Ma 4, Introduction to Mathematical Chaos

Spring 2007

FINAL EXAM

Due Wednesday May 30, 1:00pm, 2007.

You may use your notes for this course, your solutions to course homework, and the textbooks. No collaboration or a direct assistance of other people.

All problems are weighted equally.

Time limit 3.5 hours. If you need more time, you should mark which part of the work was done after 3.5 hours; you will be given a partial credit for that work.

1) Let $f : M \to M$ and $g : N \to N$ be continuous maps of metric spaces (M, d_M) and (N, d_N). Define a product metric space $M \times N$ as a space of pairs $(x, y), x \in M, y \in N$ with the metric $d_{M\times N}((x_1, y_1), (x_2, y_2)) = \max\{d_M(x_1, x_2), d_N(y_1, y_2)\}$. Define the product map $f \times g : M \times N \to M \times N$ by $f \times g(x, y) = (f(x), g(y))$.

a) Assume that f and g are transitive. Does it imply that $f \times g$ is transitive?

b) Assume that f and g have dense sets of periodic orbits. Does it imply that the same holds for $f \times g$?

c) Assume that f and g are topologically mixing. Does it imply that $f \times g$ is also topologically mixing?

2) Suppose that $\sigma_A : \Sigma_A \to \Sigma_A$ and $\sigma_B : \Sigma_B \to \Sigma_B$ are transitive topological Markov chains (i.e. there exist $m_1, m_2 \in \mathbb{N}$ such that A^{m_1} and B^{m_2} are positive). Prove that the product $\sigma_A \times \sigma_B : \Sigma_A \times \Sigma_B \to \Sigma_A \times \Sigma_B$ is topologically conjugated to some topological Markov chain which is also transitive.

3) Let $L \subset \mathbb{R}^2$ be a line $L = \{(x, y)|\ y = 0\}$. Denote by S a half circle $S = \{(x, y)|\ x^2 + (y - 1)^2 = 1, y < 1\}$. For any $x \in \mathbb{R}$ define $P(x) \in S$ as the point of intersection of S with the line that contains points $(x, 0)$ and $(0, 1)$ (i.e. $P : L \to S$ is a stereographic projection). Let $C \in [0, 1] \subset L$ be the standard Cantor set. Find the Hausdorff dimension of $P(C)$.

4) Show that $z_0 = e^{2\pi i/7}$ is a periodic point of period 3 for the map $Q_0(z) = z^2$. Is this periodic orbit attracting, repelling, or neutral?

5) Let $f : S^1 \to S^1$ be a continuous map of degree one that has a fixed point, $f(x_0) = x_0$ for some $x_0 \in S^1$. Assume also that f has a periodic point of prime period 3. Is it true that f must have periodic orbits of all periods? Prove or give a counterexample.