

# A Basic Thermodynamic Derivation of the Maximum Overburden Pressure Generated in Frost Heave

Kenneth G. Libbrecht<sup>1</sup>

Department of Physics, California Institute of Technology  
Pasadena, California 91125

---

## ABSTRACT

I describe a simple “heat-engine” derivation of the maximum overburden pressure that can be generated in frost heave. The method stems from the fact that useful work can, in principle, be extracted from the forces generated by an advancing solidification front via the frost heave mechanism. Using an idealized frost heave “engine,” together with the maximum thermodynamic efficiency of any heat engine, one can derive the maximum overburden pressure. A similar argument can also produce the maximum “thermodynamic buoyancy” force on a foreign object within a solid surrounded by a premelted layer.

## 1. A Frost Heave Engine

Frost heave is a common environmental process in which the freezing of water into ice can produce forces large enough to seriously damage roads and bridges [1]. Contrary to common belief, frost heave is not the result of the simple expansion that takes place when water freezes. Rather, it results when water migrates through a porous soil, forming ice “lenses” within the soil. This migration tends to extract water from the pores in the soil, so the final structure (containing soil plus ice lenses) has a higher volume than the initially saturated soil. Frost heave has also been observed with other materials freezing in porous media [2, 3].

The expansion that results from frost heave is known to produce a maximum overburden pressure  $P_{\max} = q_m \rho \Delta T / T_m$ , where  $q_m$  is the latent heat of fusion per unit mass,  $\rho$  is the ice density,  $T_m$  is the bulk melting temperature, and  $\Delta T = T_m - T_1$ , where  $T_1$  is the temperature at which the ice freezes. This is a well-known result that is typically derived from the modified Clausius-Clapeyron equation (e.g. [1, 3]), in which solidification gains the free-energy  $\Delta S \Delta T$ , where  $\Delta S \approx q_m \rho A L / T_m$  is the entropy change upon freezing.

We can produce a very simple derivation of this same result by constructing the frost heave “engine” shown in Figure 1, in which mechanical work is extracted from heat flow. Referring to Figure 1(a), we have a porous piston separating ice from water in an ideal (frictionless), insulating cylinder of length  $L$  and area  $A$ . As the solidification front pushes the piston down the length of the cylinder, a heat  $\delta Q \approx q_m \rho A L$  must be removed from the left end at temperature  $T_1$ . If the piston travels against an external force  $F$ , then a total work  $\delta W = F L$  is extracted. After this “power stroke,” we complete the engine cycle by warming the ice to  $T_m$ , melting the ice, lowering the water temperature back to  $T_1$ , and flipping cylinder around. We ignore the small heat needed to change the temperature from  $T_1$  to  $T_m$  and back, so the dominant heat input is again  $\delta Q$ , which is applied at a temperature  $T_m$ . Thus, after one cycle of the engine, there has been a net heat flow of  $\delta Q \approx q_m \rho A L$  from  $T_m$  to  $T_1$ , from which we extracted a work  $F L$ . Since the maximum efficiency of any heat engine is  $\varepsilon = \Delta T / T$ , we must have  $F_{\max} = q_m \rho A \Delta T / T_m$ , where  $\Delta T = T_m - T_1$ , which is valid to first order in  $\Delta T$ . This gives the maximum overburden pressure  $P_{\max} = F_{\max} / A = q_m \rho \Delta T / T_m$ , equal to the expression above.

Assuming ice does not enter the porous piston, the motion of the solidification front can be halted by applying a force  $F_{\max}$  to the piston. If the motion of the piston is quasi-static, and the only heat flow through the piston is via liquid flow, then a near-maximum efficiency can be attained, and this is seen in frost heave experiments using both

---

<sup>1</sup> e-mail address: kgl@caltech.edu

water and helium [1, 3].

