Ma 4, Introduction to Mathematical Chaos

Spring 2007

HOMEWORK # 6

Due Wednesday May 16, 1:00pm, 2007.

1) Let $Lip([0,1])$ be a space of Lipschitz functions $f : [0,1] \to \mathbb{R}$. Prove that $d(f_1, f_2) = \|f_1 - f_2\|_{Lip}$, where

$$\|g\|_{Lip} = \max_{x\in[0,1]} |g(x)| + \sup_{x\neq y} \frac{|g(x) - g(y)|}{|x - y|},$$

is a metric on $Lip([0,1])$.

2) Let $f : M \to M$, $M = S^1 \times D$, $\varphi \in S^1$, $D = \{(x,y)|\ x^2 + y^2 \leq 1\}$, be a solenoid map,

$$f(\varphi, x, y) = \left(2\varphi \,(\mathrm{mod}\ 1), \frac{x}{5} + \frac{1}{2}\cos 2\pi\varphi, \frac{y}{5} + \frac{1}{2}\sin 2\pi\varphi\right).$$

What is the number of periodic points of f of a given period n ($\#\mathrm{Per}_n(f)$)?

3) Suppose that all the eigenvalues of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ have absolute value less than 1. Does it imply that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a contraction? Prove or give a counterexample.

4) Consider $U = [-1,1] \times [-1,1] \times [-1,1] \subset \mathbb{R}^3$ and a map $f : U \to \mathbb{R}^3$,

$$f(x,y,z) = \begin{cases} \left(\frac{1}{100}x + \frac{1}{2}, \frac{1}{100}y + \frac{1}{2}, 100z - 10\right) & \text{if } \ z \geq 0 \\ \left(\frac{1}{100}x - \frac{1}{2}, \frac{1}{100}y - \frac{1}{2}, 100z + 10\right) & \text{if } \ z < 0 \end{cases}$$

Prove that the set $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$ is a non-empty hyperbolic set.

5) For the map $f : U \to \mathbb{R}^3$ from the previous problem find the number of periodic points of $f|_\Lambda : \Lambda \to \Lambda$ of a period n. Prove that $f|_\Lambda : \Lambda \to \Lambda$ is transitive.