description of the solar p-mode frequency measurements. Using this model, and the above power-law extrapolations of $\Gamma(\nu)$ and $A(\nu)$, we can estimate the lowest frequency p- or f-mode frequency observable in a 3 yr observation. Assuming modes can be observed only if $\beta \leq (2\ell + 1)$ (for m-combined power spectra), the lowest observable frequency would be $\nu \approx 0.85$ MHz for $n = 0$ and $\ell \approx 70$.

With these results one can examine the potential observability of different solar phenomena, given a model of their p-mode signal. For example, with sufficient accuracy one might consider observing solar evolution through p-mode frequency shifts. Model calculations by Christensen-Dalsgaard (1986) give $d \log \nu_{n\ell}/dt \approx 5 \times 10^{-11}$ yr^{-1}. Combining σ_ν for $0 \leq \ell \leq 100$, $n \geq 0$, and $\nu < 2$ mHz (the higher frequency modes being contaminated by solar cycle effects—Libbrecht & Woodard 1990), I find $(\sigma_\nu/\nu)_{\rm comb} \approx 10^{-7}$. Thus approximately 6000 yr is needed to achieve a 3 σ detection. Although this length of time is about 100 times shorter than that required to detect solar evolution via a change in solar radius, it is unfortunately still too long to be of any current interest.

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REFERENCES

Anderson, E. R., Duvall, T. L., Jr., & Jefferies, S. M. 1990, ApJ, 364, 699
Brandt, S. 1970, Statistical and Computational Methods in Data Analysis (Amsterdam: North Holland)
Christensen-Dalsgaard, J. 1986, in Seismology of the Sun and Distant Stars, ed. D. O. Gough (Dordrecht: Reidel), 23
Duvall, T. L., Jr. 1990, in IAU Colloq. 121, Inside the Sun, ed. B. Berthomieu & M. Cribier (Dordrecht: Kluwer), 253
Duvall, T. L., Jr., & Harvey, J. W. 1986, in Seismology of the Sun and Distant Stars, ed. D. O. Gough (Dordrecht: Reidel), 105
Harvey, J. W., & the GONG Instrument Team. 1988, in Seismology of the Sun and Sun-Like Stars, ed. E. J. Rolfe (Noordwijk: ESA), 203
Jiménez, A., et al. 1988, A&A, 192, L7
Kumar, P., & Goldreich, P. 1989, ApJ, 343, 558
Libbrecht, K. G. 1988, ApJ, 334, 510
Libbrecht, K. G., & Woodard, M. F. 1990, Nature, 345, 779
———. 1991, Science, 253, 152
Libbrecht, K. G., Woodard, M. F., & Kaufman, J. M. 1990, ApJS, 74, 1129
Scherrer, P. H., Hoeksema, J. T., & Bogart, R. S. 1988, in Seismology of the Sun and Sun-Like Stars, ed. E. J. Rolfe (Noordwijk: ESA), 375