

## Dynamical Systems

### Final Exam

1. Fix  $\alpha \notin \mathbb{Q}$ ,  $\alpha \in (0, 1)$ . Consider a map  $f : [0, 1) \rightarrow [0, 1)$ ,

$$f(x) = \begin{cases} x + 1 - \alpha, & \text{if } x \in [0, \alpha) \\ x - \alpha, & \text{if } x \in [\alpha, 1). \end{cases}$$

- Show that  $f$  preserves the Lebesgue measure on  $[0, 1)$ .
- Prove that  $f$  is ergodic.
- Find  $h_\mu(f)$ .

2. Consider the following map  $f : [-1, 1] \times [-1, 1] \times [-1, 1] \rightarrow \mathbb{R}^3$ ,

$$f(x, y, z) = \begin{cases} \left( \frac{x}{100} - \frac{1}{2}, \frac{y}{100} - \frac{1}{2}, 100z - 50 \right), & \text{if } z \in \left[ \frac{1}{3}, 1 \right], \\ \left( \frac{x}{100}, \frac{y}{100}, 100z \right), & \text{if } z \in \left[ -\frac{1}{3}, \frac{1}{3} \right], \\ \left( \frac{x}{100} + \frac{1}{2}, \frac{y}{100} + \frac{1}{2}, 100z + 50 \right), & \text{if } z \in \left[ -1, -\frac{1}{3} \right]. \end{cases}$$

Let  $\Lambda \subset [-1, 1] \times [-1, 1] \times [-1, 1]$  be a set of points where all (positive and negative) iterates of  $f$  are defined.

- Describe  $\Lambda$ , prove that  $\Lambda$  is a hyperbolic set.
- Show that periodic orbits of  $f$  are dense in  $\Lambda$ .
- Prove that  $f|_\Lambda$  is transitive.
- Find  $h_{top}(f|_\Lambda)$ .