DYNAMICAL SYSTEMS

HOMEWORK # 5

HTTP://WWW.ITS.CALTECH.EDU/∼ASGOR/DYNSYS/

1. Prove that powers of an ergodic (weakly mixing, mixing) measure-preserving map are ergodic (weakly mixing, mixing).

2. Prove that a circle rotation R_α is not weakly mixing for any $\alpha \in \mathbb{R}$.

3. Prove that a measure preserving map $f : (M, \mu) \to (M, \mu)$ is ergodic if and only if for any measurable sets A, B

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(f^{-k}(A) \cap B) = \mu(A)\mu(B).$$

4. Let x_k and y_k be a first digit of 2^k and of $3^k + 1$, correspondingly. For each $j = 1, 2, \dots, 9$ find

$$\lim_{n\to+\infty} \frac{\#\{k = 1, 2, \dots, n \,|\, x_k = j\}}{n} \quad \text{and} \quad \lim_{n\to+\infty} \frac{\#\{k = 1, 2, \dots, n \,|\, y_k = j\}}{n}.$$

5. Consider the following map $\alpha : \Sigma_2^+ \to \Sigma_2^+$,

$$(\alpha(\omega))_i = \begin{cases} 1 - \omega_i & \text{if } \ \omega_j = 1 \ \text{ for all } \ j < i, \\ \omega_i & \text{otherwise.} \end{cases}$$

Prove that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is uniquely ergodic. Describe the invariant measure.