**Ma 4, Introduction to Mathematical Chaos**

**Spring 2006**

## FINAL EXAM

*Due Thursday June 1, 2:30pm, 2006.*

*You may use your notes for this course, your solutions to course homework, and the textbooks. No collaboration or a direct assistance of other people.*

*All problems are weighted equally.*

*Time limit 3 hours. If you need more time, you should mark which part of the work was done after 3 hours; you will be given a partial credit for that work.*

1) (10) Find all the point of period three of the Bernoulli shift $\sigma : \Sigma_2 \to \Sigma_2$. Which of these points have prime period three? Which are in the same orbit?

2) (10) Does Sharkovski's Theorem apply to the topological Markov chains? Prove or give a counterexample.

3) (10) Sketch the phase curves of the vector field

$$\begin{cases} \dot{x} = & x^2 + y^2 - 1 \\ \dot{y} = & x^2 + y^2 - 1 \end{cases}$$

What is $\omega(p)$, if $p = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$? If $p = (0, 0)$? If $p = (-10, -10)$?

4) (10) Consider *the tent map* $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 2x, & \text{for } x \text{ in } [0, \frac{1}{2}]; \\ 2 - 2x, & \text{for } x \text{ in } [\frac{1}{2}, 1]. \end{cases}$$

Prove that

a) $T$ has $2^n$ periodic points of period $n$;

b) $T$ is transitive;

c) $T$ is topologically mixing;

d) $T$ has sensitive dependance on the initial conditions;

e) The map $h(x) = \frac{1}{2}\cos(\pi(1-x)) + \frac{1}{2}$ is a topological conjugacy between the map $f(x) = 4x(1-x)$ on $[0,1]$ and the tent map.

5) (10) Let $f : M \to M$ and $g : N \to N$ be continuous maps of metric spaces $(M, d_M)$ and $(N, d_N)$. Define a product metric space $M \times N$ as a space of pairs $(x, y), x \in M, y \in N$ with metric $d_{M\times N}((x_1, y_1), (x_2, y_2)) = \max\{d_M(x_1, x_2), d_N(y_1, y_2)\}$. Define the product map $f \times g : M \times N \to M \times N$ by $f \times g(x, y) = (f(x), g(y))$.

a) Assume that $f$ and $g$ are transitive. Does it imply that $f \times g$ is transitive?

b) Assume that $f$ and $g$ have dense sets of periodic orbits. Does it imply that the same holds for $f \times g$?

c) Assume that $f$ and $g$ are topologically mixing. Does it imply that $f \times g$ is also topologically mixing?