We can apply the same reasoning to the situation shown in Figure 1(b), for which we now have ice on either side of the piston, with  $T_2 > T_1$ . In this case, the motion of the piston does not change the total amount of ice, except for a negligible change in the thickness of the premelted layer. As melting and resolidification pushes the piston across the cylinder, a heat  $\delta Q \approx q_m \rho A L$  is added to one side of the piston to melt the ice at temperature  $T_2$ , and the same heat is extracted at  $T_1$  to refreeze the ice (again ignoring the small heat required to change the temperature of the water). As before, there is a flow of heat  $\delta Q \approx q_m \rho A L$  from  $T_2$  to  $T_1$ , from which we extract the useful work  $\delta W = F L$ . At maximum efficiency, this yields  $F_{\max}/A = q_m \rho \Delta T/T$ , where  $\Delta T = T_2 - T_1$ .

The idealized piston in Figure 1(b) is functionally equivalent to a foreign (nonporous) particle surrounded by a premelted layer, in which case the fluid flow is around the particle rather than through a porous piston. If the particle has a characteristic size  $R$ , then we can take  $A \approx R^2$  and  $\Delta T = R \nabla T$ , where  $\nabla T$  is the temperature gradient at the object. This gives  $F_{\max} = m_s q_m \nabla T/T_m$ , where  $m_s$  is the mass of the surrounding solid displaced by the object, which is the result reported in [4].

- [1] J. G. Dash, J. Fu, and J. S. Wettlaufer, "The Premelting of Ice and its Environmental Consequences," Rep. Prog. Phys. 58, 115-167 (1995).
- [2] J. G. Dash, "Frost Heave in Helium and Other Substances," J. Low Temp. Phys. 89, 277 (1992).
- [3] T. Mizusaki and M. Hiroi, "Frost Heave in Helium," Physica B 210, 403 (1995).
- [4] A. W. Rempel, J. S. Wettlaufer, and M. G. Worster, "Interfacial Premelting and the Thermomolecular Force: Thermodynamic Buoyancy," Phys. Rev. Lett. 87, 088501 (2001).

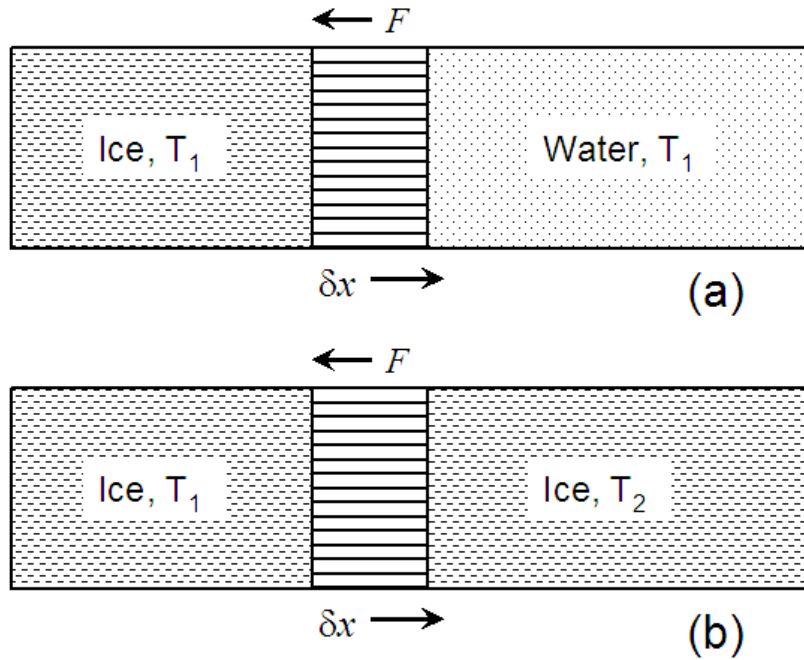


Figure 1. (a) An idealized frost-heave experiment, in which ice at a temperature  $T_1 < T_m$  is separated from water at the same temperature by a porous plug, all within a frictionless, insulating cylinder. As water permeates the plug and solidifies, the solidification front pushes the plug to the right, against an opposing force  $F$ . (b) A similar situation to (a), but with ice at different temperatures on both sides of the cylinder. Here we assume there exists a thin premelted layer on all surfaces within the plug, which allows fluid flow from one side to the other.