DYNAMICAL SYSTEMS

HOMEWORK #5

HTTP://WWW.ITS.CALTECH.EDU/~ASGOR/DYNSYS/

1. Given a measure-preserving transformation f in a finite measure space (M, μ) and a measurable subset $A \subset M$ of positive measure, define the *conditional measure* μ_A by

$$\mu_A(B) := \frac{\mu(B \cap A)}{\mu(A)}$$

The first return map $f_A : A \to A$ is defined by $f_A(x) = f^k(x)$, where $k \in \mathbb{N}$ is the smallest natural number for which $f^k(x) \in A$. Prove that $f_A : (A, \mu_A) \to (A, \mu_A)$ is a measure-preserving transformation.

Remark. The first return map is often called derivative map, or Poincare map.

2. Is it true that powers of an ergodic (mixing) measure-preserving map are ergodic (mixing)?

3. Give an example of a continuous map of the real line that does not have non-trivial finite invariant Borel measures.

4. Describe all the ergodic measures of the homeomorphism of the torus $f : \mathbb{T}^2 \to \mathbb{T}^2,\ f(x, y) = (x, x + y)(\bmod 1)$.

5. Consider the following map $\alpha : \Sigma_2^+ \to \Sigma_2^+$,

$$(\alpha(\omega))_i = \begin{cases} 1 - \omega_i, & if\ \omega_i = 1\ for\ \ all\ \ j < i, \\ \omega_i, & otherwise. \end{cases}$$

Prove that $\alpha : \Sigma_2^+ \to \Sigma_2^+$ is uniquely ergodic and describe the invariant measure.