Ma 4, Introduction to Mathematical Chaos

Spring 2006

MIDTERM EXAM

Due Thursday April 27, 2:30pm, 2006.

You may use your notes for this course, your solutions to course homework, and the textbooks. No collaboration or a direct assistance of other people.

All problems are weighted equally.

Time limit 3 hours. If you need more time, you should mark which part of the work was done after 3 hours; you will be given a partial credit for that work.

1) (10) Assume that all prime periods of periodic orbits of a continuous map $f : [0,1] \to [0,1]$ are uniformly bounded (i.e. there exists $N \in \mathbb{N}$ such that the prime period of every periodic orbit of f is smaller than N). What can you say about periods of periodic orbits of f? For example, can f have a periodic orbit of period 2006? Of period 2048?

2) (10) Under what conditions on the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ the curve

$$\gamma(t) = (\sin(\alpha_1 t + \beta_1), \sin(\alpha_2 t + \beta_2)), \quad t \in \mathbb{R}^1,$$

is dense in the square $[-1,1] \times [-1,1]$?

3) (10) Consider the following homeomorphism of the circle:

$$f(x) = \begin{cases} \frac{1}{4} + 2x \pmod 1, & \text{if } x \in [0, \frac{1}{4}]; \\ \frac{5}{8} + \frac{x}{2} \pmod 1, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}]; \\ x + \frac{1}{4} \pmod 1, & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

What is the rotation number of f?

4) (10) Consider the following map of a torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$:

$$f : \mathbb{T}^2 \to \mathbb{T}^2, \quad f(x, y) = (2x, 3y) \pmod 1.$$

Prove hat f is topologically mixing and periodic points of f are dense in $\mathbb{T}^2$. Can you also find $\#Per_n(f)$?

5) (10) Find the number of periodic points of period n for the topological Markov chain determined by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$