DYNAMICAL SYSTEMS

HOMEWORK #4

HTTP://WWW.ITS.CALTECH.EDU/~ASGOR/DYNSYS/

1. Let M be a closed manifold (e.g. $M = \mathbb{T}^2$, or $M = S^3$). For a transitive homeomorphism $f : M \to M$ define an invariant subset $T(f) \subset M$ in the following way:

$$T(f) = \{x \in M \,|\, \text{orbit of } x \text{ is dense in } M\ \}.$$

Is it possible for $T(f)$ to contain only finite number of orbits for some transitive homeomorphism f? Only countable number of orbits?

2. How many conjugacy classes of homeomorphisms of S^1 with the rotation number $\alpha \notin \mathbb{Q}$ there exist? Finite number? Countably many? Uncountably many?

3. Prove that there are no expansive homeomorphisms of S^1.

4. Let $f : S^1 \to S^1$ be an orientation-reversing homeomorphism. Prove that f has exactly two fixed points and does not have periodic points of minimal period greater than 2.

5. Let $\bar{0}$, $\bar{1}$ be the sequences in the space Σ_2 that consist entirely of zeros and ones, respectively. Prove that there exists a point $\omega \in \Sigma_2$ such that for every continuous function φ

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\sigma^k \omega) = \lim_{n\to-\infty} \frac{1}{|n|} \sum_{k=-1}^{n} \varphi(\sigma^k \omega) = \frac{1}{2}\varphi(\bar{0}) + \frac{1}{2}\varphi(\bar{1}) = \int \varphi d\mu,$$

where μ is the probability measure with $\mu(\bar{0}) = \mu(\bar{1}) = \frac{1}{2}$.

6. Given a measure-preserving transformation f in a finite measure space (M, μ) and a measurable subset $A \subset M$ of positive measure, define the *conditional measure* μ_A by

$$\mu_A(B) := \frac{\mu(B \cap A)}{\mu(A)}.$$

The *first return map* $f_A : A \to A$ is defined by $f_A(x) = f^k(x)$, where $k \in \mathbb{N}$ is the smallest natural number for which $f^k(x) \in A$. Prove that $f_A : (A, \mu_A) \to (A, \mu_A)$ is a measure-preserving transformation.

Remark. *The first return map is often called the derivative map, or the Poincare map.*